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RINGS WHOSE ADDITIVE ENDOMORPHISMS ARE RING ENDOMORPHISMS

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A ring R is said to be an AE-ring if every endomorphism of its additive group R^+ is a ring endomorphism. Clearly, the zero ring on any abelian group is an AE-ring. In a recent article, Birkenmeier and Heatherly characterised the so-called *standard* AE-rings, that is, the non-trivial AE-rings whose maximal 2-subgroup is a direct summand. The present article demonstrates the existence of non-standard AE-rings. Four questions posed by Birkenmeier and Heatherly are answered in the negative.

1. INTRODUCTION

In 1977, Sullivan posed the problem of characterising all rings R with the property that every endomorphism of its additive group R^+ is, in fact, a ring homomorphism [**9**]. It is convenient to call such a ring an *AE-ring* [**3**]. In 1981, Kim and Roush characterised all finite AE-rings [**7**], and in a recent paper Feigelstock extended this characterisation to the AE-rings R whose additive group is a torsion group [**3**]. Birkenmeier and Heatherly solved Sullivan's problem for the case that the 2-component R_2 of R^+ is a direct summand [**1**]. Without explicitly addressing the problem, they hinted that this need not always be the case [**1**, Theorem 8(ii)], and posed four questions:

QUESTION I. Are all AE-rings commutative?

QUESTION II. Is every subdirectly irreducible homomorphic image of an AE-ring also an AE-ring?

QUESTION III. Is every homomorphic image of an AE-ring an AE-ring?

QUESTION IV. If R is an AE-ring in which $x^2 = 0$ for each $x \in R$, is $R^2 = 0$?

We will show that the answer to each of these questions is negative. For this, we need to consider AE-rings R whose 2-component is not a direct summand (we will term such AE-rings *non-standard*). It is shown that any non-standard AE-ring R must be close to a zero ring in the sense that $R \cdot (tR + 2R) = 0 = (tR + 2R) \cdot R$ where tR denotes the maximal torsion subgroup of R^+, and $R^3 = 0$. An example will

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be constructed which demonstrates that non-standard AE-rings exist which are not zero rings. It should be noted that a similar construction yields an abelian group G which supports $2^{2^{\aleph_0}}$ pairwise non-isomorphic AE-rings. Thus, there is little hope for a complete solution to Sullivan's problem.

Abelian group notation will follow Fuchs' monographs [4, 5]. In particular, $o(a)$ denotes the order of an element a in the group A, and $|A|$ is the order of A; the subgroup of A consisting of all elements of 2-power order is denoted by A_2, and $A[2]$ is the subgroup consisting of all elements of order at most 2. As is customary, $2^\omega A = \bigcap_{n<\omega} 2^n A$, and $R = I \dotplus J$ denotes the ring direct sum of the ideals I and J. For ease of reference, we collect some results due to Feigelstock:

Lemma 1.1. [3] *Let R be an AE-ring. Then*

(1) *If $R^+ = A \oplus B$ then A and B are ideals of R and $R = A \dotplus B$.*

(2) *$R^2 \subseteq R[2]$.*

(3) *If $R^2 \neq 0$ then R_2 is reduced.*

2. Standard AE-Rings

Given an abelian group A, *a ring on* A is a ring R such that $R^+ = A$. The zero ring on A is the ring R on A with trivial multiplication: $R^2 = 0$. Obviously, the zero ring on any abelian group A is an AE-ring. By a *non-trivial* AE-ring we shall mean an AE-ring R with $R^2 \neq 0$.

Birkenmeier and Heatherly characterised the AE-rings R whose 2-component is a direct summand [1, Theorem 4]. We have an alternate condition:

Proposition 2.1. *Let R be an AE-ring such that $R^2 \neq 0$. Then R_2 is a direct summand of R if and only if $R^2 \not\subseteq 2^\omega R$.*

Proof: By [1, Theorem 4], R_2 being a direct summand of R implies R_2 bounded so that $2^\omega R = 0$. Conversely, assume $R^2 \not\subseteq 2^\omega R$. Frequent use will be made of 1.1. Let x and y be elements of R such that $xy \notin 2^\omega R$. Then $xy \in R[2]$ has finite 2-height $n - 1 \geqslant 0$. If $xy = 2^{n-1}c$ for some $c \in R$ then c has order 2^n and [4, p.117, 27.1] implies $R^+ = \langle c \rangle \oplus W$. Hence $R = \langle c \rangle \dotplus W$. Let p and q be integers and $v, w \in W$ such that $x = pc + v$ and $y = qc + w$. Then $2^{n-1}c = xy = pqc^2 + vw$ which implies pq is odd and $c^2 = 2^{n-1}c$. Assume, by way of contradiction, that R_2 is unbounded. Since R_2 is reduced, there exists a decomposition $W = \langle d \rangle \oplus X$ with $o(d) = 2^m$ for some $m \geqslant 2n$. It follows that there is $f \in End(R^+)$ such that $f(d) = c$ and $f(c) = 0$. Lemma 1(ii) of [1] implies $c^2 = 0$ which is a contradiction. By [1, Corollary 5], R_2 is a direct summand. ∎

Theorem 2.2. *Let R be a ring. Then R is an AE-ring with $R^2 \not\subseteq 2^\omega R$ if and*

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only if

$$R = \langle c \rangle \dotplus S \dotplus N$$

with $o(c) = 2^n$, n *a positive integer,* $c^2 = 2^{n-1}c$, S *and* N *zero rings with* $2^{n-1}S = 0$, $N_2 = 0$ *and* $N = 2N$.

