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Lifting Quasianalytic Mappings over Invariants

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Abstract. Let $\rho\colon G \to \mathrm{GL}(V)$ be a rational finite dimensional complex representation of a reductive linear algebraic group G, and let $\sigma_1, \dots, \sigma_n$ be a system of generators of the algebra of invariant polynomials $\mathbb{C}[V]^G$. We study the problem of lifting mappings $f\colon \mathbb{R}^q \supseteq U \to \sigma(V) \subseteq \mathbb{C}^n$ over the mapping of invariants $\sigma = (\sigma_1, \dots, \sigma_n)\colon V \to \sigma(V)$. Note that $\sigma(V)$ can be identified with the categorical quotient $V /\!\!/ G$ and its points correspond bijectively to the closed orbits in V. We prove that if f belongs to a quasianalytic subclass $\mathcal{C} \subseteq C^\infty$ satisfying some mild closedness properties that guarantee resolution of singularities in $\mathcal{C}$, *e.g.,* the real analytic class, then f admits a lift of the same class $\mathcal{C}$ after desingularization by local blow-ups and local power substitutions. As a consequence we show that f itself allows for a lift that belongs to $\mathrm{SBV}_{\mathrm{loc}}$, *i.e.,* special functions of bounded variation. If ρ is a real representation of a compact Lie group, we obtain stronger versions.

1 Introduction

Let G be a reductive linear algebraic group defined over $\mathbb{C}$ and let $\rho\colon G \to \mathrm{GL}(V)$ be a rational representation on a finite dimensional complex vector space V. The algebra $\mathbb{C}[V]^G$ of G-invariant polynomials on V is finitely generated. Let $V /\!\!/ G$ denote the categorical quotient, *i.e.,* the affine algebraic variety with coordinate ring $\mathbb{C}[V]^G$, and let $\pi\colon V \to V /\!\!/ G$ be the morphism defined by the embedding $\mathbb{C}[V]^G \to \mathbb{C}[V]$. Choose a system of homogeneous generators of $\mathbb{C}[V]^G$, say $\sigma_1, \dots, \sigma_n$. Then we can identify π with the mapping $\sigma = (\sigma_1, \dots, \sigma_n)\colon V \to \sigma(V) \subseteq \mathbb{C}^n$ and the categorical quotient $V /\!\!/ G$ with the image $\sigma(V)$. In each fiber of σ there lies exactly one closed orbit.

Given a mapping $f\colon \mathbb{R}^q \supseteq U \to V /\!\!/ G = \sigma(V) \subseteq \mathbb{C}^n$ possessing some type of regularity $\mathcal{F}$ (as a mapping into $\mathbb{C}^n$), it is natural to ask whether f can be lifted regularly (maybe of some weaker type $\mathcal{G}$) over the mapping of invariants σ. By a lift of f we understand a mapping $\bar f\colon U \to V$ satisfying $f = \sigma \circ \bar f$ such that the orbit $G.\bar f(x)$ through $\bar f(x)$ is closed for each $x \in U$. Lifting $\mathcal{F}$-mappings over invariants is independent of the choice of the generators σ_i as long as the set of $\mathcal{F}$-functions forms a ring under addition and multiplication, *viz.,* any two choices of generators differ by a polynomial diffeomorphism.

This question represents a generalization of the following perturbation problem for polynomials which has important applications in PDEs and in the perturbation theory of linear operators (see [22] and the references therein): How nicely can we

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choose the roots of a monic univariate polynomial whose coefficients depend on parameters in a regular way? Namely, for the standard representation of the symmetric group S_n in $\mathbb{C}^n$ by permuting the coordinates (the roots), $\mathbb{C}[\mathbb{C}^n]^{S_n}$ is generated by the elementary symmetric functions $\sigma_j(x) = \sum_{i_1<\cdots<i_j} x_{i_1} \cdots x_{i_j}$ (the coefficients up to sign, by Vieta's formulas).

To our knowledge the lifting problem in full generality has not been studied before. Some results are known about lifting curves ($q = 1$) and about lifting mappings over invariants of real compact Lie group representations. See the summary of the most important known facts in Table 1. Lifting problems with slightly different scope were treated in [3, 13, 16, 20, 23] (amongst others).

In this paper we prove that for subclasses ($C^\omega \subseteq$) $\mathcal{C} \subseteq C^\infty$ that admit resolution of singularities (for instance the real analytic class C^ω), $\mathcal{C}$-mappings can be lifted over invariants after desingularization. More precisely, let $\mathcal{C}$ be any quasianalytic subalgebra of the C^∞-functions that contains the real analytic functions and is stable under composition, derivation, division by coordinates, and taking the inverse. Due to Bierstone and Milman [5, 6] the category of $\mathcal{C}$-manifolds and $\mathcal{C}$-mappings admits resolution of singularities. Let M be a $\mathcal{C}$-manifold, $f\colon M \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$ a $\mathcal{C}$-mapping, and $K \subseteq M$ compact. We show in Theorem 4.5 that there exist

(i) a neighborhood W of K,
(ii) a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution,

such that for all k, the mapping $f \circ \pi_k$ allows a $\mathcal{C}$-lift on U_k. The analogous statement holds for holomorphic mappings (see Theorem 4.6). If G is a compact Lie group, V a real Euclidean vector space, and $\rho\colon G \to \mathrm{O}(V)$, then no local power substitutions are needed (see Theorem 5.3). A local blow-up over an open subset $U \subseteq M$ is a blow-up over U composed with the inclusion of U in M. A local power substitution is the composite of the inclusion of a coordinate chart W in M and a mapping $V \to W$ given in local coordinates by

$$(x_1, \ldots, x_q) \mapsto ((-1)^{\epsilon_1} x_1^{\gamma_1}, \ldots, (-1)^{\epsilon_q} x_q^{\gamma_q})$$

for some $\gamma = (\gamma_1, \ldots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{0, 1\}^q$. (See §§4.1 for a precise explanation of these notions.)

This "$\mathcal{C}$-lifting after desingularization" result enables us to show in Theorem 6.4 that a $\mathcal{C}$-mapping $f\colon U \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$ (where $U \subseteq \mathbb{R}^q$ open) admits a lift $\bar f$ that is "piecewise Sobolev $W^{1,1}_{\mathrm{loc}}$"; more precisely, $\bar f$ is of class $\mathcal{C}$ outside of a null set E of finite $(q-1)$-dimensional Hausdorff measure such that its classical derivative is locally integrable (we shall write $\bar f \in \mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}$, see 6.1). As a consequence, we deduce in Theorem 6.7 that the lift $\bar f$ belongs to $\mathrm{SBV}_{\mathrm{loc}}$ (SBV stands for special functions of bounded variation, see 6.4). If $\rho\colon G \to \mathrm{GL}(V)$ is coregular, *i.e.*, $\mathbb{C}[V]^G$ is generated by algebraically independent elements, then we obtain as a corollary that the mapping $\sigma\colon V \to V/\!\!/G = \sigma(V) = \mathbb{C}^n$ admits local $\mathcal{W}^{\mathcal{C}}$ (resp. SBV) sections (see Corollaries 6.5 and 6.8). Note that the regularity of $\bar f$ is best possible: in general there does not exist a lift $\bar f$ with classical derivative in L^p_{loc} for any $1 < p \leq \infty$. Moreover there is in general (for $q \geq 2$) no lift in $W^{1,1}_{\mathrm{loc}}$ and in VMO (see Remark 6.9).

The question of optimal assumptions is open. For instance, it is unknown whether a C^∞-mapping $f\colon U \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$ admits a lift in $\mathrm{SBV}_{\mathrm{loc}}$. That problem requires different methods.

In §7 we prove for real polar representations of compact Lie groups that the $\mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}$-lift $\bar{f}$ of a $\mathcal{C}$-mapping f is actually "piecewise locally Lipschitz" (see Theorem 7.2), *i.e.,* the classical derivative of $\bar{f}$ is locally bounded outside of the exceptional set E.

Notation We use $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. Let $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$ and $x = (x_1, \dots, x_q) \in \mathbb{R}^q$. We write $\alpha! = \alpha_1! \cdots \alpha_q!$, $|\alpha| = \alpha_1 + \cdots + \alpha_q$, $x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_q^{\alpha_q}$. We shall also use $\partial_i = \partial/\partial x_i$. If $\alpha, \beta \in \mathbb{N}^q$, then $\alpha \le \beta$ means $\alpha_i \le \beta_i$ for all $1 \le i \le q$.

Let $U \subseteq \mathbb{R}^q$ be open. We will use classes of real and complex valued functions $\mathcal{F}(U)$ possessing a certain regularity $\mathcal{F}$ (like $\mathcal{C}$, L^1, $W^{1,1}$, SBV, etc.). A complex valued function f is of class $\mathcal{F}$ if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are of class $\mathcal{F}$. Mappings of class $\mathcal{F}$ with values in $\mathbb{R}^p$ (or $\mathbb{C}^p$) are defined by $\mathcal{F}(U, \mathbb{R}^p) := (\mathcal{F}(U, \mathbb{R}))^p$. Each class $\mathcal{F}$ we shall use will be invariant under linear coordinate changes. So we may consider mappings $\mathcal{F}(U, V)$ with values in a finite dimensional vector space V.

All manifolds in this paper are assumed to be Hausdorff, paracompact, and finite dimensional.

2 The Setting

Throughout the paper, we work in the following setting (unless otherwise stated).

