# ANNIHILATORS AND THE CS-CONDITION

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**Abstract.** It is proved that if every cyclic right $R$-module is torsionless and $R$ is a left CS-ring then $R$ is semiperfect left continuous with $soc(R_R)$ essential in $_RR$. As a consequence every right cogenerator, left CS-ring $R$ is shown to be right pseudo-Frobenius and left continuous, and an example is given to show that $R$ need not be left selfinjective. It is also proved that if $R$ is a left CS-ring and every cyclic right $R$-module embeds in a free module, then $R$ is quasi-Frobenius if and only if $J(R) \subseteq Z(R_R)$.

**1. Introduction.** Right CS-rings with certain cogenerating (annihilator) conditions were considered by Gomez-Pardo and Guil Asensio in [**10**] and [**11**]. For example they show that if $R$ is a right CS-ring and every cyclic (finitely generated) right $R$-module embeds in a free module then $R$ is right artinian (quasi-Frobenius). They also prove that if $R$ is a right cogenerator right CS-ring then $R$ is right pseudo-Frobenius.

In this paper we consider these same classes of rings but with the left CS-condition rather than the right CS-condition. We show that if $R$ is a left CS-ring and every cyclic right $R$-module is torsionless, then $R$ is a semiperfect left continuous ring with $soc(R_R) \subseteq^{ess} {}_RR$. We use this result to show that if $R$ is a right cogenerator left CS-ring then $R$ is a right pseudo-Frobenius, left continuous ring. An example of Dischinger and Müller [**5**] shows that the ring $R$ need not be left selfinjective. We also prove that if $R$ is a left CS-ring and every cyclic right $R$-module embeds in a free module, then $R$ is quasi-Frobenius if and only if $J(R) \subseteq Z(R_R)$.

Throughout this paper every ring $R$ is associative with unity and all modules are unitary. If $M_R$ is a right $R$-module we write $J(M)$, $Z(M)$, $soc(M)$ and $E(M)$ for the Jacobson radical, the singular submodule, the socle and the injective hull of $M$, respectively. We denote the direct sum of $k$ copies of $M$ by $M^{(k)}$, and the notation $K \subseteq^{ess} M$ means that $K$ is an essential submodule of $M$.

We frequently refer to the following conditions on a module $M_R$:

*The $C_1$-condition (or the CS-condition)*: Every submodule of $M$ is essential in a direct summand of $M$.

*The $C_2$-condition*: Every submodule of $M$ that is isomorphic to a summand of $M$ is itself a summand of $M$.

The module $M_R$ is called *continuous* if $M$ satisfies both the $C_1$- and $C_2$-conditions. The ring $R$ is called *right CS* (*right continuous*) if $R_R$ is a CS-module (a continuous module).

The left (respectively right) annihilator of a subset $X$ of a ring $R$ is denoted $l(X)$ (respectively $r(X)$). The ring $R$ is called *right Kasch* if every simple right $R$-module embeds in $R$ (equivalently if $l(T) \neq 0$ for every maximal right ideal $T$ of $R$). A right $R$-module $M$ is called *torsionless* if $M$ is embedded in a direct product of copies of $R$. We often use the fact that, if $T$ is a right ideal of $R$, then $R/T$ is torsionless as a right $R$-module if and only if $rl(T) = T$. A ring $R$ is called *right cogenerator* if every right $R$-module is torsionless, and $R$ is called a *right PF-ring* (pseudo-Frobenius ring) if it is right cogenerator and right selfinjective (equivalently if it is semiperfect, right selfinjective and $soc(R_R) \subseteq^{ess} R_R$). In [15] $R$ is called a *right GPF-ring* (generalized pseudo-Frobenius ring) if $R$ is semiperfect, right P-injective and $soc(R_R) \subseteq^{ess} R_R$. Here $R$ is called right *P-injective* (*principally injective*) if every $R$-homomorphism from a principal right ideal of $R$ into $R$ is given by left multiplication.

**2. Annihilators and the CS-condition.** We begin with a basic fact about annihilators of right ideals which will be used several times.