PROOF: Again, 1.1 will be used without mention. Assume, firstly, that R is an AE-ring such that $R^2 \not\subseteq 2^\omega R$. By 2.1, we may apply Theorem 4 of [**1**]. Using the notation of [**1**], it remains to show that N is 2-divisible. Since $N_2 = 0$, the torsion subgroup tN of N is 2-divisible. Assume $N \neq 2N$. Then N/tN is a torsion-free abelian group which is not 2-divisible and as such has a quotient group isomorphic to $Z(2^n) \simeq C$. Let $g \in Hom(N^+, R_2^+)$ with $g(N) = C$. By [**1**, Theorem 4(iii)], $g(N) \cdot R_2 = 0$ contradicting $C \cdot C \neq 0$. Thus $N = 2N$. For the reverse implication, assume R is as stated. Then R_2 is a direct summand, R_2 is bounded, and the 2-divisibility of N implies $Hom(N^+, R_2^+) = 0$. By [**1**, Theorem 4] R is an AE-ring. ∎

We shall call the AE-rings described in 2.1 the ***standard*** AE-rings. Thus R is standard if $R^2 \not\subseteq 2^\omega R$. Every non-trivial torsion AE-ring is standard [**3**].

One verifies that an abelian group A which supports one standard AE-ring will support no other AE-ring except for the zero ring. This will be different in the case of non-standard AE-rings.

3. NON-STANDARD AE-RINGS

An AE-ring R is called ***non-standard*** if $R^2 \subseteq 2^\omega R$ *and* R is not trivial, that is, $R^2 \neq 0$. We have the following result.

PROPOSITION 3.1. *Let* R *be a non-standard* AE*-ring. Then necessarily* $R \cdot (tR + 2R) = 0 = (tR + 2R) \cdot R$. *In particular,* $R^3 = 0$.

PROOF: Assume, by way of contradiction, there exist elements $x \in tR$ and $r \in R$ such that $R \cdot (x + 2r) \neq 0$. Since $2R^2 = 0$ it follows that $R \cdot x \neq 0$ and $R \cdot x = R \cdot a$ for some $a \in R$ of 2-power order. Pick $c \in R_2$ of minimal order satisfying $R \cdot c \neq 0$. Let $o(c) = 2^n$. Then n is positive and, by 1.1(2), c has height zero. We claim that $\langle c \rangle$ is a direct summand of R^+. By [**4**, p.117, 27.1], it suffices to show that $\langle c \rangle \cap 2^n R = 0$. Assume there exists an integer m and an element $s \in R$ such that $2^m c = 2^n s \neq 0$. Then $1 \leqslant m < n$ which implies $c - 2^{n-m}s \in R_2$ of order at most 2^m. Since $R \cdot (c - 2^{n-m}s) = R \cdot c \neq 0$, this contradicts the minimality of the order of c. Thus $\langle c \rangle$ is a direct summand of R^+. From 1.1(1) we have $0 \neq R \cdot c \subseteq \langle c \rangle$. Since $\langle c \rangle \cap 2^\omega R = 0$, this is a contradiction. Similarly, $(tR + 2R) \cdot R = 0$. ∎

In order to demonstrate the existence of non-standard AE-rings the following theorem is needed. Given two abelian groups G and H, the set of all homomorphisms

$\phi : G \to H$ with $Im\phi$ a bounded torsion group is denoted by $Hom_b(G, H)$. Throughout, P denotes Prüfer's 2-group and $a \in P$ has the property that $\langle a \rangle = 2^\omega P \simeq Z(2)$ [4, p. 150]. The set of all endomorphisms of a group G which are integral multiplications is denoted by $\mathbf{Z} \cdot 1_G$.

Theorem 3.2. *Let G be an abelian group such that $tG = P$, $G \neq tG + 2G$, and assume that $EndG = \mathbf{Z} \cdot 1_G \oplus Hom_b(G, P)$. Then G supports a non-standard AE-ring. In fact:*

(1) *if K is a subgroup of G containing $tG + 2G$ and X is a set of elements in G which is minimal with respect to the set $\{x + K \mid x \in X\}$ being a basis of G/K, then, given any map $f : X \times X \to 2^\omega P$, there exists an AE-ring R with $R^+ = G$ such that $R \cdot K = 0 = K \cdot R$ and $x \cdot y = f(x, y)$ for all $(x, y) \in X \times X$;*

(2) *if $G/(tG + 2G)$ has infinite dimension δ over $\mathbf{Z_2}$, then there exist 2^δ pairwise non-isomorphic AE-rings on G.*

Proof: Let $\pi : G \to G/K$ denote the natural epimorphism, and consider any $\eta \in Hom(G/K \otimes G/K, 2^\omega P)$. Define $\mu : G \otimes G \to 2^\omega P$ by $\mu = \eta \circ (\pi \otimes \pi)$. Then μ is a homomorphism and, defining a multiplication $\cdot_\eta$ on G by $g \cdot_\eta h = \mu(g \otimes h)$ for all $g, h \in G$ makes G into a ring $R = R_\eta$ (which, in general, need not be associative) [5, p. 281]. In our case, $g \cdot_\eta h = \eta[(g + K) \otimes (h + K)]$ which implies $R \cdot_\eta K = 0 = K \cdot_\eta R$; by construction, $R \cdot_\eta R \subseteq Im\eta \leqslant 2^\omega P \leqslant K$ Hence, $R^3 = 0$ which implies that R, in fact, is associative. We claim that, for every $\eta \in Hom(G/K \otimes G/K, 2^\omega P)$, the ring $R = R_\eta$ is an AE-ring. Let ε be an endomorphism of $R^+ = G$. By hypothesis, $\varepsilon = n \cdot 1_G + \beta$ with $n \in \mathbf{Z}$ and $Im\beta$ bounded. It follows that $\beta(2^\omega P) = 0$ and $\beta(G) \subseteq K$. Let $g, h \in G$. Then, skipping the subscript η, we have $\varepsilon(g) \cdot \varepsilon(h) = (ng + \beta(g)) \cdot (nh + \beta(h)) = n^2 gh$, and $\varepsilon(gh) = ngh + \beta(gh) = ngh$. Since $gh \in 2^\omega P = \langle a \rangle$ and $na = n^2 a$ for all integers n, we have shown that R is an AE-ring. In order to verify (1), let $f : X \times X \to 2^\omega P$ be a map. Since the set $B = \{\pi(x) \otimes \pi(y) | (x, y) \in X \times X\}$ is a basis for the vector space $G/K \otimes G/K$ [5, p.255, (I) and (H)] and $(x, y) \mapsto \pi(x) \otimes \pi(y)$ defines a bijection between $X \times X$ and B, there exists a homomorphism $\sigma : G/K \otimes G/K \to 2^\omega P \simeq Z(2)$ such that, for all $(x, y) \in X \times X$, $\sigma[\pi(x) \otimes \pi(y)] = f(x, y)$. If $S = R_\sigma$ then S is an AE-ring on G and, for all $x, y \in X$, we have $x \cdot_\sigma y = \sigma[\pi(x) \otimes \pi(y)] = f(x, y)$ as claimed. In order to verify (2), note that

