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ISOTONIAN ALGEBRAS

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Abstract. To a pair P and Q of finite posets we attach the toric ring $K[P, Q]$ whose generators are in bijection to the isotone maps from P to Q. This class of algebras, called isotonian, are natural generalizations of the so-called Hibi rings. We determine the Krull dimension of these algebras and for particular classes of posets P and Q we show that $K[P, Q]$ is normal and that their defining ideal admits a quadratic Gröbner basis.

§1. Introduction

Let K be a field, and let L be a finite distributive lattice. In 1987, Takayuki Hibi [11] introduced the K-algebra $K[L]$ which nowadays is called the Hibi ring of the distributive lattice L. The K-algebra $K[L]$ is generated over K by the elements $\alpha \in L$ with defining relations $\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta)$ with $\alpha, \beta \in L$. In the early paper, Hibi also showed that $K[L]$ is a normal Cohen–Macaulay domain.

One remarkable fact is that $K[L]$ may be viewed as a toric ring. This can be seen by using Birkhoff's fundamental theorem from 1937 which says that each finite distributive lattice is the ideal lattice of a finite poset P. Indeed, the subposet of L, induced by the join-irreducible elements of L, is the poset P whose ideal lattice $\mathcal{I}(P)$ is the given distributive lattice L. Having the poset P of join-irreducible elements of L at our disposal, we can write $K[L]$ as the K-algebra generated over K by the monomials $u_I = \prod_{p\in I} x_p \prod_{p\notin I} y_p \in K[\{x_p, y_p\}_{p\in P}]$ with $I \in \mathcal{I}(P)$. Hibi, in his classical paper also showed that the Krull dimension of $K[L]$ is equal to $|P| + 1$, where $|P|$ is the cardinality of P.

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Birkhoff's theorem can also be phrased as follows: let P and Q be finite posets. We denote by $\mathrm{Hom}(P, Q)$ the set of order preserving maps, also called isotone maps. Observe that $\mathrm{Hom}(P, Q)$ is again a finite poset, by setting $\varphi \leqslant \psi$ if and only if $\varphi(p) \leqslant \psi(p)$ for all $p \in P$. Let L be a distributive lattice with P its subposet of join-irreducible elements. Birkhoff's theorem is equivalent to saying that there is a natural isomorphism of posets $L \cong \mathrm{Hom}(P, [2])$. Here for an integer n we denote by $[n]$ the totally ordered set $\{1 < 2 < \cdots < n\}$.

The set $\mathcal{P}$ of finite posets together with the isotone maps forms a category, first considered in [6]. In the same paper the authors introduced the ideal $L(P, Q)$ which is generated by the monomials

$$u_\varphi = \prod_{p\in P} x_{p,\varphi(p)} \quad \text{with } \varphi \in \mathrm{Hom}(P, Q).$$

In the special cases when $P = [n]$ or $Q = [n]$, the ideals $L(P, Q)$ first appeared in the work [4] of Ene, Mohammadi and Jürgen Herzog. In the sequel the algebraic and homological properties of the ideals $L(P, Q)$ have been subject of further investigations in the papers [10] and [14].

Here we are interested in the algebras $K[P, Q]$ which are the toric rings generated over K by the monomials u_φ with $\varphi \in \mathrm{Hom}(P, Q)$. We call these algebras *isotonian* because their generators are in bijection with the isotone maps from P to Q. In the special case that $Q = [2]$, we obtain the classical Hibi rings. Accordingly, one would expect that isotonian algebras share all the nice properties of Hibi rings. In Theorem 3.1 it is shown that $\dim K[P, Q] = |P|(|Q| - s) + rs - r + 1$, where r is the number of connected components of P and s is the number of connected components of Q. The proof of this theorem also shows that the algebraic variety whose coordinate ring is $K[P, Q]$ is birationally equivalent to the Segre product of suitable copies of affine spaces.

Hibi rings are normal and Cohen–Macaulay. Do the same properties hold for isotonian algebras? In Corollary 4.3 it is shown that this is the case when the Hasse diagram of P is a forest. But this is also the case when $Q = [n]$, as shown in [4, Corollary 4.3]. Corollary 4.3 is a straightforward consequence of a more general fact. Indeed, in Theorem 4.2 the following result is proved: let P' be the poset which is obtained from P by adding an element p' to P which has a unique upper or lower neighbor in P. Then $K[P, Q]$ is normal if and only if $K[P', Q]$ is normal. Based on these results and on computational evidence we are lead to conjecture that all isotonian algebras are normal Cohen–Macaulay domains. One way to prove this conjecture in general would be to show that there exists a term order such that the initial ideal of the defining ideal $J_{P,Q}$ of $K[P,Q]$ is squarefree. By a theorem of Sturmfels [17, Chapter 8] this would imply that $K[P,Q]$ is normal, and by a theorem of Hochster [13, Theorem 1] this in turn implies Cohen–Macaulayness. In Theorem 5.5 we show that if P is a chain and Q is a rooted or co-rooted poset, then $J_{P,Q}$ has a quadratic (and hence also a squarefree) Gröbner basis with respect to the reverse lexicographic order induced by a canonical labeling of the variables. In general, $J_{P,Q}$ may contain generators of arbitrarily high degree. This happens if Q contains as an induced subposet, what we call a poset cycle. On the other hand, we conjecture that $J_{P,Q}$ is quadratically generated if and only if Q does not contain any induced poset cycle of length greater than 4.

We would like to mention that Engström and Norén [5] introduced an algebra, associated to the set of homomorphisms between two finite graphs. While in our case the generators of the algebra correspond to the graph of isotone maps between posets, the generators of the Engström–Norén algebras correspond to the edge maps between the two given graphs. It is interesting that the classical Hibi rings, which are a very special case of isotonian algebras, allows also some interpretation in their theory, [5, Theorem 10.2]. Otherwise, though in spirit similar, the two theories are independent and disjoint.

