Convergence criteria for Fourier series

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The following convergence criterion of Fourier series is due to M. Izumi, S. Izumi and the author:

THEOREM. Let $\Delta \geq 1$. If

(i) $\int_0^t \phi(u)du = o(t)$, and

(ii) $\int_0^\delta |d(u^{-\alpha}\phi(u))| \leq A t^{-\alpha}$ as $t \to 0$

for an $\alpha$, $0 < \alpha < 1$ and for a $\delta$, $0 < \delta < \pi$, then the Fourier series of $\phi(t)$ is convergent at the origin.

The object of this paper is to generalize the above theorem in the Hardy-Littlewood direction.

1.

Let $\phi(t)$ be an even periodic function which is integrable $L$ and let

$$\phi(t) \sim \sum_{n=1}^\infty a_n \cos nt.$$  

Sunouchi [4] generalized the Young-Pollard [3]-convergence criterion as follows:

THEOREM A. The Fourier series of $\phi(t)$ converges at the point $t = 0$ to the value zero, provided that there is a $\Delta \geq 1$ such that

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(1.1) \[ \int_0^t \phi(u)du = o(t^\delta) \]

and

(1.2) \[ \int_0^t |d(u^\alpha \phi(u))| = o(t), 0 \leq t \leq \eta. \]

Recently we [2] proved the following theorem:

**THEOREM B.** Let \( \Delta \geq 1 \). If the condition (1.1) holds and

(1.3) \[ \int_\delta^{\gamma/\Delta} |d(u^{-\alpha} \phi(u))| = o(t^{-\alpha}) \quad \text{as} \quad t \to 0 \]

for an \( \alpha, 0 < \alpha < 1 \) and for a \( \delta, 0 < \delta < \pi \), then the Fourier series of \( \phi \) is convergent at the origin.

Hardy and Littlewood [1] generalized the condition (1.1) for the case \( \Delta = 1 \) in the form

\[ \phi_\beta(t) = o(t^\gamma), \quad \text{as} \quad t \to 0 \]

for any \( \beta > 0 \), where \( \phi_\beta(t) \) is the \( \beta \)-th integral of \( \phi(t) \).

Corresponding to this result we prove the following theorem, which generalizes Theorem B in the Hardy-Littlewood direction.

**THEOREM.** Let \( \Delta = \gamma/\beta \geq 1 \) and \( 1 > \beta > 0 \). If

(1.4) \[ \phi_\beta(t) = o(t^\gamma) \]

where \( \phi_\beta(t) \) is the \( \beta \)-th integral of \( \phi(t) \), and further if

(1.5) \[ \int_\delta^{1/\Delta} |d(u^{-\eta} \phi(u))| = o(t^{-\eta}), \quad 1 > \eta > 0, \]

and \( \Delta > 1 \), then the Fourier series of \( \phi(t) \) converges at \( t = 0 \).

2.

Proof of the theorem. To prove our theorem we need to show that
Putting \( n^{-1/\Delta} = \alpha \), we have

\[
\int_0^\delta \frac{\theta(t) \sin nt}{t} \, dt = \int_0^\alpha \frac{\theta(t) \sin nt}{t} \, dt + \int_\alpha^{\delta} \frac{\theta(t) \sin nt}{t} \, dt
\]

\[
= I + J ,
\]
say.

Putting \( \theta(t) = t^{-n} \psi(t) \), then \( \theta(t) = O(t^{-n\Delta}) \) by (1.5). Since

\[\theta(t) = \int_0^\delta \frac{\sin nu}{n t^{1-n}} \, du = O\left(\frac{1}{n t^{1-n}}\right) \text{ as } t \to 0 ,\]

we get

\[
J = [\theta(t)\theta(t)]_\alpha^\delta - \int_\alpha^\delta \theta(t) \, d\theta(t)
\]

\[
= O\left(\frac{1}{n (1-n)(1-1/\Delta)}\right) + O\left(\frac{1}{n} \int_\alpha^\delta \frac{d\theta(t)}{t^{1-n}}\right)
\]

\[
= O\left(\frac{1}{n (1-n)(1-1/\Delta)}\right) = o(1) \text{ as } n \to \infty .
\]

We shall now estimate \( I \).

Putting \( \Phi(t) = \int_0^t \theta(u) \, du \) and integrating by parts we have

\[
I = \left[ \Phi(t) \frac{\sin nt}{t} \right]_0^\alpha - \int_0^\alpha \frac{nt \cos nt - \sin nt}{t^2} \, dt
\]

\[
= I_1 + I_2 ,
\]
say. Since

\[
\Phi(t) = o(t^{1+\gamma-\beta}) = o(t)
\]

by (1.4), we get

\[
I_1 = o(1) .
\]
Finally

\[ I_2 = \int_0^\alpha \frac{nt \cos nt - \sin nt}{t^2} \, dt \int_0^t \phi_\beta(t)(t-u)^{-\beta} \, du \]

\[ = \int_0^\alpha \phi_\beta(u) \int_0^\alpha \frac{nt \cos nt - \sin nt}{t^2} \, dt \]

where the inner integral becomes

\[ n^{1+\beta} \int_0^\alpha \frac{\cos nt - \sin nt}{t^2} \, dt = o \left( \frac{n^\beta}{u} \right) . \]

Thus we get

\[ I_2 = o \left( n^\beta \int_0^\alpha \frac{\phi_\beta(u)}{u} \, du \right) \]

\[ = o \left( n^\beta \int_0^\alpha \frac{u^\gamma}{u} \, du \right) \]

\[ = o(1) . \]

This completes the proof of the theorem.

References


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