LEMMA 2.1. *Let $C$ be a class of right ideals of $R$ with the property that $T \in C$ implies $bT \in C$ for all $b \in R$. The following conditions are equivalent:*

(1) *$rl(T) = T$ for all $T \in C$.*

(2) *$r[l(T) \cap Rb] = T + r(b)$ for all $T \in C$ and all $b \in R$.*

*Proof.* Clearly (2) $\Rightarrow$ (1). Given (1), observe that $T + r(b) \subseteq r[l(T) \cap Rb]$ always holds. If $x \in r[l(T) \cap Rb]$, then $l(bT) \subseteq l(bx)$ (indeed, if $y \in l(bT)$ then $ybT = 0$, so $yb \in l(T) \cap Rb$, whence $ybx = 0$). But then $bx \in rl(bx) \subseteq rl(bT) = bT$ by hypothesis. If $bx = bt$ for $t \in T$, then $(x - t) \in r(b)$ so $x \in T + r(b)$, as required.

REMARK. The class $C$ in Lemma 2.1 could be any of the following classes of right ideals: all; finitely generated; principal; semisimple; minimal or zero; small; singular.

LEMMA 2.2. *Suppose that $rl(T) = T$ for all right ideals $T$ of $R$. If every complement left ideal of $R$ is principal, then $R$ is semiperfect.*

*Proof.* Let $T$ be a right ideal, and let $L$ be a left ideal maximal with respect to $l(T) \cap L = 0$. By hypothesis, $L = Rb, b \in R$, so $T + r(b) = R$ by Lemma 2.1. It suffices (see [14], Theorem 11.1.5) to show that $r(b)$ is minimal with respect to this property (that is, $T$ has an additive complement in $R$). But if $R = T + C$ with $C \subseteq r(b)$, then $L = Rb \subseteq lr(b) \subseteq l(C)$. Since $l(T) \cap l(C) = 0$, the choice of $L$ gives $L = l(C)$, so $C = rl(C) = r(L) = r(b)$, as required.

LEMMA 2.3. *Suppose that $R$ is a right Kasch, left CS-ring. Then $R$ is left continuous with $soc(R_R) \subseteq^{ess} {}_RR$.*

*Proof.* Every right Kasch ring satisfies the left $C_2$-condition (see [21], Lemma 1.15), so $R$ is left continuous. By the left CS-condition, let $soc(R_R) \subseteq^{ess} Re$ for $e^2 = e \in R$, so that $1 - e \in r[soc(R_R)]$. But $r[soc(R_R)] = J(R)$ because $R$ is right Kasch, so $1 - e \in J(R)$. This means that $e = 1$, and so $soc(R_R) \subseteq^{ess} {}_RR$.

PROPOSITION 2.4. *Suppose that $R$ is a left CS-ring for which every cyclic right $R$-module is torsionless. Then $R$ is a semiperfect, left continuous ring with $soc(R_R) \subseteq^{ess} {}_RR$. In particular, $R$ is left finite dimensional.*

*Proof.* We have $rl(T) = T$ for every right ideal $T$ because $R/T$ is torsionless. In particular, $R$ is right Kasch and so is left continuous with $soc(R_R) \subseteq^{ess} {}_RR$ by Lemma 2.3. Furthermore, the left CS-condition shows that every complement left ideal is a summand, and so is principal. Thus $R$ is semiperfect by Lemma 2.2, so write $R = Re_1 \oplus \cdots \oplus Re_n$ where each $e_i$ is a local idempotent. Then each $Re_i$ is a CS-module and so is uniform. Thus $R$ is left finite dimensional.