§2. Operations on posets and the K-algebra $K[P,Q]$

Let P and Q be finite posets. A map $\varphi\colon P \to Q$ is called *isotone* (order preserving), if $\varphi(p) \leqslant \varphi(p')$ for all $p, p' \in P$ with $p < p'$. The set of all isotone maps from P to Q is denoted by $\mathrm{Hom}(P,Q)$. Obviously, if P, Q and R are finite posets and $\varphi \in \mathrm{Hom}(P,Q)$ and $\psi \in \mathrm{Hom}(Q,R)$, then $\psi \circ \varphi \in \mathrm{Hom}(P,R)$. We denote by $\mathcal{P}$ the category whose objects are finite posets and whose morphisms are isotone maps. We note that $\mathrm{Hom}(P,Q)$ is again a poset with $\varphi \leqslant \psi$ for $\varphi, \psi \in \mathrm{Hom}(P,Q)$ if and only if $\varphi(p) \leqslant \psi(p)$ for all $p \in P$. Thus, $\mathrm{Hom}(P, \_)\colon \mathcal{P} \to \mathcal{P}$ is a covariant and $\mathrm{Hom}(\_, Q)\colon \mathcal{P} \to \mathcal{P}$ a contravariant functor.

Let $P \in \mathcal{P}$, and let $p_1, p_2 \in P$. One says that p_2 *covers* p_1 if $p_1 < p_2$, and there is no $p \in P$ with $p_1 < p < p_2$. We define the graph $G(P)$ on the vertex set P as follows: a 2-element subset $\{p_1, p_2\}$ is an edge of $G(P)$ if and only if p_2 covers p_1 or p_1 covers p_2. The graph $G(P)$ is the underlying graph of

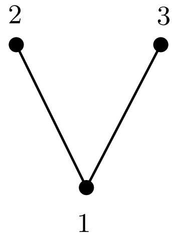

Figure 1.
Poset P.

the so-called *Hasse diagram* of P which may be viewed as a directed graph whose edges are the ordered pairs (p_1, p_2), where p_2 covers p_1.

We say that P is *connected*, if $G(P)$ is a connected graph. Given two posets P_1 and P_2, the sum $P_1 + P_2$ is defined to be the disjoint union of the elements of P_1 and P_2 with $p \leqslant q$ if and only if $p, q \in P_1$ or $p, q \in P_2$ and $p \leqslant q$ in the corresponding posets P_1 or P_2. Then it is clear that any $P \in \mathcal{P}$ can be written as $P = \sum_{i=1}^{r} P_i$ where each P_i is a connected subposet of P. The subposets P_i of P are called the *connected components* of P.

EXAMPLE 2.1. Let P be the poset displayed in Figure 1.

We identify an isotone map $\varphi\colon P \to P$ with $\varphi(p_i) = p_{j_i}$ $(i = 1, 2, 3)$ with the sequence $j_1 j_2 j_3$. With the notation introduced, the elements of $\mathrm{Hom}(P, P)$ are:

$$111, 112, 121, 113, 131, 122, 123, 132, 133, 222, 333.$$

The poset $\mathrm{Hom}(P, P)$ is displayed in Figure 2.

The *product* $P_1 \times P_2$ of P_1 and P_2 is the poset whose elements are the pairs (p_1, p_2) with $p_1 \in P_1$ and $p_2 \in P_2$. The order relations in $P_1 \times P_2$ are defined componentwise. For $P_1 \times P_2 \times \cdots \times P_s$ we also write $\prod_{i=1}^{s} P_i$.

In the next lemma we present two obvious (but useful) rules of the Hom-posets.

LEMMA 2.2. *Let $P, P_1, P_2, \ldots, P_r$ and $Q, Q_1, Q_2, \ldots, Q_s$ be finite posets, and assume that P is connected. Then:*

(a) $\mathrm{Hom}(\sum_{i=1}^{r} P_i, Q) \cong \prod_{i=1}^{r} \mathrm{Hom}(P_i, Q)$;
(b) $\mathrm{Hom}(P, \sum_{i=1}^{s} Q_i) \cong \sum_{i=1}^{s} \mathrm{Hom}(P, Q_i)$.

We now introduce the *isotonian algebra* $K[P, Q]$ attached to a pair P, Q of finite posets. For this purpose we fix a field K, and consider the polynomial

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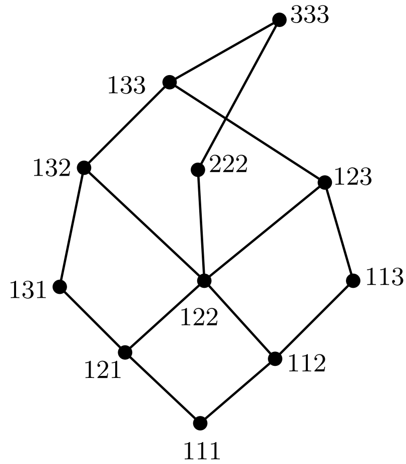

Figure 2.
Hom(P, P).

ring over K in the variables $x_{p,q}$ with $p \in P$ and $q \in Q$. Then $K[P, Q]$ is the toric ring generated over K by the monomials

$$u_{\varphi} = \prod_{p \in P} x_{p,\varphi(p)}.$$

Let $R_1 = K[f_1, \dots, f_r] \subset K[x_1, \dots, x_n]$ and $R_2 = K[g_1, \dots, g_s] \subset K[y_1, \dots, y_m]$ be two standard graded K-algebras. Then

$$R_1 \otimes R_2 = K[f_1, \dots, f_r, g_1, \dots, g_s]$$

is the tensor product of R_1 and R_2 over K, while

$$R_1 * R_2 = K[\{f_i g_j : i = 1, \dots, r, j = 1, \dots, s\}]$$

is the Segre product of R_1 and R_2.

The following isomorphisms are immediate consequences of Lemma 2.2 and the definition of isotonian algebras.

LEMMA 2.3. *Let $P, P_1, P_2, \dots, P_r$ and $Q, Q_1, Q_2, \dots, Q_s$ be finite posets and assume that P is connected. Then:*

(a) $K[\sum_{i=1}^{r} P_i, Q] \cong K[P_1, Q] * K[P_2, Q] * \cdots * K[P_r, Q]$;
(b) $K[P, \sum_{i=1}^{s} Q_i] \cong K[P, Q_1] \otimes K[P, Q_2] \otimes \cdots \otimes K[P, Q_s]$.

As a first consequence we obtain

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COROLLARY 2.4. *Let P be a finite poset with connected components $P_1, P_2, \ldots, P_r$ and let Q be a finite poset with connected components $Q_1, Q_2, \ldots, Q_s$. Then $K[P, Q]$ is normal if all $K[P_i, Q_j]$ are normal. In particular, if this is the case, then $K[P, Q]$ is also Cohen–Macaulay.*

Proof. It is a well-known fact that the Segre product or the tensor product of normal standard graded toric rings is normal. Thus the assertion follows. The Cohen–Macaulayness of $K[P, Q]$ is then the consequence of Hochster's theorem [13, Theorem 1]. □

Let $P \in \mathcal{P}$. The *dual poset* $P^\vee$ of P is the poset whose underlying set coincides with that of P and whose order relations are reversed. In other words, $p < p'$ in P is equivalent to $p' < p$ in $P^\vee$.

