Proceedings of the Edinburgh Mathematical Society (1992) 35, 511–518 ©

ASYMPTOTIC BEHAVIOUR OF IDEALS RELATIVE TO INJECTIVE MODULES OVER COMMUTATIVE NOETHERIAN RINGS II

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(Received 17th January 1991)

Let E be an injective module over a commutative Noetherian ring A (with non-zero identity), and let $\mathfrak{a}$ be an ideal of A. The submodule $(0:_E \mathfrak{a})$ of E has a secondary representation, and so we can form the finite set $\mathrm{Att}_A(0:_E \mathfrak{a})$ of its attached prime ideals. In [1, 3.1], we showed that the sequence of sets $(\mathrm{Att}_A(0:_E \mathfrak{a}^n))_{n \in \mathbb{N}}$ is ultimately constant; in [2], we introduced the integral closure $\mathfrak{a}^{*(E)}$ of $\mathfrak{a}$ relative to E, and showed that $(\mathrm{Att}_A(0:_E (\mathfrak{a}^n)^{*(E)}))_{n \in \mathbb{N}}$ is increasing and ultimately constant. In this paper, we prove that, if $\mathfrak{a}$ contains an element r such that $rE = E$, then $(\mathrm{Att}_A((0:_E \mathfrak{a}^n)/(0:_E (\mathfrak{a}^n)^{*(E)})))_{n \in \mathbb{N}}$ is ultimately constant, and we obtain information about its ultimate constant value.

1991 *Mathematics subject classification*: 13E10.

1. Introduction

Throughout this paper, A will denote a commutative Noetherian ring with non-zero identity, and E will denote an injective A-module.

In [**1**, 2.2], we showed that, for each ideal $\mathfrak{a}$ of A, the submodule $(0:_E \mathfrak{a})$ of E has a secondary representation, and so we can form the finite set $\mathrm{Att}_A(0:_E \mathfrak{a})$ of its attached prime ideals. (Accounts of the relevant theory of secondary representation of modules and attached prime ideals are available in [**5**], [**4**] and [**8**], and we shall use the terminology of [**12**] and [**5**] for these topics.) One of the main results of [**1**] is Theorem 3.1, which shows that the sequence of sets

$$(\mathrm{Att}_A(0:_E \mathfrak{a}^n))_{n \in \mathbb{N}}$$

is ultimately constant: we denote its ultimate constant value by $\mathrm{At}^*(\mathfrak{a}, E)$. This result can be viewed as a companion to [**13**, (3.1)(iii)], which shows that, for an ideal I in a commutative ring R (with identity) and an Artinian R-module N, the sequence of sets

$$(\mathrm{Att}_R(0:_N I^n))_{n \in \mathbb{N}}$$

is ultimately constant; and this result can, in turn, be viewed as dual to Brodmann's result [**3**] that, for a Noetherian A-module M, the sequence of sets

$$(\mathrm{Ass}_A(M/\mathfrak{a}^n M))_{n \in \mathbb{N}}$$

is ultimately constant. In fact, our proof of Theorem 3.1 in [1] depended heavily on Brodmann's work.

In [2], we introduced concepts of reduction and integral closure of $\mathfrak{a}$ relative to the injective A-module E, and we showed that these concepts have properties which reflect some of those of the classical concepts of reduction and integral closure introduced by Northcott and Rees in [9].

We say that the ideal $\mathfrak{a}$ of A is a *reduction* of the ideal $\mathfrak{b}$ of A *relative to* E if $\mathfrak{a}\subseteq\mathfrak{b}$ and there exists $s\in\mathbb{N}$ (we use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers) such that $(0:_E\mathfrak{a}\mathfrak{b}^s)=(0:_E\mathfrak{b}^{s+1})$. An element x of A is said to be *integrally dependent on* $\mathfrak{a}$ *relative to* E if there exists $n\in\mathbb{N}$ such that

$$\left(0:_E\sum_{i=1}^{n}x^{n-i}\mathfrak{a}^i\right)\subseteq(0:_E x^n).$$

In fact, this is the case if and only if $\mathfrak{a}$ is a reduction of $\mathfrak{a}+Ax$ relative to E [2, 2.2]; moreover,

$$\mathfrak{a}^{*(E)}:=\{y\in A: y \text{ is integrally dependent on } \mathfrak{a} \text{ relative to } E\}$$

is an ideal of A, called the *integral closure of* $\mathfrak{a}$ *relative to* E, and is the largest ideal of A which has $\mathfrak{a}$ as a reduction relative to E. The main result of [2] is Theorem 3.2, which shows that the sequence of sets