It was proved by Gomez Pardo and Guil Asensio [**11**] that, for a right cogenerator ring $R$, right CS implies right selfinjective. In other words, $R$ is a right PF-ring (that is $R$ is right cogenerator, right selfinjective) if and only if $R$ is a right cogenerator, right CS-ring. This theorem extends all the known results on the subject. On the other hand, it is well-known [**7**] that a ring $R$ is a left and right PF-ring if and only if $R$ is a right cogenerator, left selfinjective ring. So it is natural to ask whether the result of Gomez Pardo and Guil Asensio can be obtained if we replace the right CS-condition by the left CS-condition. In fact we have

Proposition 2.5. *Let $R$ be a right cogenerator ring.*
(1) *If $R$ is left CS then $R$ is left continuous and right selfinjective (and so is right PF).*
(2) *If $R \oplus R$ is CS as a left $R$-module then $R$ is left and right PF.*

*Proof.* (1) $R$ is a semiperfect, left continuous ring by Proposition 2.4. In particular $R$ has a finite number of isomorphism classes of simple right (and left) $R$-modules. Since $R$ is a right cogenerator, $R$ is right selfinjective by [**7**, Proposition 24.9], and hence is a right PF-ring.

(2) $R$ is left continuous by (1), so is left selfinjective by [**21**, Proposition 1.21]. Hence $R$ is a left PF-ring; it is right PF by (1).

Remark. A right cogenerator, left CS-ring need not be left selfinjective. In fact, an example of Dischinger and Müller [**5**] shows the existence of a local, left continuous right PF-ring which is not left selfinjective.

Before proceeding, we record a fact about embeddings which will be used later. The proof is straightforward.

Lemma 2.6. *If a uniform module $U$ can be embedded in $M_1 \oplus \cdots \oplus M_n$ where each $M_i \neq 0$, then $U$ embeds in $M_k$ for some $k = 1, \cdots, n$.*

In order to strengthen Proposition 2.5, we need the following well known lemma. Recall that a module $M$ is called *finitely embedded (finitely cogenerated)* if $M$ has a finitely generated essential socle.

Lemma 2.7. *Every finitely embedded torsionless right $R$-module $M$ embeds in a free module $R^n$ of finite rank $n$.*

*Proof.* See for example [**8**, Corollary 3.1.B] or [**1**, Propositions 10.2 and 10.7].

Theorem 2.8. *The following conditions are equivalent for a left CS-ring $R$.*
(1) *$R$ is a right PF-ring.*
(2) *$J(R) \subseteq Z(R_R)$ and every 2-generated right $R$-module is torsionless.*

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*Proof.* (1)$\Rightarrow$(2). Every right PF-ring is a right selfinjective, right cogenerator ring.

(2)$\Rightarrow$(1). By Proposition 2.4, $R$ is a semiperfect, left continuous ring with $soc(R_R) \subseteq^{ess} {}_RR$. The fact that $rl(T) = T$ for all right ideals $T$ implies that $R$ is left P-injective, so $soc({}_RR) \subseteq soc(R_R)$ by Theorem 1.14 in [**16**]. Thus, using (2),

$$soc({}_RR) \subseteq soc(R_R) \subseteq r[Z(R_R)] \subseteq r[J(R)] = soc({}_RR)$$

and it follows that $soc({}_RR) \subseteq^{ess} {}_RR$. As $R$ is left P-injective, this shows that $R$ is a left GPF-ring [**15**]. By Theorem 2.3 in [**15**], $soc(eR)$ is simple and essential in $eR$ for every local idempotent $e$ of $R$. Write

$$R = e_1R \oplus \cdots \oplus e_nR$$

where $\{e_1, \cdots, e_n\}$ is a complete set of orthogonal local idempotents of $R$. It follows that $soc(R_R) \subseteq^{ess} R_R$, so it remains to show that $R$ is right selfinjective. We do this by showing that $e_iR$ is injective for each $i = 1, \cdots, n$.

Let $a \in E(e_iR)$. Then $e_iR + aR$ is a finitely embedded torsionless right $R$-module so, by Lemma 2.7, let $\tau : e_iR + aR \to \oplus_{t=1}^{k} e_{i_t}R$ be an embedding where each $e_{i_t} \in \{e_1, \cdots, e_n\}$. Then $e_iR + aR$ embeds in $e_{i_j}R$ for some $j$ by Lemma 2.6. So let $\sigma : (e_iR + aR) \to e_{i_j}R$ be monic. If $a \notin e_iR$ then $\sigma(e_iR)$ is a proper submodule of the local module $e_{i_j}R$, so $\sigma(e_iR) \subseteq J(e_{i_j}R) \subseteq Z(R_R)$ by (2), a contradiction. Hence $a \in e_iR$ so $e_iR = E(e_iR)$ as required.