$$Hom(G/K \otimes G/K, 2^\omega P) \simeq \prod_{(x,y) \in X \times X} Hom(\langle \pi(x) \rangle \otimes \langle \pi(y) \rangle, \langle a \rangle)$$

which has cardinality 2^δ if $|X| = \delta$ is infinite. Let $\eta, \sigma \in Hom(G/K \otimes G/K, \langle a \rangle)$. Assume the resulting AE-rings R_η and R_σ are isomorphic and let α be a ring isomorphism between them. Then α would have to be an automorphism of the underlying additive group, G, in particular $\alpha = n \cdot 1_G + \beta$ with $\beta(G)$ bounded. It follows that $\beta(a) = 0$, $\beta(G) \leqslant K$ and $a = \alpha(a) = n(a)$ so that n is odd. Being a ring isomorphism, we have $\alpha(g \cdot_\eta h) = \alpha(g) \cdot_\sigma \alpha(h)$ for all $g, h \in G$ and consequently $\eta(\pi(g) \otimes \pi(h)) = n\eta(\pi(g) \otimes \pi(h)) = n(g \cdot_\eta h) = \alpha(g \cdot_\eta h) = (ng + \beta(g)) \cdot_\sigma (nh + \beta(h)) = (ng) \cdot_\sigma (nh) = n^2(g \cdot_\sigma h) = (g \cdot_\sigma h) = \sigma(\pi(g) \otimes \pi(h))$. It follows that $\eta = \sigma$. This completes the proof. ∎

The last section of our paper is devoted to the proof of

Proposition 3.3. *There exists an abelian group G satisfying the hypothesis of 3.2 such that $G/(tG + 2G)$ has rank two.*

Proposition 3.3 enables us to answer all four question posed in the Introduction in the negative:

Corollary 3.4. *Not every AE-ring is commutative.*

Corollary 3.5. *There exists an AE-ring R such that $r^2 = 0$ for all $r \in R$ but $R^2 \neq 0$.*

Proof of 3.4 and 3.5: Let G be the group of 3.3. Put $tG + 2G = K$ and let $x, y \in G$ such that $G/K = \langle x + K\rangle \oplus \langle y + K\rangle$. By 3.2, there exists an AE-ring R with $R^+ = G$ such that $x \cdot y = a$ and $y \cdot x = 0$. Thus R is not commutative, proving 3.4. Similarly, there exists and AE-ring S with $S^+ = G$ such that $x^2 = 0 = y^2$ and $x \cdot y = a = y \cdot x$. Let $r \in R$. Then $r = mx + ny + k$ for some integers m and n and some $k \in K$, and

$$r \cdot r = m^2x^2 + mnx \cdot y + nmy \cdot x + n^2y^2 = 2mna = 0$$

proving 3.5. ∎

Thus, Questions I and IV of [1] have negative answers. As pointed out by Birkenmeier and Heatherly, an affirmative answer to II would imply that every AE-ring is commutative. This follows from the fact that every subdirectly irreducible AE-ring is commutative [**1**, Theorem 13(iii)] and the well known theorem that every ring R is a subdirect sum of a product of subdirectly irreducible quotient rings of itself [**8**, p. 129, Theorem 34]. It thusly follows from 3.4 that Question II has a negative answer as well (and, hence, so does III).

In order to give a concrete example we include

Proposition 3.6. *Not every subdirectly irreducible homomorphic image of an AE-ring is an AE-ring.*

Proof: Let G be the group of 3.3. Then there exists $x \in G$ and a subgroup A of G such that $tG + 2G \leqslant A$ and $G/A = \langle x + A\rangle \simeq Z(2)$. By 3.2, there exists an AE-ring R on G with $x^2 = a$ and $R \cdot A = 0 = A \cdot R$. Let B be a basic subgroup of P such that $P/B \simeq Z(2^\infty)$. Then $(P/B)[2] = \langle a+B\rangle$. Since $x \notin A$, the order of x must be infinite so that $\langle x+B\rangle \cap P/B = 0$. Divisible subgroups are absolute direct summands. Thus $G/B = P/B \oplus C/B$ for some subgroup C of G with $B \leqslant C$ and $x \in C$. Put $A \cap C = I$. Then I is an ideal of R and $x \notin I$ since $x \notin A$. Also, since $a \in P \setminus B$, we have $a \notin C$ so that $a \notin I$. By construction, $(x+I)^2 = x^2 + I = a + I \neq 0$ which shows that R/I is not a zero ring. Since $G = A + \langle x\rangle = A + C$ we have

$$A/I = A/(A\cap C) \simeq (A+C)/C = G/C \simeq (G/B)/(C/B) \simeq P/B \simeq Z(2^\infty).$$