$$(\mathrm{Att}_A(0:_E(\mathfrak{a}^n)^{*(E)}))_{n\in\mathbb{N}}$$

is increasing and ultimately constant; we denote its ultimate constant value by $\overline{\mathrm{At}}^*(\mathfrak{a},E)$. Our proof of this result used, among other things, the result of L. J. Ratliff [10, (2.4) and (2.7)] that the sequence of sets $(\mathrm{ass}(\mathfrak{a}^n)^-)_{n\in\mathbb{N}}$ is increasing and ultimately constant, where $(\mathfrak{a}^n)^-$ denotes the classical integral closure of the ideal $\mathfrak{a}^n$. (For a proper ideal $\mathfrak{c}$ of A, we use $\mathrm{ass}\,\mathfrak{c}$ to denote the set of associated prime ideals of $\mathfrak{c}$ for primary decomposition. We interpret $\mathrm{ass}\,A$ as $\emptyset$.)

The above-mentioned results of Brodmann and Ratliff have led to a large body of research: see, for example, McAdam's book [7]. Indeed, that research provides ideas for possible directions in which the theory of asymptotic behaviour of ideals relative to injective A-modules might be pursued. For example, [7, 11.16] shows that, if the ideal $\mathfrak{a}$ of A contains a non-zerodivisor on A, then the sequence of sets

$$(\mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n))_{n\in\mathbb{N}}$$

is ultimately constant; moreover, if we denote the ultimate constant value of the above sequence by $\mathrm{Cs}^*(\mathfrak{a},A)$, and the ultimate constant values of the sequences

$$(\mathrm{ass}\,\mathfrak{a}^n)_{n\in\mathbb{N}} \quad\text{and}\quad (\mathrm{ass}(\mathfrak{a}^n)^-)_{n\in\mathbb{N}}$$

by $\mathrm{As}^*(\mathfrak{a},A)$ and $\overline{\mathrm{As}}^*(\mathfrak{a},A)$ respectively, then [7, 11.19] shows that (still assuming $\mathfrak{a}$ contains a non-zerodivisor), $\mathrm{As}^*(\mathfrak{a},A)=\overline{\mathrm{As}}^*(\mathfrak{a},A)\cup \mathrm{Cs}^*(\mathfrak{a},A)$. These results raise questions about asymptotic behaviour relative to E: under what conditions on $\mathfrak{a}$ and E can we show that the sequence of sets

$$(\mathrm{Att}_A((0:_E\mathfrak{a}^n)/(0:_E(\mathfrak{a}^n)^{*(E)})))_{n\in\mathbb{N}}$$

is ultimately constant, and, when this is the case and $\mathrm{Ct}^*(\mathfrak{a},E)$ denotes the ultimate constant value of the sequence, are we also able to show that

$$\mathrm{At}^*(\mathfrak{a},E)=\overline{\mathrm{At}}^*(\mathfrak{a},E)\cup \mathrm{Ct}^*(\mathfrak{a},E)?$$

These questions are the concern of this paper. It is an easy consequence of our methods and results in [1] and [2] that the sequence is stable and the second question has an affirmative answer when $\mathfrak{a}$ contains a non-zerodivisor (and E is an arbitrary injective A-module). However, it is more interesting and perhaps more appropriate to consider the case where it is assumed only that there exists $r\in\mathfrak{a}$ such that $rE=E$: this is automatically the case when $\mathfrak{a}$ contains a non-zerodivisor on A, but can also occur when $\mathfrak{a}$ consists entirely of zerodivisors on A. The purpose of this paper is to prove similar results for this more general situation.

2. Notation and previous results

Throughout the paper, $\mathfrak{a}$ will denote an ideal of the commutative Noetherian ring A, and E will denote an injective A-module.

Notation 2.1. (i) We shall use the notation $\mathrm{Occ}(E)$ of [**12**, Section 2] in connection with our injective A-module E: this is explained as follows. By well-known work of Matlis and Gabriel, there is a family $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda}$ of prime ideals of A for which $E\cong\bigoplus_{\alpha\in\Lambda}E(A/\mathfrak{p}_\alpha)$ (we use $E(L)$ to denote the injective envelope of an A-module L), and the set $\{\mathfrak{p}_\alpha:\alpha\in\Lambda\}$ is uniquely determined by E: we denote it by $\mathrm{Occ}(E)$ (or $\mathrm{Occ}_A(E)$).