2.1 Representations of Reductive Algebraic Groups

Let G be a reductive linear algebraic group defined over $\mathbb{C}$ and let $\rho\colon G \to \mathrm{GL}(V)$ be a rational representation on a finite dimensional complex vector space V. It is well known that the algebra $\mathbb{C}[V]^G$ of G-invariant polynomials on V is finitely generated. We consider the *categorical quotient* $V/\!\!/G$, *i.e.,* the affine algebraic variety with coordinate ring $\mathbb{C}[V]^G$, and the morphism $\pi\colon V \to V/\!\!/G$ defined by the embedding $\mathbb{C}[V]^G \to \mathbb{C}[V]$. Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{C}[V]^G$ with positive degrees $d_1, \dots, d_n$. Then we can identify π with the mapping of invariants $\sigma = (\sigma_1, \dots \sigma_n) : V \to \sigma(V) \subseteq \mathbb{C}^n$ and the categorical quotient $V/\!\!/G$ with the image $\sigma(V)$ (which we shall do consistently). Each fiber of σ contains exactly one closed orbit. If $v \in V$ and the orbit $G\,.\,v = \{g\,.\,v : g \in G\}$ through v is closed, then the isotropy group $G_v = \{g \in G : g\,.\,v = v\}$ is reductive. See [25].

2.2 Luna's Slice Theorem

We state a version of Luna's slice theorem [18]. Recall that U is a *G-saturated* subset of V if $\pi^{-1}(\pi(U)) = U$ and that a mapping between smooth complex algebraic varieties is *étale* if its differential is everywhere an isomorphism.

Theorem 2.1 ([23, 5.3]) *Let $G\,.\,v$ be a closed orbit, $v \in V$. Choose a G_v-splitting of*

Table 1: Let $f\colon \mathbb{R}^q \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$. The table provides a (non-exhaustive) summary of the most important results concerning the existence of a lift $\bar f$ of some regularity of f, given that f fulfills certain conditions. The regularity of $\bar f$ is in general best possible under the respective conditions on f, which might partly not be optimal. By the attribute "complex" (resp. "real") we refer to the setting in 2.1 (resp. 5.1). By $\mathcal{C}$ we mean a subclass of C^∞ satisfying $(A_{1'})(A_2)$–(A_6). For a definition of $\mathcal{W}^{\mathcal{C}}$ (resp. $\mathcal{L}^{\mathcal{C}}$) see 6.1 (resp. 7.1). Normal nonflatness is defined in [17]. Let $d = d(\rho) := \max_j \deg \sigma_j$. If G is finite, let $k = k(\rho) := \{d, |G|/|G_{v_j}| : 1 \le j \le l\}$, where $V = V_1 \oplus \cdots \oplus V_l$ with V_j irreducible and $v_j \in V_j \setminus \{0\}$ such that G_{v_j} is maximal. If ρ is polar (see §§2.4), then $k = k(\rho_\Sigma)$ for some Cartan subspace Σ and $\rho_\Sigma\colon W(\Sigma) \to \mathrm{GL}(\Sigma)$.

Representation	q	**Regularity of f**	$\Longrightarrow$	**Regularity of $\bar f$**	**Reference**
complex, polar	1	continuous		continuous	[17, 8.2(1)]
complex	1	C^∞ & normally nonflat		local desingularization by $x \mapsto \pm x^\gamma$ ($\gamma \in \mathbb{N}_{>0}$), AC_{loc}	[17, 3.3, 5.4]
complex	≥ 1	$\mathcal{C}$ (resp. holomorphic)		local desingularization by finitely many local blow-ups with smooth center and local power substitutions (in the sense of 4.1), $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ & SBV_{loc}	Theorem 4.5 (resp. 4.6) Theorems 6.4, 6.7
real	1	continuous		continuous	[19] (see also [10, 3.1])
real	1	C^ω (resp. $\mathcal{C}$)		locally C^ω (resp. $\mathcal{C}$)	[1] (resp. Corollary 5.4)
real	1	C^∞ & normally nonflat		C^∞	[1]
real	1	C^d		differentiable	[10]
real, polar	1	C^k (resp. C^{k+d})		C^1 (resp. twice differentiable)	[11, 12]
real, polar, G connected or a finite reflection group	≥ 1	continuous		continuous	*e.g.*, [12]
real, polar, G connected or a finite reflection group	≥ 1	C^k		locally Lipschitz	[12]
real	≥ 1	$\mathcal{C}$		local desingularization by finitely many local blow-ups with smooth center $\mathcal{W}^{\mathcal{C}}_{\text{loc}}$ & SBV_{loc}	Theorem 5.3 Theorem 7.1
real, polar	≥ 1	$\mathcal{C}$		$\mathcal{L}^{\mathcal{C}}_{\text{loc}}$	Theorem 7.2

$V \cong T_vV$ as $T_v(G\,.\,v) \oplus N_v$ and let φ denote the mapping

$$G \times_{G_v} N_v \to V, \quad [g, n] \mapsto g(v + n).$$

There is an affine open G-saturated subset U of V and an affine open G_v-saturated neighborhood S_v of 0 in N_v such that

$$\varphi : G \times_{G_v} S_v \to U \quad \text{and} \quad \bar\varphi : (G \times_{G_v} S_v) /\!\!/ G \to U /\!\!/ G$$

are étale, where $\bar\varphi$ is the mapping induced by φ. Moreover, φ and the natural mapping $G \times_{G_v} S_v \to S_v /\!\!/ G_v$ induce a G-isomorphism of $G \times_{G_v} S_v$ with $U \times_{U /\!\!/ G} S_v /\!\!/ G_v$.

Corollary 2.2 ([23, 5.4]) *Choose a G-saturated neighborhood $\overline{S}_v$ of 0 in S_v (classical topology) such that the canonical mapping $\overline{S}_v /\!\!/ G_v \to \overline{U} /\!\!/ G$ is a complex analytic isomorphism, where $\overline{U} = \pi^{-1}(\bar\varphi((G \times_{G_v} \overline{S}_v) /\!\!/ G))$. Then $\overline{U}$ is a G-saturated neighborhood of v and $\varphi : G \times_{G_v} \overline{S}_v \to \overline{U}$ is biholomorphic.*

A *slice representation* of ρ is a rational representation $G_v \to \mathrm{GL}(V/T_v(G\,.\,v))$, where $G\,.\,v$ is a closed orbit.

2.3 Luna's stratification

Let $v \in V$ and let G_v be the isotropy group of G at v. Denote by (G_v) its conjugacy class in G, also called an *isotropy class*. If (L) is an isotropy class, let $(V /\!\!/ G)_{(L)}$ denote the set of points in $V /\!\!/ G$ corresponding to closed orbits with isotropy group in (L), and put $V_{(L)} := \pi^{-1}((V /\!\!/ G)_{(L)})$. Then the collection $\{(V /\!\!/ G)_{(L)}\}$ forms a finite stratification of $V /\!\!/ G$ into locally closed irreducible smooth algebraic subvarieties. The isotropy classes are partially ordered, namely $(H) \le (L)$ if H is conjugate to a subgroup of L. If $(V /\!\!/ G)_{(L)} \neq \varnothing$, then its Zariski closure is equal to $\bigcup_{(M) \ge (L)} (V /\!\!/ G)_{(M)} = \pi(V^L)$, where V^L is the set of all $v \in V$ fixed by L. There exists a unique minimal isotropy class (H) corresponding to a closed orbit, the *principal isotropy class*. Closed orbits $G\,.\,v$ with $G_v \in (H)$ are called principal. The subset $(V /\!\!/ G)_{(H)} \subseteq V /\!\!/ G$ is Zariski open. If we set $V_{\langle H \rangle} := \{v \in V : G\,.\,v \text{ closed and } G_v = H\}$, then π restricts to a principal $(N_G(H)/H)$-bundle $V_{\langle H \rangle} \to (V /\!\!/ G)_{(H)}$, where $N_G(H)$ denotes the normalizer of H in G. See [18, 23, 25].

2.4 Polar Representations

Let $v \in V$ be such that the orbit $G\,.\,v$ is closed and consider the subspace $\Sigma_v = \{x \in V : \mathfrak{g}\,.\,x \subseteq \mathfrak{g}\,.\,v\}$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}\,.\,x = \{X\,.\,x : X \in \mathfrak{g}\} \cong T_x(G\,.\,x)$. Then for each $x \in \Sigma_v$ the orbit $G\,.\,x$ is closed. The representation ρ is called *polar* if there is a $v \in V$ with $G\,.\,v$ closed such that $\dim \Sigma_v = \dim \mathbb{C}[V]^G$. In particular, representations of finite groups are polar. Such Σ_v is called a *Cartan subspace*. Any two Cartan subspaces are conjugate. All closed orbits in V intersect Σ_v. The *generalized Weyl group*

$$W(\Sigma_v) = \{g \in G : g.\Sigma_v = \Sigma_v\}/\{g \in G : g\,.\,x = x \text{ for all } x \in \Sigma_v\}$$

is finite. Restriction to Σ_v induces an isomorphism $\mathbb{C}[V]^G \to \mathbb{C}[\Sigma_v]^{W(\Sigma_v)}$. So we have the identifications $V /\!\!/ G = \sigma(V) = \sigma_{\Sigma_v}(\Sigma_v) = \Sigma_v /\!\!/ W(\Sigma_v)$. See [7].

3 C^∞ Classes That Admit Resolution of Singularities

Following [6, §3] we discuss classes of smooth functions that admit resolution of singularities.

3.1 Classes $\mathcal{C}$ of C^∞-Functions

Let us assume that for every open $U \subseteq \mathbb{R}^q$, $q \in \mathbb{N}$, we have a subalgebra $\mathcal{C}(U)$ of $C^\infty(U) = C^\infty(U, \mathbb{R})$. Resolution of singularities in $\mathcal{C}$ requires only the following assumptions (A_1)–(A_6) for any open $U \subseteq \mathbb{R}^q$.