Thus $(R/I)^+ = G/I = A/I + \langle x+I\rangle$ and $x \notin I$ but $2x \in A\cap C = I$ which implies $(R/I)^+ = A/I \oplus \langle x+I\rangle \simeq Z(2^\infty) \oplus Z(2)$. It follows from 1.1(3) that R/I cannot be an AE-ring. In order to verify that R/I is subdirectly irreducible, it suffices to show that every nonzero ideal of R/I contains $\langle a+I\rangle$. Since $a\cdot R = 0 = R\cdot a$, $\langle a+I\rangle$ is clearly an ideal and $a+I\neq 0$. Let J be a nonzero ideal of R/I. Then there exist $y\in A$ and an integer n such that $0\neq y+nx+I\in J$, and we can choose $n=1$ or $n=0$. If $n=1$ then $(y+x+I)(x+I) = a+I \in J$ as claimed. Suppose $n=0$. Then $y\in A\setminus C$ and $G = P + C$ with $P \leqslant A$ implies $y = z + i$ with $0\neq z\in P$ and $i = y - z \in C\cap A = I$. Since $y\notin I$ we have $z\notin B$. From $(P/B)[2] = \langle a+B\rangle$ we infer $2^m z + B = a + B$ for some positive integer m. Thus

$$a + I = 2^m z + I = 2^m(z+i) + I = 2^m y + I \in J$$

completing the proof. ∎

4. The Proof of 3.3

Throughout, we let $T = \bigoplus_{n<\omega} Z(2^n)$, and $\widehat{T}$ denotes the 2-adic completion of T. Then $T \leqslant \widehat{T} \leqslant \prod_{n<\omega} Z(2^n)$, and every endomorphism ϕ of T can be extended uniquely to an endomorphism of $\widehat{T}$. Let $\pi : \widehat{T} \to \widehat{T}/T$ denote the natural epimorphism. Note that the group $\widehat{T}/T$ is divisible.

We start with a general construction using ideas from [2].

A. Let B be any torsion-free group such that $End B = \mathbf{Z}\cdot 1_B$, and let $f : B \to \widehat{T}$ be a *set* function. If $\pi\circ f \in Hom\left(B, \widehat{T}/T\right)$ is a homomorphism we say that f is *eligible*. Assume f is eligible. Define

$$B_f = \{(t + f(b), b) | t \in T, b \in B\} \subseteq \widehat{T} \oplus B.$$

One verifies the following. We will identify $\widehat{T}$ with $\widehat{T} \oplus 0$.

4.1. *B_f is a subgroup of $\widehat{T} \oplus B$ with torsion part $tB_f = T$, and $B_f/tB_f \simeq B$.*

4.2. *$End B_f = \mathbf{Z} \cdot 1_{B_f} \oplus Hom(B_f, T)$.*

4.3. *If $g : B \to \widehat{T}$ is a set function such that $\pi \circ g = \pi \circ f$ then $B_g = B_f$.*

4.4. *B_f splits if and only if there exists a homomorphism $\eta : B \to \widehat{T}$ such that $\pi \circ f = \pi \circ \eta$.*

For $\phi \in Hom(T, T)$, its unique extension to $\widehat{T}$ will be denoted by $\widehat{\phi}$. Note that f eligible implies $\widehat{\phi} \circ f$ eligible.

4.5. *Given $\phi \in Hom(T, T)$, there exists $\theta \in Hom(B_f, T)$ extending ϕ if and only if $B_{\widehat{\phi} \circ f}$ splits.*

PROOF: Suppose, firstly, that $B_{\widehat{\phi} \circ f}$ splits. It follows that there exists a homomorphism $\sigma : B_{\widehat{\phi} \circ f} \to T$ such that $\sigma \circ \iota = 1_T$ where $\iota : T \to B_{\widehat{\phi} \circ f}$ denotes the inclusion map. One verifies that $\widehat{\phi} \oplus 1_B \in End\left(\widehat{T} \oplus B\right)$ induces a homomorphism from B_f to $B_{\widehat{\phi} \circ f}$. Let $\theta = \sigma \circ \left(\widehat{\phi} \oplus 1_B\right)|B_f$. Conversely, suppose there exists a θ as stated. In particular, $\theta(t, 0) = \phi(t)$ for all $t \in T$. Define $\eta : B \to \widehat{T}$ by $\eta(b) = \widehat{\phi}(f(b)) - \theta(f(b), b)$ for all $b \in B$. Then $\pi \circ \eta = \pi \circ \widehat{\phi} \circ f$. By (4), it suffices to show η is a homomorphism. Let $b, b' \in B$. There exists $t \in T$ such that $f(b + b') = f(b) + f(b') + t$. Hence

$$\eta(b + b') = \widehat{\phi}(f(b) + f(b') + t) - \theta(f(b) + f(b') + t, b + b') = \eta(b) + \eta(b').$$

∎

B. We now specify the torsion-free group B. Let J_2 denote the ring of 2-adic integers, and let $\rho \in J_2$ be transcendental over the rational integers $\mathbf{Q}$. For each natural number n we have

$$\rho = s_n + 2^n \rho_n$$

with $s_n \in \mathbf{Z}$ and $\rho_n \in J_2$. Let

$$B = \langle \{1\} \cup \{\frac{\rho - s_n}{2^n} | n < \omega\} \rangle,$$

and let R be a subring of $\mathbf{Z}_{(2)}$, the integers localised at 2, containing $\mathbf{Z}$. The set $RB = \{rb | r \in R, b \in B\} \subseteq J_2$ is an additive group.