Since $\varphi\colon P \to Q$ is isotone if and only if $\varphi\colon P^\vee \to Q^\vee$ is isotone, we have

LEMMA 2.5. *Let P and Q be finite posets. Then the K-algebras $K[P, Q]$ and $K[P^\vee, Q^\vee]$ are isomorphic.*

§3. The dimension of $K[P, Q]$

In this section we compute the dimension of the algebra $K[P, Q]$. The result is given in

THEOREM 3.1. *Let P and Q be finite posets and let r be the number of connected components of P and s be the number of connected components of Q. Then* $\dim K[P, Q] = |P|(|Q| - s) + rs - r + 1$.

Proof. By using Lemma 2.3 and the fact that for any two standard graded K-algebras R and S, $\dim R \otimes S = \dim R + \dim S$ and $\dim R * S = \dim R + \dim S - 1$ (see [7, Theorem 4.2.3]), the desired conclusion follows once we have shown that $\dim K[P, Q] = |P|(|Q| - 1) + 1$ in the case that P and Q are connected. Let L be the quotient field of $K[P, Q]$. Since $K[P, Q]$ is an affine domain, the $\dim K[P, Q]$ is the transcendence degree of L/K.

Let L' be the field generated by the elements:

(i) $u_q = \prod_{p \in P} x_{p,q}$ for $q \in Q$;
(ii) $x_{p,q}/x_{p,q'}$ for $p \in P$, $q, q' \in Q$ and $q < q'$.

We show that $L' = L$. It is obvious that elements u_q belong to L. Let I be any poset ideal of P and let $q, q' \in Q$ with $q < q'$. Then $\varphi_I^{(q,q')} : P \to Q$ defined by

$$\varphi_I^{(q,q')}(p) = \begin{cases} q & \text{if } p \in I, \\ q' & \text{otherwise,} \end{cases}$$

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is an isotone map with image $\{q, q'\}$. Given any $p_0 \in P$, one can find two poset ideals I, J of P such that $J = I \cup \{p_0\}$ and $p_0 \notin I$. Then $u_{\varphi_I^{(q,q')}} = \prod_{p\in I} x_{p,q} \prod_{p\notin I} x_{p,q'}$ and $u_{\varphi_J^{(q,q')}} = (\prod_{p\in I} x_{p,q}) x_{p_0,q} (\prod_{p\notin I} x_{p,q'}) x_{p_0,q'}^{-1}$. It follows that $u_{\varphi_J^{(q,q')}}/u_{\varphi_I^{(q,q')}} = x_{p_0,q}/x_{p_0,q'}$. It shows that all elements in (ii) belong to L. Therefore, $L' \subset L$.

In order to prove the converse inclusion, let $P = \{p_1, \ldots, p_n\}$ and let $q_1, \ldots, q_n$ be arbitrary elements in Q. We show by induction on k that the monomial $x_{p_1,q_1} \cdots x_{p_k,q_1} x_{p_{k+1},q_{k+1}} \cdots x_{p_n,q_n}$ can be obtained as a product of $x_{p_1,q_1} \cdots x_{p_n,q_n}$ and suitable elements in (ii) and their inverses. This will imply that any monomial generator of $K[P, Q]$ is contained in L', because the element $x_{p_1,q_1} x_{p_2,q_2} \cdots x_{p_n,q_n}$ can then be written as a product of $x_{p_1,q_1} \cdots x_{p_n,q_1}$ and elements of type (ii) and their inverses. As a consequence this will imply that $L \subset L'$.

For $k = 1$, the statement is trivial. Since Q is connected, there exists a sequence of elements $\tilde{q}_1, \ldots, \tilde{q}_t$ in Q with $\tilde{q}_1 = q_1$ and $\tilde{q}_t = q_{k+1}$ and $\tilde{q}_i$ and $\tilde{q}_{i+1}$ are comparable for all i in Q. Then

$$x_{p_{k+1},q_1}/x_{p_{k+1},q_{k+1}} = \prod_{i=1}^{t} (x_{p_{k+1},\tilde{q}_i}/x_{p_{k+1},\tilde{q}_{i+1}}),$$

where each $x_{p_{k+1},\tilde{q}_i}/x_{p_{k+1},\tilde{q}_{i+1}}$ or its inverse is a monomial of type (ii). Then the monomial $x_{p_1,q_1} \cdots x_{p_k,q_1} x_{p_{k+1},q_1} x_{p_{k+2},q_{k+2}} \cdots x_{p_n,q_n}$ can be obtained as a product of the monomial $x_{p_1,q_1} \cdots x_{p_k,q_1} x_{p_{k+1},q_{k+1}} \cdots x_{p_n,q_n}$ and the monomial $x_{p_{k+1},q_1}/x_{p_{k+1},q_{k+1}}$.

Let T be a spanning tree (i.e., a maximal tree) of $G(Q)$ and choose an element $p_0 \in P$. Next we show that L is generated by the elements:

(i) $u_q = \prod_{p\in P} x_{p,q}$, for $q \in Q$;
(ii′) $x_{p,q}/x_{p,q'}$ for $p \in P \setminus \{p_0\}$ and $\{q, q'\} \in E(T)$ with $q < q'$;

where $E(T)$ is the edge set of T. For any $q < q'$ in Q, we can obtain a sequence $\tilde{q}_1, \ldots, \tilde{q}_t$ in T such that $\tilde{q}_1 = q$ and $\tilde{q}_t = q'$ and $\tilde{q}_i$ and $\tilde{q}_{i+1}$ are neighbors for all $i = 1, \ldots, t$. Consequently, we obtain

$$x_{p,q}/x_{p,q'} = \prod_{i=1}^{t-1} x_{p,\tilde{q}_i}/x_{p,\tilde{q}_{i+1}}.$$

Also note that $x_{p_0,q}/x_{p_0,q'} = (u_q/u_{q'})(\prod_{p\in P\setminus\{p_0\}}(x_{p,q'}/x_{p,q}))$ for $q < q'$, and hence $x_{p_0,q}/x_{p_0,q'}$ is a product of elements of type (i) and (ii′) and their

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inverses. This shows that all monomials of type (ii) can be obtained as product of monomials of type (i) and type (ii′) and their inverses.