(A_1) $\mathcal{C}$ *contains the restrictions of polynomial functions*. The algebra of restrictions to U of polynomial functions on $\mathbb{R}^q$ is contained in $\mathcal{C}(U)$.

(A_2) $\mathcal{C}$ *is closed under composition*. If $V \subseteq \mathbb{R}^p$ is open and $\varphi = (\varphi_1, \dots, \varphi_p) : U \to V$ is a mapping with each $\varphi_i \in \mathcal{C}(U)$, then $f \circ \varphi \in \mathcal{C}(U)$, for all $f \in \mathcal{C}(V)$.

A mapping $\varphi\colon U \to V$ is called a $\mathcal{C}$-*mapping* if $f \circ \varphi \in \mathcal{C}(U)$, for every $f \in \mathcal{C}(V)$. It follows from (A_1) and (A_2) that $\varphi = (\varphi_1, \dots, \varphi_p)$ is a $\mathcal{C}$-mapping if and only if $\varphi_i \in \mathcal{C}(U)$, for all $1 \le i \le p$.

(A_3) $\mathcal{C}$ *is closed under derivation*. If $f \in \mathcal{C}(U)$ and $1 \le i \le q$, then $\partial_i f \in \mathcal{C}(U)$.

(A_4) $\mathcal{C}$ *is quasianalytic*. If $f \in \mathcal{C}(U)$ and for $a \in U$ the Taylor series of f at a vanishes, *i.e.*, $\hat{f}_a = 0$, then f vanishes in a neighborhood of a.

(A_5) $\mathcal{C}$ *is closed under division by a coordinate*. If $f \in \mathcal{C}(U)$ is identically 0 along a hyperplane $\{x : x_i = a_i\}$, then $f(x) = (x_i - a_i)h(x)$, where $h \in \mathcal{C}(U)$.

(A_6) $\mathcal{C}$ *is closed under taking the inverse*. Let $\varphi\colon U \to V$ be a $\mathcal{C}$-mapping between open subsets U and V in $\mathbb{R}^q$. Let $a \in U$, $\varphi(a) = b$, and suppose that the Jacobian matrix $(\partial\varphi/\partial x)(a)$ is invertible. Then there exist neighborhoods U' of a, V' of b, and a $\mathcal{C}$-mapping $\psi\colon V' \to U'$ such that $\psi(b) = a$ and $\varphi \circ \psi = \mathrm{id}_{V'}$.

Property (A_6) is equivalent to the *implicit function theorem in* $\mathcal{C}$:

> Let $U \subseteq \mathbb{R}^q \times \mathbb{R}^p$ be open. Suppose that $f_1, \dots, f_p \in \mathcal{C}(U)$, $(a, b) \in U$, $f(a, b) = 0$, and $(\partial f/\partial y)(a, b)$ is invertible, where $f = (f_1, \dots, f_p)$. Then there is a neighborhood $V \times W$ of (a, b) in U and a $\mathcal{C}$-mapping $g\colon V \to W$ such that $g(a) = b$ and $f(x, g(x)) = 0$, for $x \in V$.

It follows from (A_6) that $\mathcal{C}$ *is closed under taking the reciprocal*: If $f \in \mathcal{C}(U)$ vanishes nowhere in U, then $1/f \in \mathcal{C}(U)$.

A complex valued function $f\colon U \to \mathbb{C}$ is said to be a $\mathcal{C}$-*function*, or to belong to $\mathcal{C}(U, \mathbb{C})$, if $(\operatorname{Re} f, \operatorname{Im} f)\colon U \to \mathbb{R}^2$ is a $\mathcal{C}$-mapping. It is immediately verified that (A_3)–(A_5) hold for complex valued functions $f \in \mathcal{C}(U, \mathbb{C})$ as well.

In the proof of Theorem 4.5 we shall need that $\mathcal{C}$ contains the real analytic class C^ω, so instead of (A_1) we will presuppose the following stronger condition:

$(A_{1'})$ $\mathcal{C}$ *contains the real analytic functions; i.e.*, $C^\omega(U) \subseteq \mathcal{C}(U)$.

From now on, unless otherwise stated, let $\mathcal{C}$ denote a fixed, but arbitrary, class of C^∞-functions satisfying the conditions $(A_{1'})$, (A_2)–(A_6) .

Examples 3.1 (Denjoy–Carleman classes ([15,24] and references therein)) Let $M = (M_k)_{k \in \mathbb{N}}$ be a non-decreasing sequence of real numbers with $M_0 = 1$. For $U \subseteq \mathbb{R}^q$

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open, the Denjoy–Carleman class $C^M(U)$ is the set of all $f \in C^\infty(U)$ such that for every compact $K \subseteq U$ there are constants $C, \rho > 0$ with $|\partial^\alpha f(x)| \le C\rho^{|\alpha|}|\alpha|!\, M_{|\alpha|}$ for all $\alpha \in \mathbb{N}^q$ and $x \in K$. If M is logarithmically convex ($M_k^2 \le M_{k-1} M_{k+1}$ for all k), quasianalytic ($\sum_{k=0}^\infty M_k/((k+1)M_{k+1}) = \infty$), and closed under derivations ($\sup_{k\in\mathbb{N}_{>0}}(M_{k+1}/M_k)^{1/k} < \infty$), then the Denjoy–Carleman class $\mathcal{C} = C^M$ has the properties ($A_{1'}$), (A_2)–(A_6) (see [6, §4]). In particular, this is true for the class of real analytic functions $\mathcal{C} = C^\omega$, since $C^\omega = C^{(1)_k}$. If C^M is not closed under derivations, then $\mathcal{C} = \bigcup_{j\in\mathbb{N}} C^{M^{+j}}$, where $M_k^{+j} := M_{k+j}$, has the required properties ($A_{1'}$), (A_2)–(A_6) .

3.2 Resolution of Singularities in $\mathcal{C}$

One can use the open subsets $U \subseteq \mathbb{R}^q$ and the algebras of functions $\mathcal{C}(U)$ as local models to define a category $\underline{\mathcal{C}}$ of $\mathcal{C}$-*manifolds* and $\mathcal{C}$-*mappings*. The dimension theory of $\underline{\mathcal{C}}$ follows from that of C^∞-manifolds.

The implicit function property (A_6) implies that a *smooth* (not singular) subset of a $\mathcal{C}$-manifold is a $\mathcal{C}$-submanifold: Let M be a $\mathcal{C}$-manifold. Suppose that U is open in M, $g_1, \dots, g_p \in \mathcal{C}(U)$, and the gradients ∇g_i are linearly independent at every point of the zero set $X := \{x \in U : g_i(x) = 0 \text{ for all } i\}$. Then X is a closed $\mathcal{C}$-submanifold of U of codimension p.

The category $\underline{\mathcal{C}}$ is closed under blowing up with center a closed $\mathcal{C}$-submanifold.

We shall use a simple version of the desingularization theorem of Hironaka [9] for $\mathcal{C}$-function classes due to Bierstone and Milman [5, 6]. We use the terminology therein.

Theorem 3.2 ([6, 5.12]) *Let M be a $\mathcal{C}$-manifold, X a closed $\mathcal{C}$-hypersurface in M, and K a compact subset of M. Then there is a neighborhood W of K and a surjective mapping $\varphi\colon W' \to W$ of class $\mathcal{C}$, such that the following hold:*

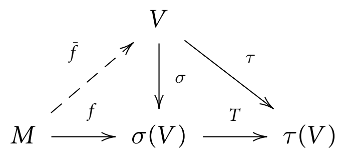
(i) *φ is a composite of finitely many $\mathcal{C}$-mappings, each of which is either a blow-up with smooth center (that is nowhere dense in the smooth points of the strict transform of X) or a surjection of the form $\bigsqcup_j U_j \to \bigcup_j U_j$, where the latter is a finite covering of the target space by coordinate charts.*

(ii) *The final strict transform X' of X is smooth, and $\varphi^{-1}(X)$ has only normal crossings. (In fact $\varphi^{-1}(X)$ and $\det d\varphi$ simultaneously have only normal crossings, where $d\varphi$ is the Jacobian matrix of φ with respect to any local coordinate system.)*

See [6, 5.9, 5.10] and [5] for stronger desingularization theorems in $\mathcal{C}$.

3.3 Lifting $\mathcal{C}$-Mappings over Invariants

Let M be a $\mathcal{C}$-manifold. Let $f\colon M \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$ be a $\mathcal{C}$-mapping, *i.e.*, with values in $\sigma(V)$ and of class $\mathcal{C}$ as mapping into $\mathbb{C}^n \cong \mathbb{R}^{2n}$. A mapping $\bar f\colon M \to V$ is called a *lift* of f (*over invariants*) to V, if $f = \sigma \circ \bar f$ and if the orbit $G.\bar f(x)$ is closed for each $x \in M$. Lifting $\mathcal{C}$-mappings over invariants is independent of the choice of generators of $\mathbb{C}[V]^G$, as any two choices σ_i and τ_j differ just by a polynomial diffeomorphism T and the set of $\mathcal{C}$-functions forms a ring under addition and multiplication (see [11, 2.2]):



4 Lifting $\mathcal{C}$ Mappings over Invariants after Desingularization

We will prove that $\mathcal{C}$-mappings admit $\mathcal{C}$-lifts after desingularization by means of local blow-ups and local power substitutions.

4.1 Local Blow-Ups and Local Power Substitutions

We introduce notation following [4, §4].