4.6. *$End(RB) = R \cdot 1_{RB}$.*

PROOF: Let ε be an endomorphism of RB and $\gamma = \varepsilon(1) \in RB$. Since, for each n, $\rho = s_n + 2^n \rho_n$, $\varepsilon(\rho) = \varepsilon(s_n) + 2^n \varepsilon(\rho_n) = s_n \varepsilon(1) + 2^n \rho'_n = s_n \gamma + 2^n \rho'_n$. Hence $\varepsilon(\rho) = \lim_{n \to \infty} s_n \gamma = \rho\gamma$. It follows that ε is the multiplication by γ. There exists a positive integer k such that $2^k \gamma = r + t\rho$ with r and t in R. Then $\varepsilon(2^k \rho) = 2^k \rho\gamma = r\rho + t\rho^2$. Since ρ is transcendental over $\mathbf{Q}$, we have $t = 0$ and $\gamma \in \mathbf{R}$. ∎

Let $\Phi = EndT \setminus Hom_b(T,T)$ denote the set of all endomorphisms of T with unbounded image. Then Φ has cardinality $2^{\aleph_0}$ so that we can fix an enumeration

$$\Phi = \{\phi_\alpha | \alpha < 2^\omega\}.$$

4.7. *For each $\alpha < 2^\omega$, there exists an eligible set function $f_\alpha : B \to \widehat{T}$ such that $\phi_\alpha \in \Phi$ cannot be extended to a homomorphism from $(B)_{f_\alpha}$ to T.*

PROOF: Let $\phi = \phi_\alpha \in \Phi$. First of all, note that there exists an element $x = x_\alpha$ in $\widehat{T}$ such that $\widehat{\phi}(x) \in \widehat{T} \setminus T$: being an unbounded subgroup of T, the image of ϕ contains a subgroup $C = \bigoplus_{i<\omega} \langle \phi(t_i) \rangle$ with $2^{2i}\phi(t_i) \neq 0$ and $o(\phi(t_i)) < o(\phi(t_{i+1}))$ for all i; if $x = \lim_{n\to\infty}(2t_1 + \cdots + 2^n t_n)$ then $\widehat{\phi}(x)$ has infinite order. Since $F = \langle 1, \rho \rangle$ is a free subgroup of B, there exists a homomorphism μ from F to $\widehat{T}$ such that $\mu(1) = 0$ and $\mu(\rho) = x$. Then $\pi \circ \mu : F \to \widehat{T}/T$ is a homomorphism which, since $\widehat{T}/T$ is divisible, can be extended to a homomorphism $\psi \in Hom\left(B, \widehat{T}/T\right)$. For each $b \in B \setminus F$ choose $t_b \in \widehat{T}$ such that $\psi(b) = \pi(t_b)$. The mapping $b \mapsto t_b$ extends μ to a function $f = f_\alpha : B \to \widehat{T}$ with $f|F = \mu$ which is eligible. Assume ϕ can be extended to a homomorphism from B_f to T. By 4.4 and 4.5, there exists $\eta \in Hom\left(B, \widehat{T}\right)$ such that $\pi \circ \widehat{\phi} \circ f = \pi \circ \eta$. Hence, $\pi\eta(1) = \pi\widehat{\phi}f(1) = \pi\widehat{\phi}(0) = 0$ which implies $\eta(1) = s \in T$; similarly, $\pi\eta(\rho) = \pi\widehat{\phi}(x)$ so that $\eta(\rho) = \widehat{\phi}(x) + t$ for some $t \in T$. As before, $\eta(\rho) = \eta(s_n + 2^n\rho_n) = s_n\eta(1) + 2^n\rho_n'$ for all n. It follows that $\widehat{\phi}(x) + t = \lim_{n\to\infty} s_n s = s\rho \in T$ which contradicts $\widehat{\phi}(x) \notin T$. ◻

C. Let K_2 denote the field of 2-adic numbers. Since K_2 has transcendence degree $2^{\aleph_0}$ over the rationals, there exists a subset $\Pi \subseteq J_2$ of cardinality $2^{\aleph_0}$ which is algebraically independent over $\mathbf{Z}$. Fix an enumeration

$$\Pi = \{\pi_\alpha \mid \alpha < 2^\omega\},$$

and let, for each α and each natural number n, $\pi_\alpha = s_n^\alpha + 2^n\rho_n^\alpha$ with $s_n^\alpha \in \mathbf{Z}$ and $\rho_n^\alpha \in J_2$. For each $\alpha < 2^\omega$, let B_α be the group B constructed in **B** with $\rho = \pi_\alpha$. Well known set theoretical arguments show the existence of a family $\mathcal{F}$ of sets of rational primes with the following properties: (i) no set in $\mathcal{F}$ contains the prime 2 or the prime 3; (ii) no set in $\mathcal{F}$ is properly contained in another one; and (iii) $\mathcal{F}$ has cardinality $2^{\aleph_0}$ [6]. (The following argument shows that every countably infinite set S has a family $\mathcal{F}$ of subsets satisfying (ii) and (iii): given $n \in \mathbf{N}$, let S_n denote the set of all functions $f : \{1, \ldots, n\} \to \mathbf{N}$. Then each S_n is countable and so is their union $S = \bigcup_{n\in\mathbf{N}} S_n$. Let $T = \mathbf{N}^\mathbf{N}$ be the set of all functions from $\mathbf{N}$ to $\mathbf{N}$. Then $|T| = 2^{\aleph_0}$. For $g \in T$, let $I(g) = \{f \in S | f|\{1, \ldots, n\} = g|\{1, \ldots, n\}$ for some $n \in \mathbf{N}\}$. Let $\mathcal{F} = \{I(g) | g \in T\}$.) Again, choose an indexing $\mathcal{F} = \{\Delta_\alpha | \alpha < 2^\omega\}$. As customary, for p a prime, $\mathbf{Q}^{(p)}$ denotes the set of all rational numbers with denominator a power of

p. For each $\alpha < 2^\omega$, let $\mathbf{Q}_\alpha = \sum\{\mathbf{Q}^{(p)} | p \in \Delta_\alpha\}$, a subring of $\mathbf{Q}$. Define a subgroup H of the external direct sum $\bigoplus_{\alpha<2^\omega} K_2$ as follows: if e_β denotes the vector with 1 in the beta-th coordinate and zeros elsewhere, we let

$$H = \bigoplus_{\alpha<2^\omega} \mathbf{Q}_\alpha B_\alpha e_\alpha + \sum_{1<\alpha<2^\omega} \mathbf{Q}^{(2)}(e_\alpha - e_0) + \mathbf{Q}^{(3)}(e_1 - e_0).$$

Let $\sigma = t(\mathbf{Q}^{(2)})$ denote the type of $\mathbf{Q}^{(2)}$, let $\tau = t(\mathbf{Q}^{(3)})$ and let $\tau_\alpha = t(\mathbf{Q}_\alpha)$. One verifies the following. For t a type, $H(t)$ denotes the (fully invariant) subgroup of H consisting of all elements of type greater than or equal to t. The pure subgroup generated by a subgroup A is denoted by A_*.