(ii) We shall use $\mathrm{As}^*(\mathfrak{a},A)$, $\overline{\mathrm{As}}^*(\mathfrak{a},A)$, $\mathrm{At}^*(\mathfrak{a},E)$ and $\overline{\mathrm{At}}^*(\mathfrak{a},E)$ to denote the ultimate constant values of the sequences of sets

$$(\mathrm{ass}\,\mathfrak{a}^n)_{n\in\mathbb{N}},\quad (\mathrm{ass}(\mathfrak{a}^n)^-)_{n\in\mathbb{N}},\quad (\mathrm{Att}_A(0:_E\mathfrak{a}^n))_{n\in\mathbb{N}}\quad\text{and}\quad (\mathrm{Att}_A(0:_E(\mathfrak{a}^n)^{*(E)}))_{n\in\mathbb{N}}$$

respectively: references for the results which show that these sequences are all ultimately constant were given in the Introduction. In the case in which $\mathfrak{a}$ contains a non-zerodivisor, we shall use $\mathrm{Cs}^*(\mathfrak{a},A)$ to denote the eventual constant value of $(\mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n))_{n\in\mathbb{N}}$: see [7, 11.16].

(iii) We shall also use the notation $\mathfrak{a}(\mathscr{P})$ of [**2**, 1.1] for a subset $\mathscr{P}$ of $\mathrm{Spec}(A)$: this denotes ($\mathfrak{a}$ if $\mathfrak{a}=A$ and), if $\mathfrak{a}$ is proper, the intersection of those primary terms in a minimal primary decomposition of $\mathfrak{a}$ which are contained in at least one member of $\mathscr{P}$. Note, in particular, that this assigns a meaning to $\mathfrak{a}(\mathrm{Occ}(E))$.

We shall need the following results from [1] and [2].

Theorem 2.2 [1, 2.1]. *Let M be a finitely generated A-module. Then the A-module $\mathrm{Hom}_A(M, E)$ has a secondary representation, and, furthermore,*

$$\mathrm{Att}_A(\mathrm{Hom}_A(M,E)) = \{\mathfrak{p}' \in \mathrm{Ass}_A(M) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

Theorem 2.3 [1, 3.1].

$$\mathrm{At}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \mathrm{As}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

Theorem 2.4 [2, 3.2].

$$\overline{\mathrm{At}}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \overline{\mathrm{As}}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

Proof. This is immediate from the proof of [2, 3.2] and the results of Ratliff [10, (2.4) and (2.7)] cited in the Introduction.

3. Consequences of results of McAdam

Remark 3.1. Let $\mathfrak{b}$ be a second ideal of A for which $\mathfrak{a} \subseteq \mathfrak{b}$. Then it follows easily from application of the exact functor $\mathrm{Hom}_A(\ ,E)$ to the canonical exact sequence

$$0 \to \mathfrak{b}/\mathfrak{a} \to A/\mathfrak{a} \to A/\mathfrak{b} \to 0$$

that

$$(0 :_E \mathfrak{a})/(0 :_E \mathfrak{b}) \cong \mathrm{Hom}_A(\mathfrak{b}/\mathfrak{a}, E).$$

Theorem 3.2. *Suppose that $\mathfrak{a}$ contains a non-zerodivisor on A.*

(i) *The sequence of sets*

$$(\mathrm{Att}_A((0 :_E \mathfrak{a}^n)/(0 :_E (\mathfrak{a}^n)^{*(E)})))_{n \in \mathbb{N}}$$

is ultimately constant. We denote its ultimate constant value by $\mathrm{Ct}^*(\mathfrak{a}, E)$.

(ii) *We have*

$$\mathrm{Ct}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \mathrm{Cs}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

(iii) *Consequently,* $\mathrm{At}^*(\mathfrak{a}, E) = \overline{\mathrm{At}}^*(\mathfrak{a}, E) \cup \mathrm{Ct}^*(\mathfrak{a}, E)$.

Proof. First note that, by [**2**, 2.6], we have $(\mathfrak{a}^n)^{*(E)} = (\mathfrak{a}^n)^-(\mathrm{Occ}(E))$, and so, by [**2**, 2.5(iii)],

$$(0:_E(\mathfrak{a}^n)^{*(E)}) = (0:_E(\mathfrak{a}^n)^-)$$

for each $n \in \mathbb{N}$. Hence, by 3.1, for each $n \in \mathbb{N}$,

$$(0:_E \mathfrak{a}^n)/(0:_E(\mathfrak{a}^n)^{*(E)}) \cong \mathrm{Hom}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n, E),$$

and it follows from 2.2 that

$$\mathrm{Att}_A(\mathrm{Hom}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n, E)) = \{\mathfrak{p}' \in \mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

It is now easy to use [7, 11.16] to prove (i) and (ii).