Let M be a $\mathcal{C}$-manifold. A family of $\mathcal{C}$-mappings $\{\pi_j \colon U_j \to M\}$ is called a *locally finite covering* of M if the images $\pi_j(U_j)$ are subordinate to a locally finite open covering $\{W_j\}$ of M, *i.e.*, $\pi_j(U_j) \subseteq W_j$ for all j, and if for each compact $K \subseteq M$ there are compact $K_j \subseteq U_j$ such that $K = \bigcup_j \pi_j(K_j)$ (the union is finite).

Locally finite coverings can be composed in the following way (see [4, 4.5]). Let $\{\pi_j \colon U_j \to M\}$ be a locally finite covering of M, and let $\{W_j\}$ be as above. For each j, suppose that $\{\pi_{ji} \colon U_{ji} \to U_j\}$ is a locally finite covering of U_j. We may assume without loss of generality that the W_j are relatively compact. (Otherwise, choose a locally finite covering $\{V_j\}$ of M by relatively compact open subsets. Then the mappings $\pi_j|_{\pi_j^{-1}(V_i)} \colon \pi_j^{-1}(V_i) \to M$ for all i and j form a locally finite covering of M.) Then for each j, there is a finite subset $I(j)$ of the set of indices i such that the $\mathcal{C}$-mappings $\pi_j \circ \pi_{ji} \colon U_{ji} \to M$ for all j and all $i \in I(j)$ form a locally finite covering of M.

We shall say that $\{\pi_j\}$ is a *finite covering* if j varies in a finite index set.

A *local blow-up* Φ over an open subset U of M means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi \colon U' \to U$ with smooth center and of the inclusion $\iota \colon U \to M$.

We denote by *local power substitution* a mapping of $\mathcal{C}$-manifolds $\Psi \colon V \to M$ of the form $\Psi = \iota \circ \psi$, where $\iota \colon W \to M$ is the inclusion of a coordinate chart W of M and $\psi \colon V \to W$ is given by

$$(y_1, \dots, y_q) = \psi_{\gamma,\epsilon}(x_1, \dots, x_q) := ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q}), \tag{4.1}$$

for some $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0, 1\}^q$, where $y_1, \dots, y_q$ denote the coordinates of W (and $q = \dim M$).

4.2 Normal Crossings

Let M be a $\mathcal{C}$-manifold and let f be a real or complex valued $\mathcal{C}$-function on M. We say that f has only *normal crossings* if each point in M admits a coordinate neighborhood

U with coordinates $x = (x_1, \dots, x_q)$ such that

$$f(x) = x^\alpha g(x), \quad x \in U,$$

where g is a non-vanishing $\mathcal{C}$-function on U, and $\alpha \in \mathbb{N}^q$.

Observation 4.1 Observe that if a product of functions has only normal crossings, then each factor has only normal crossings. Indeed, let f_1, f_2, g be $\mathcal{C}$-functions defined near $0 \in \mathbb{R}^q$ such that $f_1(x) f_2(x) = x^\alpha g(x)$ and g is non-vanishing. By quasianalyticity (A_4), $f_1 f_2|_{\{x_j=0\}} = 0$ implies $f_1|_{\{x_j=0\}} = 0$ or $f_2|_{\{x_j=0\}} = 0$. So the assertion follows from (A_5).

Observation 4.2 Let M be a $\mathcal{C}$-manifold, $K \subseteq M$ be compact, and $f \in \mathcal{C}(M, \mathbb{C})$. Then there exists a neighborhood W of K and a finite covering $\{\pi_k : U_k \to W\}$ of W by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that for each k, the function $f \circ \pi_k$ has only normal crossings. This follows from Theorem 3.2 applied to the real valued $\mathcal{C}$-function $|f|^2 = f\overline{f}$ and Observation 4.1.

Lemma 4.3 ([6, 7.7], [4, 4.7]; a proof for $\mathcal{C}$ is in [22, 6.3]) *Let $\alpha, \beta, \gamma \in \mathbb{N}^q$ and let $a(x), b(x), c(x)$ be non-vanishing germs of real or complex valued functions of class $\mathcal{C}$ at the origin of $\mathbb{R}^q$. If*

$$x^\alpha a(x) - x^\beta b(x) = x^\gamma c(x),$$

then either $\alpha \le \beta$ or $\beta \le \alpha$.

4.3 $\mathcal{C}$-Lifting after Desingularization

Lemma 4.4 (Removing fixed points) *Let V^G be the subspace of G-invariant vectors, and let V' be a G-invariant complementary subspace in V. Then $V = V^G \oplus V'$, $\mathbb{C}[V]^G = \mathbb{C}[V^G] \otimes \mathbb{C}[V']^G$, and $V /\!\!/ G = V^G \times V' /\!\!/ G$. Any $\mathcal{C}$-lift of a $\mathcal{C}$-mapping $f = (f_0, f_1)$ in $V^G \times V' /\!\!/ G \subseteq \mathbb{C}^n$ has the form $\bar{f} = (f_0, \bar{f}_1)$, where $\bar{f}_1$ is a $\mathcal{C}$-lift of f_1 to V'.*

Proof This is obvious; see [1, 3.2]. ■

Theorem 4.5 ($\mathcal{C}$-lifting after desingularization) *Let M be a $\mathcal{C}$-manifold. Consider a $\mathcal{C}$-mapping $f\colon M \to V /\!\!/ G = \sigma(V) \subseteq \mathbb{C}^n$. Let $K \subseteq M$ be compact. Then there exist*

(i) *a neighborhood W of K,*

(ii) *a finite covering $\{\pi_k\colon U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution,*

such that for all k the mapping $f \circ \pi_k$ allows a $\mathcal{C}$-lift on U_k.

Proof Since the statement is local, we may assume without loss of generality that M is an open neighborhood of $0 \in \mathbb{R}^q$. Let $v \in \sigma^{-1}(f(0))$ be such that $G\,.\,v$ is a closed orbit. We show that there exists a neighborhood of $0 \in \mathbb{R}^q$ and a finite covering $\{\pi_k\}$ of that neighborhood such that each $f \circ \pi_k$ admits a $\mathcal{C}$-lift $\bar{f}_k$ through v, *i.e.,*

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if $\pi_k^{-1}(0) \neq \varnothing$ then $\bar{f}_k(\pi_k^{-1}(0)) = \{v\}$. Let us proceed by induction over isotropy classes (slice representations).

If $(G_v) = (H)$ is the principal isotropy class, then a $\mathcal{C}$-lift $\bar{f}$ of f to $V_{\langle H \rangle}$ with $\bar{f}(0) = v$ exists locally near 0 since $V_{\langle H \rangle} \to (V /\!\!/ G)_{(H)}$ is a principal $(N_G(H)/H)$-bundle (see §§2.3) (and by $(\mathrm{A}_{1'})$ and (A_2)).

Let $(G_v) > (H)$; in particular, $f(0)$ is not principal. Assume that the assertion is shown for all rational finite dimensional complex representations of L, where $L = G_w$ is a proper isotropy subgroup of G such that the orbit $G \,.\, w$ is closed (with respect to ρ). All such L are reductive.

If $V^G \neq \{0\}$, we first remove fixed points, by Lemma 4.4. So we can assume that $V^G = \{0\}$. Let us consider the slice representation $G_v \to \mathrm{GL}(N_v)$. By Luna's slice Theorem 2.2 (and $(\mathrm{A}_{1'})$ and (A_2)), the lifting problem reduces to the group G_v acting on N_v. Closed G_v-orbits in N_v correspond to closed G-orbits in V. The stratification of $V /\!\!/ G$ in a neighborhood of $f(0)$ is naturally isomorphic to the stratification of $N_v /\!\!/ G_v$ in a neighborhood of 0.

If $f(0) \neq 0$, then G_v is a proper subgroup of G, since $V^G = \{0\}$. In that case we are done by induction.

Suppose that $f(0) = 0$. If $f = 0$ (identically), we choose the lift $\bar{f} = 0$ and are done. Otherwise, we set $D = \prod_{j=1}^n d_j$ (with $d_j = \deg \sigma_j$, see §§2.1) and define the $\mathcal{C}$-functions (where $f = (f_1, \dots, f_n)$)

$$F_j(x) = f_j(x)^{\frac{D}{d_j}}, \quad (\text{for } 1 \leq j \leq n). \tag{4.2}$$

By Theorem 3.2 (and Observation 4.2), we find a finite covering $\{\pi_k \colon U_k \to U\}$ of a neighborhood U of 0 by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center, such that for each k the non-zero $F_j \circ \pi_k$ (for $1 \leq j \leq n$) and its pairwise non-zero differences $F_i \circ \pi_k - F_j \circ \pi_k$ (for $1 \leq i < j \leq n$) simultaneously have only normal crossings.

Let k be fixed and let $x_0 \in U_k$. Then x_0 admits a neighborhood W_k with suitable coordinates in which $x_0 = 0$ and such that (for $1 \leq j \leq n$) either $F_j \circ \pi_k = 0$ or

$$(F_j \circ \pi_k)(x) = x^{\alpha_j} F_j^k(x),$$

where F_j^k is a non-vanishing $\mathcal{C}$-function on W_k, and $\alpha_j \in \mathbb{N}^q$. The collection of the multi-indices $\{\alpha_j \colon F_j \circ \pi_k \neq 0, 1 \leq j \leq n\}$ is totally ordered by Lemma 4.3. Let α denote its minimum.