4.8. *The following hold:*

(1) $H(\sigma) = \left(\sum_{1<\alpha<2^\omega} \mathbf{Q}^{(2)}(e_\alpha - e_0)\right)_*$.

(2) $H(\tau) = \left(\mathbf{Q}^{(3)}(e_1 - e_0)\right)_*$.

(3) *For each* $\alpha < 2^\omega$, $H(\tau_\alpha) = \mathbf{Q}_\alpha B_\alpha e_\alpha$.

PROOF OF (1): Put $A = \sum_{1<\alpha<2^\omega} \mathbf{Q}^{(2)}(e_\alpha - e_0)$. Then $A \leqslant H(\sigma)$. Let $w \in H$ be an element of infinite 2-height. Without loss of generality, we may assume that w is an element of $\bigoplus_{\alpha<2^\omega} \mathbf{Q}_\alpha B_\alpha e_\alpha + \mathbf{Q}^{(3)}(e_1 - e_0)$. In generalised vector notation,

$$w = \left(r_\alpha + t_\alpha \rho^\alpha_{n_\alpha} + p_\alpha\right)$$

with $r_\alpha, t_\alpha \in \mathbf{Q}_\alpha$, n_α positive integers, $p_\alpha \in \mathbf{Q}^{(3)}$, $p_0 + p_1 = 0$ and $p_\alpha = 0$ for all $\alpha > 1$. In fact, we may assume that all of $r_\alpha, t_\alpha, p_\alpha$ are integers, and $p_\alpha = 0$ for all α. Also, since only finitely many components of w are nonzero, it is possible to write w such that $n_\alpha = n_\beta = n \in \mathbf{N}$ for all α and β. By hypothesis, given any $m \in \mathbf{N}$, there exists $y_m \in H$ such that $2^m y_m = w$, and

$$y_m = \left(r^m_\alpha + t^m_\alpha \rho^\alpha_{n_m} + q^m_\alpha + p^m_\alpha\right)$$

with $r^m_\alpha, t^m_\alpha \in \mathbf{Q}_\alpha$, $q^m_\alpha \in \mathbf{Q}^{(2)}$, $q^m_1 = 0$, $\sum_\alpha q^m_\alpha = 0$, $p^m_\alpha \in \mathbf{Q}^{(3)}$, $p^m_0 + p^m_1 = 0$ and $p^m_\alpha = 0$ for all $\alpha > 1$. Thus, for all α and all m,

$$2^m\left(r^m_\alpha + t^m_\alpha \rho^\alpha_{n_m} + q^m_\alpha + p^m_\alpha\right) = r_\alpha + t_\alpha \rho^\alpha_n$$

which implies

$$2^{m+n+n_m}(r^m_\alpha + q^m_\alpha + p^m_\alpha) + 2^{m+n} t^m_\alpha\left(\pi_\alpha - s^\alpha_{n_m}\right) = 2^{n+n_m} r_\alpha + 2^{n_m} t_\alpha(\pi_\alpha - s^\alpha_n).$$

The linear independence of the π_α over $\mathbf{Q}$ implies, for all α and all m,

(i) $$2^{m+n} t^m_\alpha = 2^{n_m} t_\alpha$$

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and, substituting,

(ii) $$2^{m+n+n_m}(r_\alpha^m + q_\alpha^m + p_\alpha^m) - 2^{n_m} t_\alpha s_{n_m}^\alpha = 2^{n+n_m} r_\alpha - 2^{n_m} t_\alpha s_n^\alpha .$$

Assume, by way of contradiction, that $t_\beta \neq 0$ for some β. Then $t_\beta = (2^k p)/q$ for some integers $k \geqslant 0$ and p and q odd. Since t_β^m has odd denominator, (i) implies $n_m + k \geqslant m + n$. Note that n and k are fixed. Thus

(iii) $$\lim_{m\to\infty} n_m = \infty .$$

From (ii) we obtain

(iv) $$2^{m+n} q_\alpha^m = 2^n r_\alpha - t_\alpha s_n^\alpha + t_\alpha s_{n_m}^\alpha - 2^{m+n}(r_\alpha^m + p_\alpha^m).$$

Since, for all m, the denominator of $r_\alpha^m + p_\alpha^m$ is odd, (iii) and (iv) imply

(v) $$\lim_{m\to\infty} (2^{n+m} q_\alpha^m) = 2^n r_\alpha - t_\alpha s_n^\alpha + t_\alpha \pi_\alpha .$$