Let $\mathcal{A}$ be the set of exponent vectors (in $\mathbb{Q}^{P\times Q}$) of the monomials of type (i) and $\mathcal{B}$ be the set of exponent vectors of the monomials of type (ii′). We show the set of vectors $\mathcal{A}\cup\mathcal{B}$ is linearly independent. Then [18, Proposition 7.1.17] implies that the Krull dimension of $K[P,Q]$ is equal to the cardinality of the set $\mathcal{A}\cup\mathcal{B}$.

Note that vectors in $\mathcal{A}$ are linearly independent because their support is pairwise disjoint. Also, the vectors in $\mathcal{B}$ are linearly independent. To see this, we let $\mathcal{B}_p\subset\mathcal{B}$ be the set of exponent vectors of monomials $x_{p,q}/x_{p,q'}$ in (ii′) with p fixed. Then $\mathcal{B}$ is the disjoint union of the sets $\mathcal{B}_p$ with $p\in P\setminus\{p_0\}$. Moreover, for distinct $p,p'\in P\setminus\{p_0\}$ the vectors in $\mathcal{B}_p$ and $\mathcal{B}_{p'}$ have disjoint support. Thus it suffices to show that for a fixed $p\in P\setminus\{p_0\}$ the vectors in $\mathcal{B}_p$ are linearly independent. Now we fix such $p\in P\setminus\{p_0\}$. Then the matrix formed by the vectors of $\mathcal{B}_p$ is the incidence matrix of the tree T which is known to be of maximal rank (see for example [18, Lemma 8.3.2]). Thus the vectors of $\mathcal{B}_p$ are linearly independent, as desired.

Finally, to show that $\mathcal{A}\cup\mathcal{B}$ is a set of linearly independent vectors it suffices to show $V\cap W=\{0\}$, where V is the $\mathbb{Q}$-vector space spanned by $\mathcal{A}$ and W is the $\mathbb{Q}$-vector space spanned by $\mathcal{B}$. Thus we have to show that if $v\in V$ and $w\in W$ with $v=w$, then $v=w=0$. This is equivalent to say that if u is a product of elements of (i) and its inverses and u' is a product of elements of (ii′) and its inverses, and if $u=u'$, then $u=u'=1$. This is indeed the case, because if $u\neq 1$, then u contains factors of the form $x_{p_0,q}$, while u' does not.

Now we determine the cardinality of $\mathcal{A}\cup\mathcal{B}$ (which coincides with $\dim K[P,Q]$). Observe that $|\mathcal{A}|=|Q|$ and $|\mathcal{B}|=(|P|-1)(|Q|-1)$, so that $|\mathcal{A}\cup\mathcal{B}|=(|P|-1)(|Q|-1)+|Q|=|P|(|Q|-1)+1$. ∎

COROLLARY 3.2. *Let P and Q be finite connected posets with $|P|=n$ and $|Q|=m$, and let $X_{P,Q}$ be the irreducible variety given by the defining ideal $J_{P,Q}$ of $K[P,Q]$. Then $X_{P,Q}$ is birationally equivalent to the variety $Y_{n,m}$ whose coordinate ring is the n-fold Segre product of the m-dimensional polynomial ring over K.*

Proof. Let $P=\{p_1,\ldots,p_n\}$, and let L be the quotient field of $K[P,Q]$ and T be the toric ring whose generators are of the form $\prod_{j=1}^{n}x_{p_j,q_j}$ with $q_j\in Q$. Note that

$$T=S_1*S_2*\cdots*S_n,$$

where $S_i = K[x_{p_i,q}\colon q \in Q]$ for $i = 1, \ldots, n$. It was shown in the proof of Theorem 3.1 that L is also the quotient field of T. This yields the desired conclusion. ∎

§4. Normality of $K[P, Q]$

In this section we prove normality of $K[P, Q]$ in certain cases. As $K[P, Q]$ is a toric ring, normality of $K[P, Q]$, by a theorem of Hochster [13, Theorem 1], implies that $K[P, Q]$ is Cohen–Macaulay as well.

Let H be an affine semigroup and $\mathbb{Z}H$ be the associated group of H. The semigroup H is normal, if whenever $da \in H$ for $a \in \mathbb{Z}H$ and some $d \in \mathbb{N}$, then $a \in H$. By [2, Theorem 6.1.4], $K[H]$ is normal if and only if H is normal. We apply this criterion to $K[P, Q]$. Since $K[P, Q]$ is a toric ring, there exists an affine semigroup H such that $K[H] = K[P, Q]$. It follows that $K[P, Q]$ is normal if and only if H is normal. In our particular situation, when $P = \{p_1, \ldots, p_n\}$, the monomials in $K[\mathbb{Z}H]$ corresponding to the elements in $\mathbb{Z}H$ are of the form $u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1}$ with $u_i = x_{p_1,q_{i1}} x_{p_2,q_{i2}} \cdots x_{p_n,q_{in}}$ for $i = 1, \ldots, s$, $s \geqslant 1$ and $q_{ij} \in Q$. Thus it will follow that $K[P, Q]$ is normal if whenever $(u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1})^d \in K[P, Q]$, then $u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1} \in K[P, Q]$.

LEMMA 4.1. *Let P, Q be two finite posets with $P = \{p_1, \ldots, p_n\}$, and let as before $S_i = K[x_{p_i,q}\colon q \in Q]$ for $i = 1, \ldots, n$. Assume that for any integer $d > 1$ and any sequence of monomials $v_1, \ldots, v_t \in S_1 * S_2 * \cdots * S_n$ with $(v_1 v_2 \cdots v_t)^d \in K[P, Q]$ it follows that $v_1 v_2 \cdots v_t \in K[P, Q]$. Then the toric ring $K[P, Q]$ is normal.*

Proof. Suppose $(u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1})^d \in K[P, Q]$. We show that there exist monomials $v_1, \ldots, v_t \in S_1 * S_2 * \cdots * S_n$ such that $(u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1})^d = (v_1 \cdots v_t)^d$. Then our assumption implies that $v_1 \cdots v_t \in K[P, Q]$. Since $(u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1})^d = (v_1 \cdots v_t)^d$ and since $K[P, Q]$ is a toric ring, we have $u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1} = v_1 \cdots v_t$, and so the desired conclusion follows.