Recall [**1**] that a right artinian ring $R$ is QF if and only if $soc({}_RR) = soc(R_R)$ and $soc(eR)$ and $soc(Re)$ are simple for every local idempotent $e$ of $R$. A result of Gomez Pardo and Guil Asensio [**11**] asserts that if $R$ is a right CS-ring and every cyclic right $R$-module embeds in a free module, then $R$ is right artinian. A ring $R$ is called *right mininjective* if each $R$-homomorphism from a simple right ideal to $R$ is given left multiplication. See [**16**].

THEOREM 2.9. *Let $R$ be a left CS-ring such that every cyclic right $R$-module embeds in a free module. The following conditions are equivalent.*

(1) *$R$ is QF.*

(2) $J(R) \subseteq Z(R_R)$.

(3) $soc(R_R) \subseteq soc({}_RR)$.

(4) *$R$ is right mininjective.*

*Proof.* (1) $\Rightarrow$ (2). This is clear since $R$ is right selfinjective.

(2) $\Rightarrow$(3). $R$ is semiperfect by Proposition 2.4 and so $r(J) = soc({}_RR)$. Hence (2) gives $soc(R_R) \subseteq r[Z(R_R)] \subseteq r(J) = soc({}_RR)$.

(3) $\Rightarrow$ (1). Observe first that $soc(R_R) \subseteq^{ess} {}_RR$ by Proposition 2.4, so $soc({}_RR) \subseteq soc(R_R)$. Hence $soc({}_RR) = soc(R_R)$ by (3). Next $R$ is semiperfect by Proposition 2.4, so write

$$R = Re_1 \oplus \cdots \oplus Re_n$$

where $\{e_1, \cdots, e_n\}$ is a complete set of local orthogonal idempotents. Since $R$ is a left CS-ring with $soc({}_RR) \subseteq^{ess} {}_RR$, each $Re_i$ is uniform and so $soc(Re_i)$ is simple (and essential in $Re_i$) for each $i$.

On the other hand, $R$ is left P-injective because $rl(T) = T$ for all right ideals $T$ of $R$, so $Re_i \not\cong Re_j$ implies that $soc(Re_i) \not\cong soc(Re_j)$ by Lemma 3.4 in [**16**]. It follows that $R$ is left Kasch. Hence

$$soc(e_iR) = e_i soc(R_R) = e_i soc({}_RR) = e_i r(J) \cong (Re_i/Je_i)^* \neq 0$$

where $M^*$ denotes the dual module. But $R$ is right continuous by Theorem 1.7 in [**21**] because $r(T) = T$ for all right ideals $T$ of $R$. Hence $soc(e_iR)$ is simple and essential in $e_iR$ for each $i$ because $e_iR$ is uniform. Moreover, this shows that $R_R$ is finitely embedded. Thus every principal right $R$-module is finitely embedded (it is embedded in $R^{(n)}$ for some $n$) so $R$ is right artinian and hence QF.

(4) $\Rightarrow$ (3). This follows from Theorem 1.14 in [**16**].

(3) $\Rightarrow$ (4). This is clear because (3) $\Rightarrow$ (1).

**3. Johns rings.** According to Menal and Faith [**8**] a ring $R$ is called *right Johns* if $R$ is a right noetherian ring in which every right ideal is an annihilator. In [**9**] $R$ is called *strongly right Johns* if every $n \times n$ matrix ring $M_n(R)$ is right Johns. Strongly right Johns rings were characterized by Faith and Menal (see [**9**], Theorem 1.1) as the right noetherian rings that are left FP-injective (that is $R$-homomorphisms from a finitely generated submodule of a free left $R$-module $F$ into $R$ can be extended to $F$). It is not known if strongly right Johns rings are QF.

Several properties of right Johns rings were highlighted in [**8**] and a number of necessary and sufficient conditions for a strongly right Johns ring to be QF were collected in [**9**]. In the next proposition we deduce several new properties of these rings.