(iii) By 2.4 and (ii) above,

$$\overline{\mathrm{At}}^*(\mathfrak{a}, E) \cup \mathrm{Ct}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \overline{\mathrm{As}}^*(\mathfrak{a}, A) \cup \mathrm{Cs}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

But $\overline{\mathrm{As}}^*(\mathfrak{a}, A) \cup \mathrm{Cs}^*(\mathfrak{a}, A) = \mathrm{As}^*(\mathfrak{a}, A)$, by [7, 11.19], and so, in view of 2.3 above,

$$\begin{aligned}\overline{\mathrm{At}}^*(\mathfrak{a}, E) \cup \mathrm{Ct}^*(\mathfrak{a}, E) &= \{\mathfrak{p}' \in \mathrm{As}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\} \\ &= \mathrm{At}^*(\mathfrak{a}, E).\end{aligned}$$

This completes the proof.

Theorem 3.2 above was proved under the hypothesis that the ideal $\mathfrak{a}$ contains a non-zerodivisor on A. However, in the context of secondary representation, it is, in our view, more appropriate to work under the weaker condition that $\mathfrak{a}$ contains an element r for which $rE = E$. (It should be noted that if $\mathfrak{a}$ contains a non-zerodivisor r' on A, then by [**14**, Proposition 2.6], $r'E = E$.) Thus we would like to obtain the results of 3.2(i) and (iii) under the weaker hypothesis that $\mathfrak{a}$ contains an element r such that $rE = E$. We shall achieve this in Section 4 below.

4. The result

Theorem 4.1. *Suppose that $\mathfrak{a}$ contains an element r such that $rE = E$.*

(i) *The sequence of sets*

$$(\mathrm{Att}_A((0:_E \mathfrak{a}^n)/(0:_E(\mathfrak{a}^n)^{*(E)})))_{n \in \mathbb{N}}$$

https://doi.org/10.1017/S0013091500005782 Published online by Cambridge University Press

is ultimately constant. We denote its ultimate constant value by $\mathrm{Ct}^*(\mathfrak{a}, E)$.

(ii) *We have* $\mathrm{At}^*(\mathfrak{a}, E) = \overline{\mathrm{At}}^*(\mathfrak{a}, E) \cup \mathrm{Ct}^*(\mathfrak{a}, E)$.

Proof. First reason as in the proof of 3.2, using [**2**, 2.6], [**2**, 2.5(iii)], 3.1 and 2.2, to see that, for each $n \in \mathbb{N}$,

$$\mathrm{Att}_A((0:_E \mathfrak{a}^n)/(0:_E (\mathfrak{a}^n)^{*(E)})) = \{\mathfrak{p}' \in \mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

Note that $\mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n) \subseteq \mathrm{Ass}_A(A/\mathfrak{a}^n)$, and recall from 2.3 and 2.4 that

$$\mathrm{At}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \mathrm{As}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}$$

and

$$\overline{\mathrm{At}}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \overline{\mathrm{As}}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathrm{Occ}(E)\}.$$

Let $\mathrm{At}^*(\mathfrak{a}, E) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, and, for each $i = 1, \ldots, t$, choose $\mathfrak{p}_i \in \mathrm{Occ}(E)$ such that $\mathfrak{q}_i \subseteq \mathfrak{p}_i$. Set

$$E' = \bigoplus_{i=1}^{t} E(A/\mathfrak{p}_i),$$

and note that, since, for each $i = 1, \ldots, t$, E has a direct summand isomorphic to $E(A/\mathfrak{p}_i)$, it follows that $rE' = E'$.

Suppose that $h \in \mathbb{N}$ is such that $\mathrm{ass}(\mathfrak{a}^n) = \mathrm{As}^*(\mathfrak{a}, A)$ for all $n \geq h$. It follows from the equations displayed in the first paragraph of this proof (and the fact that $\overline{\mathrm{As}}^*(\mathfrak{a}, A) \subseteq \mathrm{As}^*(\mathfrak{a}, A)$) that $\mathrm{At}^*(\mathfrak{a}, E) = \mathrm{At}^*(\mathfrak{a}, E')$, $\overline{\mathrm{At}}^*(\mathfrak{a}, E) = \overline{\mathrm{At}}^*(\mathfrak{a}, E')$ and

$$\mathrm{Att}_A((0:_E \mathfrak{a}^n)/(0:_E (\mathfrak{a}^n)^{*(E)})) = \mathrm{Att}_A((0:_{E'} \mathfrak{a}^n)/(0:_{E'} (\mathfrak{a}^n)^{*(E')})) \quad \forall n \geq h.$$

It is therefore enough for us to prove the results under the additional assumption that $E = \bigoplus_{i=1}^{t} E(A/\mathfrak{p}_i)$. We shall make this assumption for the remainder of the proof. Note that $\mathrm{Occ}(E) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ is a finite set. Clearly, we can assume that $t \geq 1$.