Without loss of generality we may assume that

$$(u_1^{\pm 1} u_2^{\pm 1} \cdots u_s^{\pm 1})^d = (u_1^{-1} \cdots u_r^{-1})^d (u_{r+1} \cdots u_s)^d$$

for some $r \leqslant s$. Let $u_i = \prod_{j=1}^n x_{p_j,q_{ij}}$ for $1 \leqslant i \leqslant s$. Then

$$(u_1^{-1} \cdots u_r^{-1})^d (u_{r+1} \cdots u_s)^d = \prod_{i=1}^n \left[\left(\prod_{k=1}^r x_{p_i,q_{ki}}^{-d} \right) \left(\prod_{k=r+1}^s x_{p_i,q_{ki}}^{d} \right) \right]$$

belongs to $K[P, Q]$. Since the elements in $K[P, Q]$ have no negative powers, it follows that each factor in $\prod_{k=1}^r x_{p_i,q_{ki}}^{-d}$ cancels against a factor in

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$\prod_{k=r+1}^{s} x_{p_i,q_{ki}}^d$. Without loss of generality we may assume that for each i, $\prod_{k=1}^{r} x_{p_i,q_{ki}}^{-d}$ cancels against $\prod_{k=r+1}^{2r} x_{p_i,q_{ki}}^d$. Then

$$(u_1^{\pm 1}u_2^{\pm 1}\cdots u_s^{\pm 1})^d = \prod_{i=1}^{n}\left(\prod_{k=2r+1}^{s} x_{p_i,q_{ki}}^d\right) = (v_1\cdots v_{s-2r})^d,$$

where $v_j = \prod_{i=1}^{n} x_{p_i,q_{2r+j,i}}$. ∎

Based on the criterion given in Lemma 4.1 we show

THEOREM 4.2. *Let P and Q be finite posets, and let P' be a poset which is obtained from P by adding an element p' to P with the property that p' has a unique neighbor in $G(P')$. Then $K[P',Q]$ is normal if and only if $K[P,Q]$ is normal.*

Proof. By Corollary 2.4 we may assume that P is connected. Let $P=\{p_1,\ldots,p_n\}$ and set $p_{n+1}=p'$. We may assume that p_n is the unique neighbor of p_{n+1} in $G(P)$ and that $p_n < p_{n+1}$. In other words, p_n is the unique element in P covered by p_{n+1}.

Suppose first that $K[P',Q]$ is normal but $K[P,Q]$ is not normal. Then there exist monomials $u_1,\ldots,u_s \in S_1 * \cdots * S_n$ such that $(u_1\cdots u_s)^d \in K[P,Q]$ but $u_1\cdots u_s \notin K[P,Q]$.

We write φ_i for the (not necessarily isotone) map $P \to Q$ corresponding to u_i, and define the map $\varphi_i' : P' \to Q$ by setting

$$\varphi_i'(p) = \begin{cases} \varphi_i(p) & \text{if } p \neq p_{n+1},\\ \varphi_i(p_n) & \text{if } p = p_{n+1}.\end{cases}$$

Let u_i' be the monomial corresponding to φ_i'. Then we have $(u_1'\cdots u_s')^d \in K[P',Q]$ but $u_1'\cdots u_s' \notin K[P',Q]$, a contradiction.

Conversely, suppose that $K[P,Q]$ is normal. Let $u_1,\ldots,u_s \in S_1 * S_1 * \cdots * S_{n+1}$ with $u_i = \prod_{j=1}^{n+1} x_{p_j,q_{ij}}$ for $i=1,\ldots,s$. By Lemma 4.1 it is enough to show that if $(u_1u_2\cdots u_s)^d \in K[P',Q]$ for some $d\in\mathbb{N}$, then $u_1u_2\cdots u_s \in K[P',Q]$.

In order to prove this, we first observe that $\varphi : P' \to Q$ is isotone, if and only if $\varphi(p) \leqslant \varphi(q)$ for all pairs $p,q\in P'$ for which q covers p. Hence, $\varphi : P' \to Q$ is isotone if and only if the restriction of φ to P is isotone and $\varphi(p_n) \leqslant \varphi(p_{n+1})$.

As a consequence of this observation we obtain the following statement: let $v_1,\ldots,v_t \in S_1 * S_1 * \cdots * S_{n+1}$ with $v_i = \prod_{j=1}^{n+1} x_{p_j,q'_{ij}}$ for $i=1,\ldots,t$,

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and set $\tilde{v}_i = \prod_{j=1}^{n} x_{p_j,q'_{ij}}$ for $i = 1, \ldots, t$. Then $v_1 \cdots v_t \in K[P', Q]$ if and only if:

(i) $\tilde{v}_1 \cdots \tilde{v}_t \in K[P, Q]$;
(ii) there exists a permutation $\pi\colon [t] \to [t]$ such that $q'_{in} \leqslant q'_{\pi(i),n+1}$ for all i.

Condition (ii) can be rephrased as follows: let G be the bipartite graph with bipartition $V(G) = V_1 \cup V_2$, where $V_1 = \{x_1, \ldots, x_t\}$ and $V_2 = \{y_1, \ldots, y_t\}$. We let $\{x_i, y_j\}$ be an edge of G if and only if $q'_{in} \leqslant q'_{j,n+1}$. We call G the graph attached to $v_1, \ldots, v_t$. By definition, a perfect matching of G is a disjoint union of t edges of G.

Now condition (ii) is equivalent to the following condition:

(ii′) The graph G attached to $v_1, \ldots, v_t$ admits a perfect matching.

Now we come back to the sequence $u_1, \ldots, u_s \in S_1 * S_1 * \cdots * S_{n+1}$ with $u_i = \prod_{j=1}^{n+1} x_{p_j,q_{ij}}$ for $i = 1, \ldots, s$ for which $(u_1 u_2 \cdots u_s)^d \in K[P', Q]$.

Then (i) implies that $(\tilde{u}_1 \cdots \tilde{u}_s)^d \in K[P, Q]$. By assumption, $K[P, Q]$ is normal. This implies that $\tilde{u}_1 \cdots \tilde{u}_s \in K[P, Q]$, where $\tilde{u}_i = \prod_{j=1}^{n} x_{p_j,q_{ij}}$. Thus it will follow that $u_1 \cdots u_s \in K[P', Q]$ if the graph G attached to $u_1, \ldots, u_s$ admits a perfect matching.

We let $G^{(d)}$ be the graph attached to d copies of $u_1, \ldots, u_s$:

$$\underbrace{u_1, \ldots, u_s, \ldots, u_1, \ldots, u_s}_{d \text{ times}}.$$

Since $(u_1 u_2 \cdots u_s)^d \in K[P', Q]$ it follows from (ii′) that $G^{(d)}$ admits a perfect matching.