Proposition 3.1. (1) *If $R$ is a right Johns ring then $soc({}_RR) = soc(R_R) \subseteq^{ess} {}_RR$.*

(2) *If $R$ is strongly right Johns then the following properties hold.*

(a) *$lr(Rk) = Rk$ for all minimal left ideals $Rk$ of $R$.*

(b) *$R$ is right mininjective.*

(c) *$Rk$ is a minimal left ideal of $R$ if and only if $kR$ is a minimal right ideal of $R$.*

(d) *The dual of every simple right $R$-module is simple. In particular $l(T)$ is simple for every maximal right ideal $T$ of $R$.*

*Proof.* (1). By Lemma 2.2 in [**8**], it suffices to show that $l(J) \subseteq^{ess} {}_RR$ where we write $J = J(R)$. If $Rb \cap l(J) = 0$ for some $b \in R$, then $R = r[Rb \cap l(J)] = r(b) + J$ by Lemma 2.1. Hence $r(b) = R$, so $b = 0$ as required.

(2a). $R$ is left mininjective because it is left FP-injective. Hence, if $Rk$ is a minimal left ideal of $R$ then $kR$ is a minimal right ideal of $R$ by [**16**, Theorem 1.14]. This means $J \subseteq r(k)$ so $lr(k) \subseteq l(J) = S$ by Lemma 2.2 in [**8**], where we write $S = soc({}_RR) = soc(R_R)$. Thus $lr(k)$ is a semisimple left $R$-module containing $Rk$, so it suffices to show that $Rk \subseteq^{ess} lr(k)$. Suppose that $0 \neq y \in lr(k)$. Observe first that $r(k) \subseteq r(y) \neq R$, whence $r(k) = r(y)$ and $lr(k) = lr(y)$. Now suppose to the contrary that $Rk \cap Ry = 0$. Then $R = r(Rk \cap Ry) = r(k) + r(y)$ because $R$ is left FP-injective (see [**7**], Proposition 23.21). This implies that $0 = l[r(k) + r(y)] = lr(k) \cap lr(y) = lr(k)$, a contradiction.

(2b). We have $rl(T) = T$ for all right ideals $T$; in particular this holds for minimal right ideals. This with (2a) proves (2b) by [**16**, Corollary 2.6], because $soc({}_RR) = soc(R_R)$ (see [**8**], Lemma 2.2).

(2c). This follows from Theorem 1.14 in [**16**].

(2d). $R$ is right mininjective by (2b) and it is right Kasch because $rl(T) = T$ for all right ideals $T$. Hence the dual of every simple right $R$-module is simple by Proposition 2.2 in [**16**]. The last statement now follows because $l(T) \cong (R/T)^d$.

It was shown by Faith and Menal [**8**] that a right Johns, left finite dimensional ring $R$ is right artinian. In fact we can say more: By Lemma 6 in [**4**] and Lemma 2.2 in [**8**], $R$ is also left artinian. Moreover, by Theorem 1.7 in [**21**], $R$ is right continuous. On the other hand, the example provided by Faith and Menal in [**8**] shows that right Johns rings need not be left or right continuous. Indeed, Example 3.8 below provides a left and right artinian, left Johns, left continuous ring which is not right continuous. However, adding a left CS-condition to a right Johns ring forces it to be quasi-Frobenius.

THEOREM 3.2. *The following conditions on a ring $R$ are equivalent.*

(1) *$R$ is quasi-Frobenius.*

(2) *$R$ is a right Johns, left CS-ring.*

*Proof.* (1) $\Rightarrow$ (2) is well known. Assume (2); we show that $soc({}_RR) = soc(R_R)$ and that $soc(Re)$ and $soc(eR)$ are both simple for all primitive idempotents $e \in R$. Since $R$ is semiperfect by Lemma 2.2, let $R = Re_1 \oplus \cdots \oplus Re_n$ where $\{e_1, \cdots, e_n\}$ is a complete set of primitive orthogonal idempotents of $R$. Since $R$ is a left CS-ring, each $Re_i$ is uniform. Hence $R$ is left finite dimensional so, by the remark preceding this theorem, $R$ is left and right artinian. Moreover, Lemma 2.2 in [**8**] shows that $soc({}_RR) = soc(R_R)$.