For each m, let $s_m = \sum_\alpha 2^{n+m} q_\alpha^m$. Since $\sum_\alpha q_\alpha^m = 0$, each s_m is zero. Using (v) one verifies

$$0 = \lim_{m\to\infty} s_m = \sum_\alpha (2^n r_\alpha - t_\alpha s_n^\alpha + t_\alpha \pi_\alpha),$$

and the linear independence of the $\{\pi_\alpha\}$ over $\mathbf{Q}$ implies $t_\alpha = 0$ for each α. Because of (i), all t_α^m are zero and from (iv) we infer $0 = 2^m \sum_\alpha q_\alpha^m = \sum_\alpha r_\alpha - 2^m \sum_\alpha (r_\alpha^m + p_\alpha^m)$. It follows that $\sum_\alpha r_\alpha$ has infinite 2-height in the ring $\mathbf{Q}^{(3)} + \sum_\alpha \mathbf{Q}_\alpha$ which implies $\sum_\alpha r_\alpha = 0$. Thus, $w = \sum_\alpha r_\alpha e_\alpha = \sum_\alpha r_\alpha (e_\alpha - e_0) = z + k(e_1 - e_0)$ with $z = \sum_{1<\alpha<2^\omega} r_\alpha (e_\alpha - e_0) \in \sum_{1<\alpha<2^\omega} \mathbf{Z}(e_\alpha - e_0) \leqslant H(\sigma)$ and $k = r_1 \in \mathbf{Z}$. Hence, for all m, $k/2^m \in \mathbf{Q}_1 + \mathbf{Q}^{(3)}$ which implies $k = 0$. We have shown that $w \in A$. □

Proof of (2): Obviously, $\mathbf{Q}^{(3)}(e_1 - e_0) \leqslant H(\tau)$. Let $w = (r_\alpha + t_\alpha \rho_n^\alpha + q_\alpha) \in H(\tau)$ with integers $r_\alpha, t_\alpha, q_\alpha$; we may assume each q_α is zero. We use the same notation as before: there exist $y_m = (r_\alpha^m + t_\alpha^m \rho_{n_m}^\alpha + q_\alpha^m + p_\alpha^m) \in H$ such that $3^m y_m = w$. Corresponding to (i) we obtain $3^m 2^n t_\alpha^m = 2^{n_m} t_\alpha$ which shows that each t_α has infinite 3-height in $\mathbf{Q}_\alpha + \mathbf{Q}^{(2)}$. Hence $t_\alpha = 0$ for each α and $w = (r_\alpha)$, $y_m = (r_\alpha^m + q_\alpha^m + p_\alpha^m)$. For $\alpha > 1$, $p_\alpha^m = 0$ which implies that $r_\alpha = 3^m(r_\alpha^m + q_\alpha^m)$ has infinite 3-height in $\mathbf{Q}_\alpha + \mathbf{Q}^{(2)}$. It follows that $r_\alpha = 0$ for $\alpha > 1$. Hence $r_0 + r_1 = \sum_\alpha r_\alpha = \sum_\alpha 3^m (r_\alpha^m + q_\alpha^m + p_\alpha^m) = 3^m \left(\sum_\alpha r_\alpha^m + \sum_\alpha q_\alpha^m + \sum_\alpha p_\alpha^m \right) = 3^m \sum_\alpha r_\alpha^m$

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has infinite 3-height in $\sum_{\alpha} \mathbf{Q}_\alpha$. Thus, $r_0 + r_1 = 0$ and $w = r_1(e_1 - e_0) \in \mathbf{Z}(e_1 - e_0)$ as desired. ∎

PROOF OF (3): Fix $\beta < 2^\omega$ and let $w \in H(\tau_\beta)$. Assume, by way of contradiction, there exists an $\alpha \neq \beta$ belonging to the support of w. By hypothesis, there exists a prime p such that $p \in \Delta_\beta$ but $p \notin \Delta_\alpha$, and $p \neq 2, 3$. For each positive integer m, there exists $y_m \in H$ such that $p^m y_m = w$. Using the same notation as before, letting $w = (r_\alpha + t_\alpha \rho_n^\alpha + q_\alpha + p_\alpha)$ and $y_m = \left(r_\alpha^m + t_\alpha^m \rho_{n_m}^\alpha + q_\alpha^m + p_\alpha^m\right)$, the equation corresponding to (i) is $p^m 2^n t_\alpha^m = 2^{n_m} t_\alpha$ for all m which shows that $t_\alpha = 0 = t_\alpha^m$, and $r_\alpha + q_\alpha + p_\alpha = p^m(r_\alpha^m + q_\alpha^m + p_\alpha^m)$ has infinite p-height in $\mathbf{Q}_\alpha + \mathbf{Q}^{(2)} + \mathbf{Q}^{(3)}$. Thus, the α-th component of w is zero and $w = \left(r_\beta^m + t_\beta^m \rho_n^\beta + q_\beta^m + p_\beta^m\right) e_\beta$. Since, for $\alpha \neq \beta$, $r_\alpha + q_\alpha + p_\alpha = 0$, both q_α and p_α are integers. Hence, so are $q_\beta = -\sum_{\alpha \neq \beta} q_\alpha$ and $p_\beta = -\sum_{\alpha \neq \beta} p_\alpha$. It follows that $w \in \mathbf{Q}_\beta B_\beta e_\beta$. ∎

4.9. *$EndH = \mathbf{Z} \cdot 1_H$.*

PROOF: Let $\varepsilon \in EndH$. By 4.8(3), for each α, ε induces an endomorphism in $\mathbf{Q}_\alpha B_\alpha e_\alpha$ which, by 4.6, must be the multiplication by some $r_\alpha \in \mathbf{Q}_\alpha$. Pick $\beta > 1$. By 4.8(1), we have

$$\varepsilon(e_\beta - e_0) = r_\beta e_\beta - r_0 e_0 \in \left(\sum_{1 < \alpha < 2^\omega} \mathbf{Q}^{(2)}(e_\alpha - e_0) \right) * .$$

Thus, there exists a nonzero integer n such that $n(r_\beta e_\beta - r_0 e_0) = \sum_{1<\alpha} q_\alpha(e_\alpha - e_0)$. It follows that $r_\beta = n^{-1} q_\beta = r_0$. Similarly, using 4.8(2), $r_1 = r_0$. It follows that ε restricted to $\bigoplus_{\alpha < 2^\omega} \mathbf{Q}_\alpha B_\alpha e_\alpha$ is the multiplication by $r_0 \in \mathbf{Z}$. The latter subgroup being full in H shows $\varepsilon = r_0 \cdot 1_H$. ∎