Note that $G^{(d)}$ may be viewed as the bipartite graph with vertex decomposition

$$V(G^{(d)}) = V_1^{(d)} \cup V_2^{(d)},$$

where

$$V_1^{(d)} = \{x_i^{(k)} : i = 1, \ldots, s, k = 1, \ldots, d\} \quad \text{and}$$
$$V_2^{(d)} = \{y_i^{(k)} : i = 1, \ldots, s, k = 1, \ldots, d\}.$$

The edges of $G^{(d)}$ are $\{x_i^{(k)}, y_j^{(l)}\}$ with $q_{in} \leqslant q_{j,n+1}$ and $k, l \in [d]$.

Suppose that G has no perfect matching. Then Hall's marriage theorem (see for example [9, Lemma 9.1.2]) implies that there exists a set $S \subset V_1$

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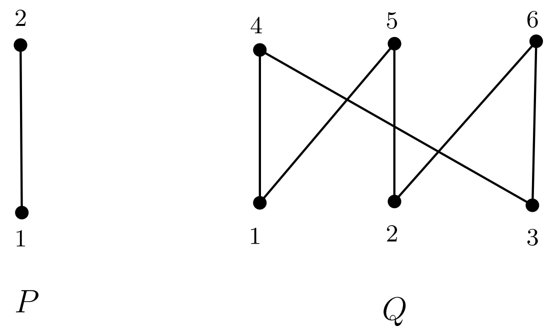

Figure 3.
$J_{P,Q}$ is not quadratic.

such that $|N_G(S)| < |S|$. Here

$$N_G(S) = \{y_j : \{x_i, y_j\} \text{ with } x_i \in S\}$$

is the set of neighbors of S.

Let $S^{(d)} = \{x_i^{(k)} : x_i \in S, 1 \leqslant k \leqslant d\}$. Then $S^{(d)} \subset V_1^{(d)}$ and

$$|N_{G^{(d)}}(S^{(d)})| = d|N_G(S)| < d|S| = |S^{(d)}|.$$

Thus, $G^{(d)}$ does not have a perfect matching, a contradiction. ∎

By using Theorem 4.2 and an obvious induction argument, we obtain

COROLLARY 4.3. *Let P and Q be finite posets. Then $K[P, Q]$ is a normal Cohen–Macaulay domain if $G(P)$ is a tree.*

Corollary 4.3 implies in particular that $K[[n], Q]$ is normal. On the other hand, the normality of $K[P, [n]]$ has been proved in [4, Corollary 4.2]. These types of algebras are called *letterplace algebras*. Thus we have

COROLLARY 4.4. *All letterplace algebras are normal Cohen–Macaulay domains.*

It seems that $K[P, Q]$ is normal, not only when $G(P)$ is a tree. For example, it can be shown by using Normaliz [3] that the isotonian algebra $K[Q, Q]$ is normal for the poset Q shown in Figure 3.

Further computational evidence and the above special cases lead us to the following

CONJECTURE 4.5. *Let P and Q be finite posets. Then $K[P, Q]$ is a normal Cohen–Macaulay domain.*

§5. Gröbner basis

Let P and Q be finite posets. Let $S = K[y_\varphi \colon \varphi \in \mathrm{Hom}(P, Q)]$ be the polynomial ring in the variables y_φ, and let $\pi : S \to K[P, Q]$ be the K-algebra homomorphism defined by $y_\varphi \mapsto \prod_{p\in P} x_{p,\varphi(p)}$. We denote the kernel of π by $J_{P,Q}$.

Let $P = \{p_1, p_2, \ldots, p_n\}$. We may assume that the labeling of the elements of P is chosen such that $p_i < p_j$ implies $i < j$. Similarly, $Q = \{q_1, q_2, \ldots, q_m\}$ is labeled. Having fixed this labeling we sometimes write $y_{(j_1,j_2,\ldots,j_n)}$ for y_φ if $\varphi(p_i) = q_{j_i}$ for $i = 1, \ldots, n$.

For $Q = [2]$ the ideal $J_{P,Q}$ is the defining ideal of the Hibi ring associated with the distributive lattice $L = \mathrm{Hom}(P, [2])$. In this case it is known that $J_{P,Q}$ has a quadratic Gröbner basis, [9, Theorem 10.1.3]. For arbitrary posets P and Q the ideal $J_{P,Q}$ is not always generated in degree 2. The simplest example is given by the posets P and Q displayed in Figure 3.

In this example $J_{P,Q}$ is generated by the binomial $y_{(1,4)}y_{(2,5)}y_{(3,6)} - y_{(1,5)}y_{(2,6)}y_{(3,4)}$.

It is known that in general $J_{P,Q}$ is generated by binomials. Consult [12] for fundamental materials on toric ideals and Gröbner basis.

We identify each $\varphi \in \mathrm{Hom}(P, Q)$ with the sequence $(j_1^{(\varphi)}, \ldots, j_n^{(\varphi)})$, where $\varphi(p_i) = q_{j_i^{(\varphi)}}$ for $1 \leqslant i \leqslant n$. We introduce the total ordering $<$ of the variables y_φ with $\varphi \in \mathrm{Hom}(P, Q)$ by setting $y_\varphi < y_\psi$ if $j_{i_0}^{(\varphi)} < j_{i_0}^{(\psi)}$, where i_0 is the smallest integer for which $j_{i_0}^{(\varphi)} \neq j_{i_0}^{(\psi)}$. Let $<_{\mathrm{rev}}$ denote the reverse lexicographic order on S induced by the above ordering $<$ of the variables y_φ with $\varphi \in \mathrm{Hom}(P, Q)$.

EXAMPLE 5.1. Let $P = \{p_1, p_2, p_3\}$, where $p_1 < p_2$ and $p_1 < p_3$, and $Q = \{q_1, q_2, q_3\}$, where $q_1 < q_3$ and $q_2 < q_3$. Then the total ordering $<$ on the variables y_φ with $\varphi \in \mathrm{Hom}(P, Q)$ is

$$y_{(1,1,1)} < y_{(1,1,3)} < y_{(1,3,3)} < y_{(2,2,2)} < y_{(2,2,3)} < y_{(2,3,3)} < y_{(3,3,3)}.$$

We say that $u_\varphi u_\psi$ is *nonstandard* with respect to $<_{\mathrm{rev}}$ if there exist φ' and ψ' for which $u_\varphi u_\psi = u_{\varphi'} u_{\psi'}$ and $y_{\varphi'} y_{\psi'} <_{\mathrm{rev}} y_\varphi y_\psi$. An expression $w = u_{\varphi_1} u_{\varphi_2} \cdots u_{\varphi_s}$ of a monomial w belonging to $K[P, Q]$ is called *standard* if no $u_{\varphi_i} u_{\varphi_j}$, where $1 \leqslant i < j \leqslant s$, is nonstandard. It follows that each monomial possesses a standard expression. However, a standard expression of a monomial may not be unique.