Since $R$ is a left CS-ring, $soc(Re)$ is simple and essential in $Re$ for every local idempotent $e$ of $R$. If $0 \neq k \in soc(eR)$, then $l(k) \supseteq R(1-e) + J$, a maximal left ideal of $R$. Thus $l(k) = R(1-e) + J$, so

$$kR = rl(k) = eR \cap r(J) = eR \cap soc(R_R) = soc(eR)$$

Thus $soc(eR)$ is simple, completing the proof.

The right Johns rings are the right noetherian rings in which every right ideal is an annihilator. We now consider the artinian case.

Recall that a ring $R$ is called a *right* CEP-*ring* if every cyclic right $R$-module is essentially embedded in a projective module. These rings are right artinian [**11**] and right continuous [**21**]. If $M$ and $N$ are right $R$-modules we say (see [**13**]) that $M$ is *weakly $N$-injective* if, for every monomorphism $\sigma : N/K \to E(M)$, there exists $X_R \subseteq E(M)$ such that $X \cong M$ and $\sigma(N/K) \subseteq X$. It was shown by Jain and López-Permouth [**13**] that $R$ is a semiperfect right CEP-ring if and only if $R$ is right artinian and every indecomposable projective right $R$-module is weakly $R$-injective.

PROPOSITION 3.3. *The following conditions are equivalent for a ring $R$.*

(1) *$R$ is right artinian and $rl(T) = T$ for all right ideals $T$ of $R$.*

(2) *$R$ is right artinian and $rl(T) = T$ for all small right ideals $T$ of $R$.*

(3) *$R$ is right continuous and every cyclic right $R$-module embeds in a free module.*

(4) *$R$ is right artinian and every indecomposable projective right $R$-module is weakly $R$-injective.*

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(5) *$R$ is a right* CEP-*ring.*

(6) *$R$ is right perfect and every cyclic right $R$-module embeds in a free module.*

(7) *$R$ is semiperfect with essential left socle and every cyclic right $R$-module embeds in a free module.*

*Proof.* (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (3). By Lemma 1.5 in **[21]**, we have $lr(T) = T$ for all right ideals $T$ of $R$. Hence $R$ is right continuous by Theorem 1.7 in **[21]**. If $M = R/T$ is a cyclic right $R$-module then $M$ is torsionless because $rl(T) = T$, and $M$ is finitely embedded because $R$ is right artinian. Hence $M$ embeds in a free module by Lemma 2.7.

(3) $\Rightarrow$ (4). $R$ is right artinian by Corollary 2.9 in **[11]**, so let $\{e_1, \cdots, e_n\}$ be a basic set of local idempotents. If $1 \leq i \leq n$, we must show that $P = e_iR$ is weakly $R$-injective. Hence let $T$ be a right ideal of $R$, and let $\sigma : R/T \to E(P)$ be an embedding. Since $R$ is right CS, $soc(P)$ is simple and essential in $P$, and so $P$ is uniform. It follows that $R/T$ is uniform. On the other hand, there is (by hypothesis) an embedding $R/T \hookrightarrow \oplus_{i=1}^{m} e_{t_i}R$ where $1 \leq t_i \leq n$ for each $i$. By Lemma 2.6 there is an embedding $\phi : R/T \to e_{t_k}R$ for some $t_k \in \{1, \cdots, n\}$, and we have

$$soc(e_{t_k}R) = soc(\phi(R/T)) \cong soc(R/T) \cong soc(E(P)) = soc(P)$$

It follows from Theorem 3.16 in **[16]** that $e_{t_k}R \cong P$. Now consider the following diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & R/T & \xrightarrow{\phi} & e_{t_k}R \\ & & \sigma\downarrow & \swarrow\hat{\sigma} & \\ & & E(P) & & \end{array}$$