If $\alpha = 0$, then $(F_j \circ \pi_k)(x_0) = F_j^k(x_0) \neq 0$ for some $1 \leq j \leq n$. So by (4.2), we have $(f \circ \pi_k)(x_0) \neq 0$. Let $w \in \sigma^{-1}((f \circ \pi_k)(x_0))$ be such that the orbit $G \,.\, w$ is closed. The stabilizer G_w is a proper subgroup of G, since $V^G = \{0\}$. By the induction hypothesis (and reduction to the slice representation $G_w \to \mathrm{GL}(N_w)$), there exists a finite covering $\{\pi_{kl} \colon W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (ii) such that for all l, the mapping $f \circ \pi_k \circ \pi_{kl}$ allows a $\mathcal{C}$-lift through w on W_{kl}.

Let us assume that $\alpha \neq 0$. Then there exist $\mathcal{C}$-functions $\tilde{F}_j^k$ (some of them 0) such that for all $1 \leq j \leq n$

$$(F_j \circ \pi_k)(x) = x^{\alpha} \tilde{F}_j^k(x), \tag{4.3}$$

and $\tilde{F}^k_j(x_0) = F^k_j(x_0) \neq 0$ for some $1 \le j \le n$. Let us write

$$\frac{\alpha}{D} = \Big(\frac{\alpha_1}{D}, \dots, \frac{\alpha_q}{D}\Big) = \Big(\frac{\beta_1}{\gamma_1}, \dots, \frac{\beta_q}{\gamma_q}\Big),$$

where $\beta_i, \gamma_i \in \mathbb{N}$ are relatively prime (and $\gamma_i > 0$) for all $1 \le i \le q$. Put $\beta = (\beta_1, \dots, \beta_q)$ and $\gamma = (\gamma_1, \dots, \gamma_q)$. Then (by (4.2) and (4.3)) for each $1 \le j \le n$ and $\epsilon \in \{0, 1\}^q$ the $\mathcal{C}$-function $f_j \circ \pi_k \circ \psi_{\gamma,\epsilon}$ is divisible by $x^{d_j\beta}$, where $\psi_{\gamma,\epsilon}$ is defined by (4.1). By (A_5) there exist $\mathcal{C}$-functions $f^{k,\gamma,\epsilon}_j$ such that

$$(f_j \circ \pi_k \circ \psi_{\gamma,\epsilon})(x) = x^{d_j\beta} f^{k,\gamma,\epsilon}_j(x), \quad (\text{for } 1 \le j \le n).$$

By construction, for some $1 \le j \le n$, we have $f^{k,\gamma,\epsilon}_j(0) \neq 0$, independently of ϵ. So there exist a local power substitution $\psi_k \colon V_k \to W_k$ given in local coordinates by $\psi_{\gamma,\epsilon}$ (for $\epsilon \in \{0, 1\}^q$) and functions f^k_j given in local coordinates by $f^{k,\gamma,\epsilon}_j$ (for $\epsilon \in \{0, 1\}^q$) such that

$$(f_j \circ \pi_k \circ \psi_k)(x) = x^{d_j\beta} f^k_j(x), \quad (\text{for } 1 \le j \le n).$$

Let us consider the $\mathcal{C}$-mapping $f^k = (f^k_1, \dots, f^k_n)$. The image of f^k lies in $\sigma(V)$, since σ_j is homogeneous of degree d_j. Let $y_0 := \psi_k^{-1}(x_0) \in V_k$. By construction $f^k(y_0) \neq 0$. Let $w \in \sigma^{-1}(f^k(y_0))$ such that the orbit $G.w$ is closed. The stabilizer G_w is a proper subgroup of G, since $V^G = \{0\}$. By the induction hypothesis (and reduction to the slice representation $G_w \to \mathrm{GL}(N_w)$), there exists a finite covering $\{\pi_{kl} \colon V_{kl} \to V_k\}$ of V_k (possibly shrinking V_k) of the type described in (ii) such that for all l, the mapping $f^k \circ \pi_{kl}$ admits a $\mathcal{C}$-lift $\bar{f}^{kl}$ through w on V_{kl}. Since a lift of f^k provides a lift of $f \circ \pi_k \circ \psi_k$ by multiplying by the monomial factor $m(x) := x^\beta$, the $\mathcal{C}$-mapping $x \mapsto m(\pi_{kl}(x)) \cdot \bar{f}^{kl}(x)$ forms a lift through 0 of $x \mapsto (f \circ \pi_k \circ \psi_k \circ \pi_{kl})(x)$ for $x \in V_{kl}$.

Since k and x_0 were arbitrary, the assertion of the theorem follows from §§ 4.1. ∎

The same proof (with obvious minor modifications) applies to holomorphic mappings. In this situation a local power substitution is (in local coordinates) simply a mapping $(z_1, \dots, z_q) \mapsto (z_1^{\gamma_1}, \dots, z_q^{\gamma_q})$ (without different sign combinations):

Theorem 4.6 (Holomorphic lifting after desingularization) *Let M be a holomorphic manifold. Consider a holomorphic mapping $f \colon M \to V /\!\!/ G = \sigma(V) \subseteq \mathbb{C}^n$. Let $K \subseteq M$ be compact. Then there exist*

(i) *a neighborhood W of K,*

(ii) *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up with smooth center or a local power substitution,*

such that for all k the mapping $f \circ \pi_k$ allows a holomorphic lift on U_k.

5 $\mathcal{C}$-Lifting in the Real Case

If G is a compact Lie group and the representation $\rho \colon G \to \mathrm{O}(V)$ is real, then no local power substitutions are needed.

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5.1 Representations of Compact Lie Groups

See [21, 23]. Let G be a compact Lie group and let $G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle\ |\ \rangle$. The algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ with positive degrees $d_1, \dots, d_n$; without loss of generality assume that $\sigma_1(v) = \langle v \mid v \rangle$. The image $\sigma(V)$ of the mapping $\sigma = (\sigma_1, \dots, \sigma_n)\colon V \to \mathbb{R}^n$ is a semialgebraic set in $Z := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$, where I is the ideal of relations among $\sigma_1, \dots, \sigma_n$. Since G is compact, σ is proper, open, and separates orbits of G. It thus induces a homeomorphism between the orbit space V/G and the image $\sigma(V)$. Note that here each orbit is closed.

Let $\langle\ |\ \rangle$ denote also the G-invariant dual inner product on V^*. The differentials $d\sigma_i\colon V \to V^*$ are G-equivariant, and the polynomials $v \mapsto \langle d\sigma_i(v) \mid d\sigma_j(v) \rangle$ are G-invariant. They are entries of an $n \times n$ symmetric matrix-valued polynomial

$$B(v) := \begin{pmatrix} \langle d\sigma_1(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_1(v) \mid d\sigma_n(v) \rangle \\ \vdots & \ddots & \vdots \\ \langle d\sigma_n(v) \mid d\sigma_1(v) \rangle & \cdots & \langle d\sigma_n(v) \mid d\sigma_n(v) \rangle \end{pmatrix}.$$

There is a unique matrix-valued polynomial $\tilde{B}$ on Z such that $B = \tilde{B} \circ \sigma$.

Theorem 5.1 (Procesi and Schwarz [21]) *We have*

$$\sigma(V) = \{z \in Z : \tilde{B}(z) \text{ is positive semidefinite}\}.$$

This theorem provides finitely many equations and inequalities describing $\sigma(V)$. Changing the choice of generators may change the equations and inequalities, but not the set they describe.

The isotropy classes in G induce a stratification of the orbit space V/G, analogously to §§ 2.3, which is isomorphic to the primary Whitney stratification of the semialgebraic set $\sigma(V)$ via the homeomorphism of V/G and $\sigma(V)$ induced by σ, by [3]. These facts are essentially consequences of the differentiable slice theorem; see [23].

5.2 $\mathcal{C}$-Liftings after Desingularization—Real Version

Lemma 5.2 *Let $\rho\colon G \to \mathrm{O}(V)$ be an orthogonal finite dimensional representation of a compact Lie group G with $V^G = \{0\}$. Let $U \subseteq \mathbb{R}^q$ be an open neighborhood of 0. Consider a $\mathcal{C}$-mapping $f\colon U \to V/G = \sigma(V) \subseteq \mathbb{R}^n$. Assume that $f_1 \neq 0$ (identically) and that for all j, $f_j \neq 0$ implies $f_j(x) = x^{\alpha_j} g_j(x)$, where $g_j \in \mathcal{C}(U, \mathbb{R})$ is non-vanishing and $\alpha_j \in \mathbb{N}^q$. Then there exists a $\delta \in \mathbb{N}^q$ such that $\alpha_1 = 2\delta$ and $\alpha_j \geq d_j \delta$ for those j with $f_j \neq 0$.*

Proof We have $\alpha_1 = 2\delta$ for some $\delta \in \mathbb{N}^q$, since $\sigma_1(v) = \langle v \mid v \rangle$ and thus $f_1 \geq 0$. If $\delta = 0$ the assertion is trivial. Let us assume that $\delta \neq 0$.