4.10. *$H = \langle e_0 \rangle + \langle e_1 \rangle + 2H$, and $H/2H$ has rank two.*

PROOF: Let R be a subring of $\mathbf{Q}$ such that every element in R has odd denominator. Then $R = \mathbf{Z} + 2R$. Since $\rho_n^\alpha = 2\rho_{n+1}^\alpha$, it follows that $\mathbf{Q}^{(3)}(e_1 - e_0) \subseteq \langle e_0 \rangle + \langle e_1 \rangle + 2H$ and, for each $\alpha < 2^\omega$,

$$\mathbf{Q}_\alpha B_\alpha e_\alpha = \mathbf{Q}_\alpha \langle 1, \rho_n^\alpha \rangle e_\alpha \subseteq \mathbf{Q}_\alpha e_\alpha + 2H \subseteq \langle e_\alpha \rangle + 2H.$$

If $\alpha > 1$, $e_\alpha = e_0 + (e_\alpha - e_0) \in \langle e_0 \rangle + 2H$. Thus, $H = \langle e_0 \rangle + \langle e_1 \rangle + 2H$. In order to show e_0 and e_1 are linearly independent modulo $2H$, let a and b be integers such that $ae_0 + be_1 \in 2H$. Using the same symbolism as above, $ae_0 + be_1 = 2(r_\alpha + t_\alpha \rho_n^\alpha + q_\alpha + p_\alpha)$ and as before we must have $t_\alpha = 0$ for all α and $r_\alpha + q_\alpha = 0$ if $\alpha > 1$ so that q_α

must be an integer. This implies $q_0 = -\sum_{1\langle\alpha} q_\alpha$ is an integer and $2q_0$ is even. Since $q_1 = 0$, $2p_1 = b - 2r_1 \in \mathbf{Q}^{(3)} \cap \mathbf{Q}_1 = \mathbf{Z}$ which implies $p_1 = -p_0$ is an integer. Since $r_0 = (a - 2q_0 - 2p_0)/2$ has odd denominator, a must be even. Similarly, b must be even. ∎

4.11. *For every $\phi \in Hom(H,T)$, $Im\phi$ is bounded.*

PROOF: There exists a positive integer n such that $2^n\phi(e_i) = 0$ for $i = 1,2$. By 4.10, $\phi(H) = \langle\phi(e_0)\rangle + \langle\phi(e_1)\rangle + 2\phi(H)$ which implies $2^n\phi(H) = 2^{n+1}\phi(H) = 0$ since T is reduced. ∎

D. We are getting ready to construct our group G. By 4.7, for each $\alpha < 2^\omega$, there exists an eligible map $f_\alpha : B_\alpha \to \widehat{T}$ such that $\phi_\alpha \in \Phi$ does not extend to a homomorphism from $(B_\alpha)_{f_\alpha}$ to T. It follows that $\oplus f_\alpha : \bigoplus_{\alpha<2^\omega} B_\alpha \to \widehat{T}$ is an eligible map which, in turn, extends to an eligible map $f : H \to \widehat{T}$ since $\widehat{T}/T$ is divisible, hence injective. By 4.9, H satisfies the same hypotheses as the group B in part A. Let $K = H_f$. Then

4.12. *$EndK = \mathbf{Z}\cdot 1_K \oplus Hom_b(K,T)$.*

PROOF: By 4.2, it suffices to show every homomorphism from K to T has bounded image. Let $\phi \in Hom(K,T)$. By 4.1, $tK = T$; let $\psi = \phi|tK$ be the restriction map and assume, by way of contradiction, that ψ has unbounded image. Then $\psi = \phi_\alpha$ for some $\alpha < 2_\omega$. By construction, ϕ_α cannot be extended to $(B_\alpha)_{f_\alpha}$. Since $T \leqslant (B_\alpha)_{f_\alpha}$ and

$$(B_\alpha)_{f_\alpha} = \{(t + f(b), b) | t \in T, b \in B_\alpha\} \leqslant H_f = K,$$

ϕ_α does not extend to K which is a contradiction. Hence $2^n\phi(T) = 0$ and $2^n\phi$ induces a homomorphism $\eta : K/T \to T$ given by $\eta(x + T) = 2^n x$. It follows from 4.1 and 4.11 that η is bounded and, hence, so is ϕ. ∎

Let P be the Prüfer 2-group with $2^\omega P = \langle a\rangle$ as above. Then there is an exact sequence $0 \to \langle a\rangle \to P \to T \to 0$ the epimorphism of which induces an epimorphism $Ext(H,P) \to Ext(H,T)$. Thus, there exists a group G and homomorphisms such that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \eta\downarrow & & \| & & \\
0 & \longrightarrow & T & \longrightarrow & K & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

By the Five Lemma, η is an epimorphism, and $\langle a\rangle$ is the kernel of η. Hence $K \simeq G/\langle a\rangle$. One verifies that $G/(tG + 2G) \simeq H/2H$ which, by 4.10, has rank two. The group G will satisfy the hypothesis of 3.2 if it has the property that $EndG = \mathbf{Z}\cdot 1_G \oplus Hom_b(G,P)$.

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Let ε be an endomorphism of G. Since $\langle a\rangle = 2^{\omega}tG$ is fully invariant in G, ε induces an endomorphism $\overline{\varepsilon}$ in $G/\langle a\rangle \simeq K$. By 4.12, there exist integers m and n such that $2^m\overline{\varepsilon} = 2^m n \cdot 1_{G/\langle a\rangle}$ and m is positive. Hence $2^{m+1}(\varepsilon - n \cdot 1_G) = 0$ as desired.

Remark. *A more elaborate construction along the same lines yields a group G with $G/(tG + 2G)$ of dimension $2^{\aleph_0}$.*

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