There exists $\hat{\sigma} : e_{t_k}R \to E(P)$ such that $\hat{\sigma} \circ \phi = \sigma$. Then $\hat{\sigma}$ is an embedding because $\phi$ is an essential embedding, and $\sigma(R/T) = \hat{\sigma}(\phi(R/T)) \subseteq \hat{\sigma}(e_{t_k}R) \subseteq E(P)$. So if we take $X = \hat{\sigma}(e_{t_k}R)$, we have $\sigma(R/T) \subseteq X \subseteq E(P)$ and $X = \hat{\sigma}(e_{t_k}R) \cong e_{t_k}R \cong P$. This proves that $P$ is weakly $R$-injective.

(4) $\Rightarrow$ (5). This follows from Theorem 5.2 in **[13]**.

(5) $\Rightarrow$ (6). This is because right CEP-rings are right artinian by **[11]**.

(6) $\Rightarrow$ (7). This is clear.

(7) $\Rightarrow$ (1). We have $rl(T) = T$ for all right ideals $T$ of $R$ because $R/T$ embeds in a free module. Hence $R$ is left P-injective so, by Theorem 2.3 in **[15]**, $R$ has a finitely generated essential right socle. Since every cyclic right $R$-module is embedded in a free module, $R$ is right artinian. This proves (1).

Strengthening condition (3) in Proposition 3.3 leads to a new characterization of quasi-Frobenius rings.

THEOREM 3.4. *The following conditions are equivalent for a ring $R$.*

(1) *$R$ is quasi-Frobenius.*

(2) *$R$ is right CS and every 2-generated right $R$-module embeds in a free module.*

(3) *Every 2-generated right $R$-module is essentially embedded in a projective module.*

(4) *R is right perfect and every 2-generated right R-module is embedded in a free module.*

(5) *R is semiperfect with essential left socle and every 2-generated right R-module is embedded in a free module.*

*Proof.* It is well known that (1) implies each of the other statements, (4) $\Rightarrow$ (5) is clear, and (3) $\Rightarrow$ (2) by Proposition 1.10 in [**21**].

(2) $\Rightarrow$ (5). By [**11**], $R$ is right artinian, and hence semiperfect with essential left socle.

(5) $\Rightarrow$ (1). $R$ is right artinian and right continuous by Proposition 3.3. In particular, $R$ is semiregular with $J = Z(R_R)$ by Utumi [**20**]. Hence $R$ is QF by Proposition 1.24 in [**21**].

Statement (5) in Theorem 3.4 extends a result of Rutter [**19**] in two directions: it replaces "right perfect" by "semiperfect with essential left socle", and it replaces "every finitely generated right $R$-module is embedded in a free module" by "every 2-generated right $R$-module is embedded in a free module".

If we replace right artinian by left artinian in Proposition 3.3, we get the following result:

PROPOSITION 3.5. *Suppose R is left artinian and $rl(T) = T$ for all finitely generated, small right ideals T of R. Then R is a right artinian, right continuous ring in which every right ideal is an annihilator. In particular, every cyclic right R-module embeds in a free module.*

*Proof.* Let $K$ be any small right ideal of $R$. Since $R$ is left artinian, $l(K) = l\{k_1, \cdots, k_n\}$ for a finite subset $\{k_1, \cdots, k_n\} \subseteq K$. Thus $rl(K) = rl\{k_1, \cdots, k_n\} = \sum_{i=1}^{n} k_iR \subseteq K$, and so $rl(K) = K$. Hence $rl(T) = T$ for all right ideals of $R$ by Lemma 1.5 in [**21**], and $R$ is right continuous by Theorem 1.7 in [**21**]. Thus $R$ is right noetherian because it has ACC on right annihilators (it has DCC on left annihilators). Hence $R$ is right artinian by Hopkin's theorem. Finally, the last assertion follows because $rl(T) = T$ for all right ideals of $R$.

It is well known that if $R$ is a left selfinjective ring then $R$ is left continuous and satisfies the following two conditions:

(A1) $rl(T) = T$ for all finitely generated right ideal $T$ of $R$; and

(A2) $r(A \cap B) = r(A) + r(B)$ for all left ideals $A$ and $B$ of $R$.