Set $\mu = (\mu_1, \dots, \mu_q)$, where

$$\mu_i := \min\Big\{ \frac{(\alpha_j)_i}{d_j} : f_j \neq 0 \Big\}. \tag{5.1}$$

For contradiction, assume that there is an i_0 such that $\mu_{i_0} < \delta_{i_0}$. Consider

$$\tilde{f}(x) := (x^{-d_1\mu} f_1(x), \dots, x^{-d_n\mu} f_n(x)).$$

If all $x_i \geq 0$, then $\tilde{f}$ is continuous (by (5.1)), and if all $x_i > 0$, then $\tilde{f}(x) \in \sigma(V)$ (by the homogeneity of the σ_j). Since $\sigma(V)$ is closed (by Theorem 5.1), $\tilde{f}(x) \in \sigma(V)$ if all $x_i \geq 0$. Since $(\alpha_1)_{i_0} - d_1\mu_{i_0} = (\alpha_1)_{i_0} - 2\mu_{i_0} = 2\delta_{i_0} - 2\mu_{i_0} > 0$, we find that the first component of $\tilde{f}$ vanishes on $\{x_{i_0} = 0\}$. Thus $\tilde{f}$ must vanish on $\{x_{i_0} = 0\}$, since $\sigma_1(v) = \langle v \mid v \rangle$. This is a contradiction for those j with $(\alpha_j)_{i_0} = d_j\mu_{i_0}$. ■

Theorem 5.3 ($\mathcal{C}$-lifting after desingularization — real version) *Let $\rho\colon G \to \mathrm{O}(V)$ be an orthogonal finite dimensional representation of a compact Lie group G. Let M be a $\mathcal{C}$-manifold. Consider a $\mathcal{C}$-mapping $f\colon M \to V/G = \sigma(V) \subseteq \mathbb{R}^n$. Let $K \subseteq M$ be compact. Then there exist*

(i) *a neighborhood W of K,*
(ii) *a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many local blow-ups with smooth center,*

such that for all k the mapping $f \circ \pi_k$ allows a $\mathcal{C}$-lift on U_k.

Proof It suffices to modify the proof of Theorem 4.5 so that no local power substitution is needed. No changes are required up to the case that $f(0) = 0$.

So assume that $V^G = \{0\}$ and $f(0) = 0$. We may suppose that $f_1 \neq 0$ (otherwise $f = 0$, as $\sigma_1(v) = \langle v \mid v \rangle$, and the lifting problem is trivial). By Theorem 3.2, we find a finite covering $\{\pi_k\colon U_k \to U\}$ of a neighborhood U of 0 by $\mathcal{C}$-mappings π_k, each of which is a composite of finitely many local blow-ups with smooth center such that for each k the non-zero $f_j \circ \pi_k$ (for $1 \leq j \leq n$) simultaneously have only normal crossings.

Let k be fixed and let $x_0 \in U_k$. Then x_0 admits a neighborhood W_k with suitable coordinates in which $x_0 = 0$ and such that (for $1 \leq j \leq n$) either $f_j \circ \pi_k = 0$ or

$$(f_j \circ \pi_k)(x) = x^{\alpha_j} f_j^k(x),$$

where f_j^k is a non-vanishing $\mathcal{C}$-function on W_k, and $\alpha_j \in \mathbb{N}^q$. By Lemma 5.2, there exists a $\delta \in \mathbb{N}^q$ such that $\alpha_1 = 2\delta$.

If $\delta = 0$, then $(f_1 \circ \pi_k)(x_0) = f_1^k(x_0) \neq 0$ and hence $(f \circ \pi_k)(x_0) \neq 0$. Let $w \in \sigma^{-1}((f \circ \pi_k)(x_0))$. The stabilizer G_w is a proper subgroup of G, since $V^G = \{0\}$. By the induction hypothesis (and reduction to the slice representation $G_w \to \mathrm{GL}(N_w)$), there exists a finite covering $\{\pi_{kl}\colon W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (ii) such that for all l, the mapping $f \circ \pi_k \circ \pi_{kl}$ allows a $\mathcal{C}$-lift through w on W_{kl}.

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Assume then that $\delta \neq 0$. By Lemma 5.2, we have $\alpha_j \geq d_j\delta$ for those $1 \leq j \leq n$ with $f_j \circ \pi_k \neq 0$. Then

$$\tilde{f}^k(x) := (x^{-d_1\delta} f_1(\pi_k(x)), \dots, x^{-d_n\delta} f_n(\pi_k(x)))$$

is a $\mathcal{C}$-mapping whose image lies in $\sigma(V)$. Since $\alpha_1 = 2\delta = d_1\delta$ and $f_1^k(x_0) \neq 0$, we have $\tilde{f}^k(x_0) \neq 0$. Let $w \in \sigma^{-1}(\tilde{f}^k(x_0))$. The stabilizer G_w is a proper subgroup of G, since $V^G = \{0\}$. By the induction hypothesis (and reduction to the slice representation $G_w \to \mathrm{GL}(N_w)$), there exists a finite covering $\{\pi_{kl} \colon W_{kl} \to W_k\}$ of W_k (possibly shrinking W_k) of the type described in (ii) such that for all l, the mapping $\tilde{f}^k \circ \pi_{kl}$ admits a $\mathcal{C}$-lift $\bar{f}^{kl}$ through w on W_{kl}. Since a lift of $\tilde{f}^k$ provides a lift of $f \circ \pi_k$ by multiplying by the monomial factor $m(x) := x^\delta$, the $\mathcal{C}$-mapping $x \mapsto m(\pi_{kl}(x)) \cdot \bar{f}^{kl}(x)$ forms a lift through 0 of $x \mapsto (f \circ \pi_k \circ \pi_{kl})(x)$ for $x \in W_{kl}$.

Since k and x_0 were arbitrary, the assertion of the theorem follows from §§ 4.1. ■

Corollary 5.4 ($\mathcal{C}$-lifting of curves — real version) *A $\mathcal{C}$-curve $c \colon \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ admits a $\mathcal{C}$-lift $\bar{c}$ locally near each $x_0 \in \mathbb{R}$. If ρ is polar, there exists a global orthogonal $\mathcal{C}$-lift which is unique up to the action of a constant in G.*

Proof The local statement follows immediately from Theorem 5.3. (Each local blow-up is the identity map, and, in fact, each non-zero component c_j of c automatically has only normal crossings.)

The proof of the remaining assertions is (almost literally) the same as in [1, 4.2] where the real analytic case is treated. ■

6 Weak Lifting over Invariants

Let M be a $\mathcal{C}$-manifold of dimension q equipped with a C^∞ Riemannian metric. Consider a $\mathcal{C}$-mapping $f \colon M \to V /\!\!/ G = \sigma(V) \subseteq \mathbb{C}^n$. We show in this section that f admits a lift $\bar{f}$ that is "piecewise Sobolev $W^{1,1}_{\mathrm{loc}}$", *i.e.*, there exists a closed nullset $E \subseteq M$ of finite $(q-1)$-dimensional Hausdorff measure such that $\bar{f}$ belongs to $W^{1,1}(K \setminus E, V)$ for all compact subsets $K \subseteq M$. In particular, the classical derivative $d\bar{f}$ exists almost everywhere and belongs to L^1_{loc}, which is best possible among L^p spaces (see Remark 6.9). The distributional derivative of $\bar{f}$ may not be locally integrable. In fact, in general f does not allow for $W^{1,1}_{\mathrm{loc}}$-lifts (by example [22, 7.17]). However, we shall conclude that the lift $\bar{f}$ belongs to $\mathrm{SBV}_{\mathrm{loc}}$, *i.e.*, special functions of bounded variation (see §§6.4).

We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. It depends on the metric but not on the ambient space. For a Lipschitz mapping $f \colon \mathbb{R}^q \supseteq U \to \mathbb{R}^p$ we have

$$\mathcal{H}^k(f(E)) \leq (\mathrm{Lip}(f))^k \mathcal{H}^k(E), \qquad \text{for all } E \subseteq U,$$

where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. The q-dimensional Hausdorff measure $\mathcal{H}^q$ and the q-dimensional Lebesgue measure $\mathcal{L}^q$ coincide in $\mathbb{R}^q$. If B is a subset of a k-plane in $\mathbb{R}^q$, then $\mathcal{H}^k(B) = \mathcal{L}^k(B)$.

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6.1 The Class $\mathcal{W}^{\mathcal{C}}$

Let M be a $\mathcal{C}$-manifold of dimension q equipped with a C^∞ Riemannian metric g. We denote by $\mathcal{W}^{\mathcal{C}}(M)$ the class of all real or complex valued functions f with the following properties:

($\mathcal{W}_1$) f is defined and of class $\mathcal{C}$ on the complement $M \setminus E_{M,f}$ of a closed set $E_{M,f}$ with $\mathcal{H}^q(E_{M,f}) = 0$ and $\mathcal{H}^{q-1}(E_{M,f}) < \infty$.
($\mathcal{W}_2$) f is bounded on $M \setminus E_{M,f}$.
($\mathcal{W}_3$) ∇f belongs to $L^1(M \setminus E_{M,f}) = L^1(M)$.

For example, the Heaviside function belongs to $\mathcal{W}^{\mathcal{C}}((-1,1))$, but the function $f(x) := \sin 1/|x|$ does not. A $\mathcal{W}^{\mathcal{C}}$-function f may or may not be defined on $E_{M,f}$. Note that if the volume of M is finite, then

$$f \in \mathcal{W}^{\mathcal{C}}(M) \Longrightarrow f \in L^\infty(M \setminus E_{M,f}) \cap W^{1,1}(M \setminus E_{M,f}). \tag{6.1}$$

We shall also use the notations $\mathcal{W}^{\mathcal{C}}_{\text{loc}}(M)$ and $\mathcal{W}^{\mathcal{C}}(M, \mathbb{C}^n) = (\mathcal{W}^{\mathcal{C}}(M, \mathbb{C}))^n$ with the obvious meanings. Since $\mathcal{W}^{\mathcal{C}}$ is preserved by linear coordinate changes, we can consider $\mathcal{W}^{\mathcal{C}}(M, V)$ for vector spaces V.