By [**12**, 2.6] (and prime avoidance), $\mathrm{Att}\, E = \{\mathfrak{p}' \in \mathrm{Ass}\, A : \mathfrak{p}' \subseteq \bigcup_{i=1}^{t} \mathfrak{p}_i\}$, and so it follows that, if we use S to denote the multiplicatively closed subset $A \setminus \bigcup_{i=1}^{t} \mathfrak{p}_i$, then $r/1 \in S^{-1}A$ is a non-zerodivisor in this ring of fractions. It therefore follows from [**7**, 11.16 and 11.19] that the sequence of sets

$$(\mathrm{Ass}_{S^{-1}A}(S^{-1}((\mathfrak{a}^n)^-/\mathfrak{a}^n)))_{n \in \mathbb{N}}$$

is ultimately constant, and that, if we denote its ultimate constant value by $\mathrm{Cs}^*(S^{-1}\mathfrak{a}, S^{-1}A)$, then

$$\mathrm{As}^*(S^{-1}\mathfrak{a}, S^{-1}A) = \overline{\mathrm{As}}^*(S^{-1}\mathfrak{a}, S^{-1}A) \cup \mathrm{Cs}^*(S^{-1}\mathfrak{a}, S^{-1}A).$$

Thus the sequence of sets

$$\left(\left\{\mathfrak{p}' \in \mathrm{Ass}_A((\mathfrak{a}^n)^-/\mathfrak{a}^n) : \mathfrak{p}' \subseteq \bigcup_{i=1}^{t} \mathfrak{p}_i\right\}\right)_{n \in \mathbb{N}}$$

is ultimately constant, that is (in view of the first four lines of this proof)

$$(\mathrm{Att}_A((0 :_E \mathfrak{a}^n)/(0 :_E (\mathfrak{a}^n)^{*(E)})))_{n \in \mathbb{N}}$$

is ultimately constant; also, if we denote its ultimate constant value by $\mathrm{Ct}^*(\mathfrak{a}, E)$, then the preceding paragraph shows that

$$\left\{\mathfrak{p}' \in \mathrm{As}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \bigcup_{i=1}^{t} \mathfrak{p}_i\right\} = \left\{\mathfrak{p}' \in \overline{\mathrm{As}}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \bigcup_{i=1}^{t} \mathfrak{p}_i\right\} \cup \mathrm{Ct}^*(\mathfrak{a}, E).$$

The result now follows from a further recourse to the first paragraph of this proof.

Acknowledgment. We are grateful to the referee for pointing out that our original proof of Theorem 4.1 above could be shortened.

REFERENCES

1. H. Ansari Toroghy and R. Y. Sharp, Asymptotic behaviour of ideals relative to injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.* (2) **34** (1991), 155–160.

2. H. Ansari Toroghy and R. Y. Sharp, Integral closures of ideals relative to injective modules over commutative Noetherian rings, *Oxford Quart. J. Math.* (2) **42** (1991), 393–402.

3. M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^nM)$, *Proc. Amer. Math. Soc.* **74** (1979), 16–18.

4. D. Kirby, Coprimary decomposition of Artinian modules, *J. London Math. Soc.* (2) **6** (1973), 571–576.

5. I. G. Macdonald, Secondary representation of modules over a commutative ring, in *Symposia Matematica* **11** (Istituto Nazionale di alta Matematica, Roma, 1973), 23–43.

6. E. Matlis, Injective modules over Noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.

7. S. McAdam, *Asymptotic prime divisors* (Lecture Notes in Mathematics **1023**, Springer, Berlin, 1983).

8. D. G. Northcott, Generalized Koszul complexes and Artinian modules, *Quart. J. Math. Oxford* (2) **23** (1972), 289–297.

9. D. G. Northcott and D. Rees, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.

10. L. J. Ratliff, Jr., On asymptotic prime divisors, *Pacific J. Math.* **111** (1984), 395–413.

11. R. Y. Sharp, The Cousin complex for a module over a commutative Noetherian ring, *Math. Z.* **112** (1969), 340–356.

12. R. Y. Sharp, Secondary representations for injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.* (2) **20** (1976), 143–151.

13. R. Y. Sharp, Asymptotic behaviour of certain sets of attached prime ideals, *J. London Math. Soc.* (2) **34** (1986), 212–218.

14. D. W. Sharpe and P. Vámos, *Injective modules* (Cambridge University Press, 1972).

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