In [**21**] several classes of non-injective semiperfect rings are given with annihilator conditions which guarantee the continuity of the ring. More precisely the following result was proved (see [**21**], Lemma 1.3 and Theorem 1.7).

LEMMA 3.6. (1) *Suppose R is a semiperfect ring in which $soc({}_RR) \subseteq^{ess} {}_RR$, $soc(R_R) \subseteq^{ess} R_R$ and $lr(K) = K$ for all minimal left ideals K of R. If L is a left ideal of R with $L \subseteq^{ess} lr(L)$, then $L \subseteq^{ess} Rf$ for some $f^2 = f \in R$.*

(2) *Suppose R is a semiperfect ring in which $soc(R_R) \subseteq^{ess} R_R$ and $lr(K) = K$ for all small left ideals K of R. Then R is left continuous and $lr(T) = T$ for all left ideals T of R.*

It is natural to ask whether the annihilator condition in (2) of Lemma 3.6 can be replaced with condition (A2) above. Note that if $R$ satisfies (A2) then $L \subseteq^{ess} lr(L)$ for all left ideals $L$ of $R$. [Indeed, if $x \in lr(L) - L$ and $Rx \cap L = 0$, then $r(L) \subseteq r(x)$ and $r(Rx \cap L) = r(x) + r(L) = R$, so $x = 0$, a contradiction.] In particular every complement (closed) left ideal of $R$ is an annihilator.

The next proposition is now an immediate consequence of the above remarks, Lemma 3.6 and Lemmas 1.1 and 1.2 in [**21**].

PROPOSITION 3.7. *Suppose that $R$ is a semiperfect ring in which $soc({}_RR) \subseteq^{ess} {}_RR$, $soc(R_R) \subseteq^{ess} R_R$, and $lr(K) = K$ for all minimal left ideals $K$ of $R$. Then $R$ is left continuous if either of the following conditions hold:*

(1) *Every complement left ideal of $R$ is an annihilator.*

(2) *$r(A \cap B) = r(A) + r(B)$ for all left ideals $A$ and $B$ of $R$.*

EXAMPLE 3.8. Let $K$ be a field and $\sigma$ an isomorphism of $K$ onto a subfield $L$ where $[K : L] = n > 1$. Let $K[X; \sigma]$ denote the ring of twisted left polynomials over $K$. Thus $K[X; \sigma]$ is the set of all formal polynomials in the indeterminate $X$ with coefficients from $K$ written on the left and with multiplication defined by $Xa = \sigma(a)X$ for all $a \in K$. Let $R = K[X; \sigma]/(X^2)$ and write $x$ for the coset determined by $X$. It can easily be verified that the only left ideals of $R$ are 0, $J(R) = Rx = Kx$ and $R$. Thus $R$ is a two-sided artinian left continuous ring which is not right continuous. The ring $R$ satisfies the following annihilator conditions: $lr(A) = A$ and $r(A \cap B) = r(A) + r(B)$ for all left ideals $A$ and $B$ of $R$. However $R$ is not right continuous and hence does not satisfy the annihilator condition $l(T \cap S) = l(T) + l(S)$ for all right ideals $T$ and $S$ of $R$.

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NOTE ADDED IN PROOF. Proposition 3.1 can be used to prove the following theorem which settles an open question of Faith and Menal.

THEOREM. *Every strongly right Johns ring is quasi-Frobenius.*

*Proof.* Let $R$ be a strongly right Johns ring. Then $S_r$ is an essential right ideal of $R$ and $l(S_r) = J$ by Faith and Menal [**8**, Lemma 2.2]. Moreover it suffices to show that $R$ is semilocal by [**9**, Corollary 1.3]. Since $R$ is right noetherian, we have $S_r = k_1R \oplus \ldots \oplus k_nR$, where each $k_iR$ is simple. Hence

$$J = l(S_r) = l(k_1R + \oplus + k_nR) = \cap_{i=1}^{n} l(k_i).$$

The result now follows from Proposition 3.1.

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