In general $\mathcal{W}^{\mathcal{C}}(M)$ depends on the Riemannian metric g. It is easy to see that $\mathcal{W}^{\mathcal{C}}(U)$ is independent of g for any relatively compact open subset $U \subseteq M$. Thus $\mathcal{W}^{\mathcal{C}}_{\text{loc}}(M)$ is also independent of g. If (U,u) is a relatively compact coordinate chart and g^u_{ij} is the coordinate expression of g, then there exists a constant C such that $(1/C)\delta_{ij} \le g^u_{ij} \le C\delta_{ij}$ as bilinear forms.

From now on, given a $\mathcal{C}$-manifold M, we tacitly choose a C^∞ Riemannian metric g on M and consider $\mathcal{W}^{\mathcal{C}}(M)$ with respect to g.

6.2 Notations

Let us introduce the following notation: For $\rho = (\rho_1, \dots, \rho_q) \in (\mathbb{R}_{>0})^q$, $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$, and $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0,1\}^q$, set

$$\Omega(\rho) := \{x \in \mathbb{R}^q : |x_j| < \rho_j \text{ for all } j\},$$
$$\Omega_\epsilon(\rho) := \{x \in \mathbb{R}^q : 0 < (-1)^{\epsilon_j} x_j < \rho_j \text{ for all } j\}.$$

The power transformation

$$\psi_{\gamma,\epsilon} \colon \mathbb{R}^q \to \mathbb{R}^q \colon (x_1, \dots, x_q) \mapsto ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q})$$

maps $\Omega_\mu(\rho)$ onto $\Omega_\nu(\rho^\gamma)$, where $\nu = (\nu_1, \dots, \nu_q)$ is such that $\nu_j \equiv \epsilon_j + \gamma_j \mu_j \mod 2$ for all j. The range of the j-th coordinate behaves differently depending on whether γ_j is even or odd. So let us consider

$$\bar\psi_{\gamma,\epsilon} \colon \Omega_\epsilon(\rho) \to \Omega_\epsilon(\rho^\gamma) : (x_1, \dots, x_q) \mapsto ((-1)^{\epsilon_1}|x_1|^{\gamma_1}, \dots, (-1)^{\epsilon_q}|x_q|^{\gamma_q}),$$

and its inverse mapping

$$\bar\psi^{-1}_{\gamma,\epsilon} \colon \Omega_\epsilon(\rho^\gamma) \to \Omega_\epsilon(\rho) \colon (x_1, \dots, x_q) \mapsto ((-1)^{\epsilon_1}|x_1|^{\frac{1}{\gamma_1}}, \dots, (-1)^{\epsilon_q}|x_q|^{\frac{1}{\gamma_q}}).$$

Then we have $\bar{\psi}_{\gamma,\epsilon} \circ \bar{\psi}_{\gamma,\epsilon}^{-1} = \mathrm{id}_{\Omega_\epsilon(\rho^\gamma)}$ and $\bar{\psi}_{\gamma,\epsilon}^{-1} \circ \bar{\psi}_{\gamma,\epsilon} = \mathrm{id}_{\Omega_\epsilon(\rho)}$ for all $\gamma \in (\mathbb{R}_{>0})^q$ and $\epsilon \in \{0,1\}^q$. Note that

$$\{\bar{\psi}_{\gamma,\epsilon} : \epsilon \in \{0,1\}^q\} \subseteq \{\psi_{\gamma,\mu}|_{\Omega_\epsilon(\rho)} : \epsilon, \mu \in \{0,1\}^q\}. \tag{6.2}$$

Let us define $\bar{\psi}_\gamma^{-1} \colon \Omega(\rho^\gamma) \to \Omega(\rho)$ by setting $\bar{\psi}_\gamma^{-1}|_{\Omega_\epsilon(\rho^\gamma)} := \bar{\psi}_{\gamma,\epsilon}^{-1}$, for $\epsilon \in \{0,1\}^q$, and by extending it continuously to $\Omega(\rho^\gamma)$. Analogously, define $\bar{\psi}_\gamma \colon \Omega(\rho) \to \Omega(\rho^\gamma)$ such that $\bar{\psi}_\gamma \circ \bar{\psi}_\gamma^{-1} = \mathrm{id}_{\Omega(\rho^\gamma)}$ and $\bar{\psi}_\gamma^{-1} \circ \bar{\psi}_\gamma = \mathrm{id}_{\Omega(\rho)}$.

6.3 $\mathcal{W}^{\mathcal{C}}$-Lifting

Lemma 6.1 ([22, 7.6]) *If $f \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho))$, then $f \circ \bar{\psi}_\gamma^{-1} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho^\gamma))$.*

Lemma 6.2 ([22, 7.9]) *Let $\varphi \colon M' \to M$ be a blow-up of a $\mathcal{C}$-manifold M with center a closed $\mathcal{C}$-submanifold C of M. If $f \in \mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}(M')$, then $f \circ (\varphi|_{M' \setminus \varphi^{-1}(C)})^{-1} \in \mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}(M)$.*

Lemma 6.3 ([22, 7.10]) *Let M be a $\mathcal{C}$-manifold. Let $K \subseteq M$ be compact, let $\{(U_j, u_j) : 1 \le j \le N\}$ be a finite collection of connected relatively compact coordinate charts covering K, and let $f_j \in \mathcal{W}^{\mathcal{C}}(U_j)$. Then after shrinking the U_j slightly so that they still cover K, there exists a function $f \in \mathcal{W}^{\mathcal{C}}(\bigcup_j U_j)$ satisfying the following condition: if $x \in \bigcup_j U_j$, then either $x \in E_{\bigcup_j U_j}$ or $f(x) = f_j(x)$ for some $j \in \{i : x \in U_i\}$.*

Theorem 6.4 ($\mathcal{W}^{\mathcal{C}}$-lifting) *Let M be a $\mathcal{C}$-manifold. Consider a $\mathcal{C}$-mapping $f \colon M \to V /\!\!/ G = \sigma(V) \subseteq \mathbb{C}^n$. For any compact subset $K \subseteq M$ there exists a relatively compact neighborhood W of K and a lift $\bar{f}$ of f on W which belongs to $\mathcal{W}^{\mathcal{C}}(W,V)$. In particular, we have that $d\bar{f}$ is L^1.*

Proof By Theorem 4.5, there exists a neighborhood W of K and a finite covering $\{\pi_k \colon U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings, each of which is either a local blow-up Φ with smooth center or a local power substitution Ψ (see §§4.1) such that for all k the mapping $f \circ \pi_k$ allows a $\mathcal{C}$-lift on U_k.

In view of Lemma 6.3 the proof of the theorem will be complete once the following assertions are proved:

(i) Let $\Psi = \iota \circ \psi \colon W' \to W \to M$ be a local power substitution. If $f \circ \Psi$ allows a lift of class $\mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}$, then so does $f|_W$.

(ii) Let $\Phi = \iota \circ \varphi \colon U' \to U \to M$ be a local blow-up with smooth center. If $f \circ \Phi$ allows a lift of class $\mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}$, then so does $f|_U$.

Assertion (ii) follows easily from Lemma 6.2. To prove (i), let $\bar{f}^{\Psi} = \bar{f}^{\psi_{\gamma,\epsilon}}$ (for some $\gamma \in (\mathbb{N}_{>0})^q$ and all $\epsilon \in \{0,1\}^q$, see §§4.1) be a lift of $f \circ \Psi$ which belongs to $\mathcal{W}^{\mathcal{C}}_{\mathrm{loc}}(W', V)$.

We can assume without loss of generality (possibly shrinking W') that for some $\rho \in (\mathbb{R}_{>0})^q$ we have $W' = \Omega(\rho)$, $W = \Omega(\rho^\gamma)$, and $\bar{f}^{\psi_{\gamma,\epsilon}} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho), V)$. Let us define a mapping $\bar{f}^{\bar{\psi}_\gamma} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho), V)$ by setting (in view of (6.2))

$$\bar{f}^{\bar{\psi}_\gamma}|_{\Omega_\epsilon(\rho)} := \bar{f}^{\bar{\psi}_{\gamma,\epsilon}}|_{\Omega_\epsilon(\rho)}, \quad \epsilon \in \{0,1\}^q.$$

On the set $\{x \in \Omega(\rho) : \prod_j x_j = 0\}$, we may define $\bar{f}^{\bar{\psi}_\gamma}$ arbitrarily such that it forms a lift of $f \circ \iota \circ \bar{\psi}_\gamma$. By Lemma 6.1,

$$\bar{f} := \bar{f}^{\bar{\psi}_\gamma} \circ \bar{\psi}_\gamma^{-1} \in \mathcal{W}^{\mathcal{C}}(\Omega(\rho^\gamma), V) = \mathcal{W}^{\mathcal{C}}(W, V).$$

Clearly, $\bar{f}$ forms a lift of $f|_W$. Thus the proof of (i) is complete. ∎

Corollary 6.5 (Local $\mathcal{W}^{\mathcal{C}}$-sections) *Assume that* $\rho\colon G \to \mathrm{GL}(V)$ *is coregular,* i.e., $\mathbb{C}[V]^G$ *is generated by algebraically independent elements. Then* $\sigma\colon V \to V/\!\!/G = \sigma(V) = \mathbb{C}^n$ *admits local* $\mathcal{W}^{\mathcal{C}}$*-sections (which map into the union of the closed orbits), for* $\mathcal{C}$ *any class of* C^∞*-functions satisfying* $(A_{1'})$, (A_2)–(A_6).