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EXAMPLE 5.2. Let $P=[2]$ and Q be as in Figure 3. Then each of the expressions $u_{(1,4)}u_{(2,5)}u_{(3,6)}$ and $u_{(1,5)}u_{(2,6)}u_{(3,4)}$ of the monomial $x_{11}x_{12}x_{13}x_{24}x_{25}x_{26}$ is standard.

LEMMA 5.3. *Let $P=\{p_1,\ldots,p_n\}$ be an arbitrary finite poset and $Q=[2]$. Then every monomial belonging to $K[P,Q]$ possesses a unique standard expression.*

Proof. Let $w=u_{\varphi_1}\cdots u_{\varphi_s}$ with $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$. Suppose that there exist $2\leqslant i_0\leqslant n$ and $1\leqslant k<k'\leqslant s$ for which $\varphi_k(p_{i_0})=2$ and $\varphi_{k'}(p_{i_0})=1$. Since $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$, it follows that there is $1\leqslant i'<i_0$ with $\varphi_k(p_{i'})=1$ and $\varphi_{k'}(p_{i'})=2$. Since each of the inverse images $\varphi_k^{-1}(1)$ and $\varphi_{k'}^{-1}(1)$ is a poset ideal of P, it follows that each of the maps $\psi_k:P\to Q$ and $\psi_{k'}:P\to Q$ defined by setting

$$\psi_k(p_i)=\min\{\varphi_k(p_i),\varphi_{k'}(p_i)\}\quad\text{and}\quad\psi_{k'}(p_i)=\max\{\varphi_k(p_i),\varphi_{k'}(p_i)\}$$

for $1\leqslant i\leqslant n$ is isotone. Furthermore, one has $y_{\psi_k}y_{\psi_{k'}}<_{\text{rev}}y_{\varphi_k}y_{\varphi_{k'}}$ and $u_{\varphi_k}u_{\varphi_{k'}}=u_{\psi_k}u_{\psi_{k'}}$. Thus $w=u_{\varphi_1}\cdots u_{\varphi_s}$ cannot be standard. Since $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$, it follows that

$$\varphi_1(p_1)\leqslant\varphi_2(p_1)\leqslant\cdots\leqslant\varphi_s(p_1).$$

Hence, if $w=u_{\varphi_1}\cdots u_{\varphi_s}$ is standard with $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$, then

$$\varphi_1(p_i)\leqslant\varphi_2(p_i)\leqslant\cdots\leqslant\varphi_s(p_i)$$

for $1\leqslant i\leqslant n$. This guarantees that a standard expression of each monomial belonging to $K[P,Q]$ is unique, as desired. □

A finite poset Q is called a *rooted tree* if whenever α,β and γ belong to Q with $\beta<\alpha$ and $\gamma<\alpha$, then either $\beta\leqslant\gamma$ or $\gamma\leqslant\beta$. In other words, a finite poset Q is a rooted tree if a maximal chain of Q descending from each $\alpha\in Q$ is unique. A finite poset is a *co-rooted tree* if its dual poset is a rooted tree.

LEMMA 5.4. *Let $P=[2]$ and let Q be a co-rooted tree. Then each monomial belonging to $K[P,Q]$ possesses a unique standard expression.*

Proof. Let u be a monomial belonging to $K[P,Q]$. Let $u=u_{\varphi_1}\cdots u_{\varphi_s}$ and $u=u_{\psi_1}\cdots u_{\psi_s}$ be standard expressions of u with $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$ and $y_{\psi_1}\leqslant\cdots\leqslant y_{\psi_s}$. Then one has $\varphi_k(1)=\psi_k(1)$ for $1\leqslant k\leqslant s$. Furthermore, $\varphi_k(1)\leqslant\varphi_k(2)$ and $\psi_k(1)\leqslant\psi_k(2)$ for $1\leqslant k\leqslant s$. In order to show that the

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standard expressions $u_{\varphi_1}\cdots u_{\varphi_s}$ and $u_{\psi_1}\cdots u_{\psi_s}$ coincide, we may assume without loss of generality that $\varphi_i \neq \psi_j$ for all i and j. Since Q is a co-rooted tree and since $\varphi_1(1)\leqslant\varphi_1(2)$ and $\psi_1(1)\leqslant\psi_1(2)$ with $\varphi_1(1)=\psi_1(1)$, it follows that $\varphi_1(2)$ and $\psi_1(2)$ are comparable. Let, say, $\varphi_1(2)<\psi_1(2)$. Since $u_{\varphi_1}\cdots u_{\varphi_s}=u_{\psi_1}\cdots u_{\psi_s}$, there is $2\leqslant k_0\leqslant s$ with $\psi_{k_0}(2)=\varphi_1(2)$. Hence $\psi_{k_0}(1)\leqslant\psi_{k_0}(2)<\psi_1(2)$ and $\psi_1(1)\leqslant\psi_{k_0}(2)<\psi_1(2)$. One can then define ψ'_1 and ψ'_{k_0} belonging to $\mathrm{Hom}(P,Q)$ by setting

$$\psi'_1(1)=\psi_1(1),\quad \psi'_1(2)=\psi_{k_0}(2),\quad \psi'_{k_0}(1)=\psi_{k_0}(1),\quad \psi'_{k_0}(2)=\psi_1(2).$$

Then $u_{\psi_1}u_{\psi_{k_0}}=u_{\psi'_1}u_{\psi'_{k_0}}$ with $y_{\psi'_1}<y_{\psi_1}\leqslant y_{\psi_{k_0}}$. Furthermore, $y_{\psi'_{k_0}}\geqslant y_{\psi'_1}$. Hence $y_{\psi'_{k_0}}y_{\psi'_1}<_{\mathrm{rev}}y_{\psi_{k_0}}y_{\psi_1}$. It then follows that $u_{\psi_1}u_{\psi_{k_0}}$ cannot be standard. Hence a standard expression of each monomial belonging to $K[P,Q]$ is unique, as desired. □