Proof Apply Theorem 6.4 to the identity mapping on $V/\!\!/G = \sigma(V) = \mathbb{C}^n = \mathbb{R}^{2n}$ (which is of class $\mathcal{C}$ by $(A_{1'})$). ∎

6.4 Special Functions of Bounded Variation

Let $U \subseteq \mathbb{R}^q$ be open. A real valued function $f \in L^1(U)$ is said to have *bounded variation*, or to belong to $\mathrm{BV}(U)$, if its distributional derivative is representable by a finite Radon measure Df in U. (See [2].) For $f \in \mathrm{BV}(U)$ we have the decomposition $Df = D^a f + D^j f + D^c f$ in the *absolutely continuous part* $D^a f$, the *jump part* $D^j f$, and the *Cantor part* $D^c f$. We say that $f \in \mathrm{BV}(U)$ is a *special function of bounded variation*, and we write $f \in \mathrm{SBV}(U)$, if the Cantor part of its derivative $D^c f$ is zero. This notion is due to [8]. A complex valued function $f\colon U \to \mathbb{C}$ is in $\mathrm{BV}(U, \mathbb{C})$ (resp. $\mathrm{SBV}(U, \mathbb{C})$), if $(\mathrm{Re}\, f, \mathrm{Im}\, f) \in (\mathrm{BV}(U))^2$ (resp. $(\mathrm{SBV}(U))^2$); similarly for vector valued functions.

6.5 SBV-Lifting

Proposition 6.6 ([2, 4.4]) *Let* $U \subseteq \mathbb{R}^q$ *be open and bounded,* $E \subseteq \mathbb{R}^q$ *closed, and* $\mathcal{H}^{q-1}(E \cap U) < \infty$. *Then any function* $f\colon U \to \mathbb{R}$ *that belongs to* $L^\infty(U \setminus E) \cap W^{1,1}(U \setminus E)$ *belongs also to* $\mathrm{SBV}(U)$.

Theorem 6.7 (*SBV*-lifting) *Let* $U \subseteq \mathbb{R}^q$ *be open. Consider a* $\mathcal{C}$*-mapping* $f\colon U \to V/\!\!/G = \sigma(V) \subseteq \mathbb{C}^n$. *For any compact subset* $K \subseteq U$ *there exists a relatively compact neighborhood* W *of* K *and a lift* $\bar{f}$ *of* f *on* W *that belongs to* $\mathrm{SBV}(W, V)$.

Proof It follows immediately from Theorem 6.4, Proposition 6.6, and (6.1). ∎

Corollary 6.8 (Local *SBV*-sections) *Assume that* $\rho\colon G \to \mathrm{GL}(V)$ *is coregular. Then* $\sigma\colon V \to V/\!\!/G = \sigma(V) = \mathbb{C}^n$ *admits local* SBV*-sections (which map into the union of the closed orbits).*

Proof Combine Corollary 6.5 with Proposition 6.6 or apply Theorem 6.7 to the identity mapping on $V/\!\!/G = \sigma(V) = \mathbb{C}^n = \mathbb{R}^{2n}$. ∎

Remark 6.9 In general a $\mathcal{C}$ (even polynomial) mapping f into $V/\!\!/G = \sigma(V)$ does not allow a lift $\bar{f}$ with $d\bar{f} \in L^p_{\mathrm{loc}}$ for any $1 < p \le \infty$ (see [22, 7.13]). Moreover, there is in general (for $q \ge 2$) no lift in $W^{1,1}_{\mathrm{loc}}$ and in VMO (see [22, 7.17 and 7.18]). If $q = 1$, then locally absolutely continuous lifts exist, even under milder conditions [17].

7 Weak Lifting in the Real Case

For the sake of completeness, in Theorem 7.1 we list the conclusions for $\mathcal{W}^{\mathcal{C}}$ (resp. SBV) lifting over invariants of compact Lie group representations. For polar representations of compact Lie groups in Theorem 7.2 we show that $\mathcal{C}$-mappings actually admit lifts that are "piecewise locally Lipschitz". We do not know whether that is true when the representation is not polar.

Theorem 7.1 (Weak lifting — real version) *Let $\rho\colon G \to \mathrm{O}(V)$ be an orthogonal finite-dimensional representation of a compact Lie group G. Let M be a $\mathcal{C}$-manifold. Consider a $\mathcal{C}$-mapping $f\colon M \to V/G = \sigma(V) \subseteq \mathbb{R}^n$. For any compact subset $K \subseteq M$ there exists a relatively compact neighborhood W of K and a lift $\bar f$ of f on W such that:*

(i) *$\bar f$ belongs to $\mathcal{W}^{\mathcal{C}}(W,V)$.*
(ii) *If M is open in $\mathbb{R}^q$, then $\bar f$ belongs to $SBV(W,V)$.*

Proof The proofs are essentially the same as in §6; instead of Theorem 4.5 we use Theorem 5.3 and we do not have to deal with local power substitutions. ■

Due to [12], if G is finite, then any continuous lift $\bar f$ of f is actually locally Lipschitz, given that f is C^k with k sufficiently large (namely, $k = k(\rho)$ in Table 1). But continuous lifts do not exist in general (for instance, if G is a finite rotation group). Sufficient for the existence of continuous and thus locally Lipschitz lifts is that G is a finite reflection group or that G is connected and ρ is polar.

Evidently, if there are no continuous lifts, we cannot hope for locally Lipschitz lifts. However, there might exist lifts that are "piecewise locally Lipschitz".

7.1 The Class $\mathcal{L}^{\mathcal{C}}$

Let M be a $\mathcal{C}$-manifold equipped with a C^∞ Riemannian metric g. We denote by $\mathcal{L}^{\mathcal{C}}(M)$ the class of all real functions f with the properties $(\mathcal{W}_1)$, $(\mathcal{W}_2)$ from §§6.1 and

$(\mathcal{L}_3)$ ∇f is bounded on $M \setminus E_{M,f}$.

For example, the Heaviside function belongs to $\mathcal{L}^{\mathcal{C}}((-1,1))$ (as does any step function), but the function $f(x) := |x|^\alpha$, for $0 < \alpha < 1$, does not. If the volume of M is finite, then $\mathcal{L}^{\mathcal{C}}(M) \subseteq \mathcal{W}^{\mathcal{C}}(M)$. An $\mathcal{L}^{\mathcal{C}}$-function f may or may not be defined on $E_{M,f}$. We shall also use $\mathcal{L}^{\mathcal{C}}_{\mathrm{loc}}(M)$, $\mathcal{L}^{\mathcal{C}}(M,\mathbb{R}^n) = (\mathcal{L}^{\mathcal{C}}(M,\mathbb{R}))^n$, and $\mathcal{L}^{\mathcal{C}}(M,V)$ for vector spaces V with the obvious meanings.

For relatively compact open subsets $U \subseteq M$, the set $\mathcal{L}^{\mathcal{C}}(U)$ is independent of g.

7.2 $\mathcal{L}^{\mathcal{C}}$-Lifting—Real Version

Theorem 7.2 ($\mathcal{L}^{\mathcal{C}}$-lifting — real version) *Let $\rho\colon G \to \mathrm{O}(V)$ be a polar orthogonal real finite dimensional representation of a compact Lie group G. Let M be a $\mathcal{C}$-manifold. Consider a $\mathcal{C}$-mapping $f\colon M \to V/G = \sigma(V) \subseteq \mathbb{R}^n$. For any compact subset $K \subseteq M$ there exists a relatively compact neighborhood W of K and a lift $\bar f$ of f on W which belongs to $\mathcal{L}^{\mathcal{C}}(W,V)$.*

Proof Without loss of generality we may assume that G is finite, since, by §§2.4, we can reduce to the representation $W(\Sigma) \to \mathrm{O}(\Sigma)$ for a Cartan subspace Σ.

By Theorem 7.1 there exists a lift $\bar{f}$ of f on W that belongs to $\mathcal{W}^{\mathcal{C}}(W, V)$. We claim that $\bar{f}$ is actually in $\mathcal{L}^{\mathcal{C}}(W, V)$. We have to check that $d\bar{f}$ is bounded on $W \setminus E_{W,\bar{f}}$. For contradiction suppose that there exists a sequence $(x_k) \subseteq W \setminus E_{W,\bar{f}}$ with $x_k \to x_\infty \in E_{W,\bar{f}}$ such that $d\bar{f}(x_k)$ is unbounded. Without loss of generality we may assume that W is open in $\mathbb{R}^q$, (by passing to a subsequence) that x_k converges fast to x_∞, *i.e.,* for all n the sequence $k^n(x_k - x_\infty)$ is bounded, and that there is a sequence $(v_k) \subseteq \mathbb{R}^q$ that converges fast to 0 such that $\|d_{v_k}\bar{f}(x_k)\| \to \infty$. By the general curve lemma [14, 12.2], for $s_k \geq 0$ reals with $\sum_k s_k < \infty$ there exist a C^∞-curve c and a converging sequence of reals t_k such that $c(t + t_k) = (x_k - x_\infty) + tv_k$ for $|t| < s_k$ for all k. For the shifted curve $\tilde{c}(t) := c(t) + x_\infty$, we thus have

$$\|(\bar{f} \circ \tilde{c})'(t_k)\| = \|d_{v_k}\bar{f}(x_k)\| \to \infty.$$

Now $\bar{f} \circ \tilde{c}$ represents a lift of the C^∞-curve $f \circ \tilde{c}$. By [11, 4.2, 8.1], $f \circ \tilde{c}$ admits a C^1-lift $\overline{f \circ \tilde{c}}$, and, by [11, 3.4], there exist $g_k \in G$ such that $(\bar{f} \circ \tilde{c})'(t_k) = g_k.(\overline{f \circ \tilde{c}})'(t_k)$. So $\|(\bar{f} \circ \tilde{c})'(t_k)\| = \|(\overline{f \circ \tilde{c}})'(t_k)\|$ is bounded, which is a contradiction. ■

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