THEOREM 5.5. *Let P be a chain and Q a co-rooted tree. Then each monomial belonging to $K[P,Q]$ possesses a unique standard expression.*

Proof. We may assume that $P=[n]$. Then for each $\varphi\in\mathrm{Hom}(P,Q)$, one has $q_{j_1^{(\varphi)}}\leqslant\cdots\leqslant q_{j_n^{(\varphi)}}$. In other words, the image $\varphi([n])$ is a multichain (chain with repetitions) of Q of length n. It then follows that $u_\varphi u_\psi$ with $y_\varphi<y_\psi$ is nonstandard if and only if there is $2\leqslant i\leqslant n$ with $\varphi(i)>\psi(i)$ such that $\varphi(i-1)\leqslant\psi(i)$ and $\psi(i-1)\leqslant\varphi(i)$. In fact, one has $y_{\psi'}y_{\varphi'}<_{\mathrm{rev}}y_\psi y_\varphi$, where

$$\varphi'=(j_\varphi^{(1)},\ldots,j_\varphi^{(i-1)},j_\psi^{(i)},\ldots,j_\psi^{(n)})\quad\text{and}$$
$$\psi'=(j_\psi^{(1)},\ldots,j_\psi^{(i-1)},j_\varphi^{(i)},\ldots,j_\varphi^{(n)}).$$

Given $\varphi\in\mathrm{Hom}(P,Q)$, we introduce $\varphi^*\in\mathrm{Hom}(P\setminus\{n\},Q)$ by setting $\varphi^*(i)=\varphi(i)$ for $i\in[n-1]$. Let u be a monomial belonging to $K[P,Q]$. Let $u=u_{\varphi_1}\cdots u_{\varphi_s}$ and $u=u_{\psi_1}\cdots u_{\psi_s}$ be standard expressions of u with $y_{\varphi_1}\leqslant\cdots\leqslant y_{\varphi_s}$ and $y_{\psi_1}\leqslant\cdots\leqslant y_{\psi_s}$. The above observation guarantees that each of $u_{\varphi_1^*}\cdots u_{\varphi_s^*}$ and $u_{\psi_1^*}\cdots u_{\psi_s^*}$ is a standard expression. Thus, working on induction on n, it follows that $\varphi_k^*=\psi_k^*$ for $1\leqslant k\leqslant s$. In particular $\varphi_k(n-1)=\psi_k(n-1)$ for $1\leqslant k\leqslant s$.

Now, in order to show that the standard expressions $u_{\varphi_1}\cdots u_{\varphi_s}$ and $u_{\psi_1}\cdots u_{\psi_s}$ coincide, one must show that $\varphi_k(n)=\psi_k(n)$ for $1\leqslant k\leqslant s$. This can be done by using the same technique as in the proof of Lemma 5.4. □

COROLLARY 5.6. *Let P be a chain and suppose that each connected component of Q is either a rooted or a co-rooted poset. Then the toric ideal $J_{P,Q}$ possesses a quadratic Gröbner basis.*

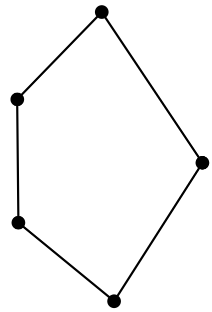

Figure 4.
A cycle but not a poset cycle.

Proof. Let Q be a co-rooted tree. Let $\mathcal{G}$ denote the set of quadratic binomials of S of the form $y_\varphi y_\psi - y_{\varphi'} y_{\psi'}$ with $y_{\varphi'} y_{\psi'} <_{\text{rev}} y_\varphi y_\psi$. Furthermore, write $\text{in}_{<_{\text{rev}}}(\mathcal{G})$ for the monomial ideal of S which is generated by those quadratic monomials $y_\varphi y_\psi$ for which $u_\varphi u_\psi$ are nonstandard. Let w and w' be monomials of S with $w \neq w'$ such that neither w nor w' belongs to $\text{in}_{<_{\text{rev}}}(\mathcal{G})$. It then follows from Theorem 5.5 that $\pi(w) \neq \pi(w')$. In other words, the set of those monomials $\pi(w)$ of $K[P, Q]$ with $w \notin \text{in}_{<_{\text{rev}}}(\mathcal{G})$ is linearly independent. By virtue of the well-known technique ([1, Lemma 1.1] and [16, (0.1)]) on initial ideals, this fact guarantees that $\mathcal{G}$ is a quadratic Gröbner basis of $J_{P,Q}$ with respect to $<_{\text{rev}}$, as desired.

Let Q be a rooted tree. Lemma 2.5 says that $K[P^\vee, Q^\vee]$ is isomorphic to $K[P, Q]$. Since $P^\vee = P$ and since $Q^\vee$ is a co-rooted tree, it follows that $J_{P,Q}$ possesses a quadratic Gröbner basis, as required.

Now assume that each component Q_i $(i = 1, \ldots, m)$ of Q is either rooted or co-rooted. As seen before, the toric ideal of each $K[P, Q_i]$ has a quadratic Gröbner basis. Since by Lemma 2.3(b), $K[P, Q]$ is the tensor product of the K-algebras $K[P, Q_i]$ it follows that the Gröbner basis of $J_{P,Q}$ is the union of the Gröbner basis of the J_{P,Q_i}. Thus the desired result follows. ∎

As we have seen at the beginning of this section, $J_{P,Q}$ is not always generated by quadratic binomials. We say that C is a *poset ℓ-cycle*, if the vertices of C are $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell$ whose covering relations are $a_i < b_i$, $a_i < b_{i+1}$ for $1 \leqslant i \leqslant \ell$ where $b_{\ell+1} = b_1$. The poset Q in Figure 3 is a poset 6-cycle. Note that not any cycle of $G(P)$ is a poset cycle (see for example Figure 4).

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Based on Theorem 5.5, Corollary 5.6 and experimental evidence we propose

CONJECTURE 5.7. *Let $P, Q \in \mathcal{P}$, and assume that Q does not contain any induced poset cycle of length greater than 4. Then $J_{P,Q}$ is generated by quadratic binomials.*

The poset Q given in Figure 3 is a poset cycle. As expected by our conjecture, $J_{[2],Q}$ is not generated by quadrics. Actually, $J_{[2],Q}$ does not even possess a quadratic Gröbner basis with respect to the reverse lexicographic term order induced by the natural order of the variables.

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