This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X25101661

Inequalities for the sequences with unified form

Li-Mei Dou[1], Zhen-Yu Gao[2] and Larry X.W. Wang[3]
[1,2,3]Center for Combinatorics, LPMC
Nankai University
Tianjin 300071, P. R. China
Email: [1]2120220002@mail.nankai.edu.cn,
[2]1120230007@mail.nankai.edu.cn, [3]wsw82@nankai.edu.cn

Abstract. We consider a class of sequence $c(n)$ with the following asymptotic form

$$c(n) \sim \frac{C}{n^{\kappa}} \exp\left(\sum_{\lambda\in\mathcal{S}} A_{\lambda} n^{\lambda}\right) \sum_{\mu\in\mathcal{T}} \frac{\beta_{\mu}}{n^{\mu}} \qquad (n \to \infty).$$

We give criteria for the Turán inequality of any order, the double Turán inequality, the Laguerre inequality of any order of $c(n)$ for sufficiently large n. We also give the companion inequalities for the Turán inequality and the Laguerre inequality of any order for $c(n)$. As applications, we will show the numbers of commuting ℓ-tuples in S_n, the partition without sequence, the plane partition, the partition into k-gonal numbers, the finite-dimensional representations of groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$ and the coefficients of infinite product generating functions asymptotically satisfy these inequalities. Some of them settle open problems proposed by Bringmann, Franke and Heim.

Keywords: Turán inequality, Laguerre inequality, symmetric group, plane partition, finite representation of group.

AMS Classification: 11P82, 11E45, 11M41, 05A20

1 Introduction

The main purpose of this paper is to consider sequences $c(n)$ of real numbers of the form

$$c(n) \sim \frac{C}{n^{\kappa}} \exp\left(\sum_{\lambda\in\mathcal{S}} A_{\lambda} n^{\lambda}\right) \sum_{\mu\in\mathcal{T}} \frac{\beta_{\mu}}{n^{\mu}} \qquad (n \to \infty). \tag{1.1}$$

Here $\kappa \in \mathbb{R}$, $\mathcal{S} \subset \mathbb{Q}^{+} \cap (0,1)$ is finite, $\mathcal{T} \subset \mathbb{Q}_0^{+}$ is finite, C, A_{λ}, $\beta_{\mu} \in \mathbb{R}$ with $\beta_0 = 1$ if $0 \in \mathcal{T}$. This general form was introduced by Bringmann, Franke and Heim [5], including the asymptotical expression of the numbers of the unimodal sequences [2], the strongly unimodal sequences [23], the partition without sequence [16], the plane partition [28], the partition into k-gonal numbers [3], the finite-dimensional representations of groups $\mathfrak{su}(3)$ [4] and $\mathfrak{so}(5)$ [3] and the coefficients of the infinite product generating functions [5]. Bringmann, Franke and Heim [5] proved $c(n)$ satisfies the Turán inequality for sufficiently large n and put forward an open problem concerning with the Turán inequalities of higher order.

Recall that a real entire function

$$\psi(x)=\sum_{k=0}^{\infty}\gamma_k\frac{x^k}{k!} \tag{1.2}$$

is said to be in the Laguerre-Pólya class, denoted by $\psi(x)\in\mathcal{LP}$, if it can be represented in the form

$$\psi(x)=cx^me^{-\alpha x^2+\beta x}\prod_{k=1}^{\infty}\left(1+x/x_k\right)e^{-x/x_k},$$

where c, β, x_k are real numbers, $\alpha\geq 0$, m is a nonnegative integer and $\sum x_k^{-2}<\infty$. The $\mathcal{LP}$ class has attracted much attention in view of its connection with Riemann hypothesis.

The Riemann Ξ-function is defined as

$$\Xi(z):=\frac{1}{2}\left(-z^2-\frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

Following [8], the Riemann Ξ-function can be written in Taylor series form as

$$F(z):=\frac{1}{8}\Xi\left(\frac{z}{2}\right)=\sum_{n=0}^{\infty}\frac{\gamma(n)}{n!}z^{2n}.$$

We say that a polynomial with real coefficients is hyperbolic if all of its zeros are real. The d-th associated Jensen polynomial with shift n of an arbitrary sequence $\{\alpha(0),\alpha(1),\alpha(2),...\}$ of real numbers is the polynomial

$$J_{\alpha}^{d,n}(x)=\sum_{k=0}^{d}\binom{d}{k}\alpha_{k+n}x^k. \tag{1.3}$$

The hyperbolicity of degree d Jensen polynomial associated with the sequence $\{\alpha(n)\}$ is equivalent to the Turán inequality for $\{\alpha(n)\}$ of order d. Pólya and Schur proved that the Riemann hypothesis holds if and only if the function $F(z)$ belongs to $\mathcal{LP}$ class, i.e., having only real zeros, or equivalently, all Jensen polynomials having only real zeros.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Griffin, Ono, Rolen and Zagier [12] proved that under some conditions, the degree d Jensen polynomials associated with the sequence $\{\alpha(n)\}$ are hyperbolic for any d and sufficiently large n. In fact, they established the following theorem.

Theorem 1.1 (Griffin, Ono, Rolen and Zagier)**.** *Let $\alpha(n)$, $A(n)$, and $\delta(n)$ be sequences of positive real numbers with $\delta(n)$ tending to zero and satisfying*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + o\left(\delta(n)^d\right) \quad \text{as} \quad n \to \infty$$

for some $d \geq 1$, all $0 \leq j \leq d$ and some $g_i(n) = o\left(\delta(n)^i\right)$. Then, we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X-1}{\exp(A(n))}\right)\right) = H_d(X)$$

uniformly for X in any compact subset of $\mathbb{R}$, where $H_d(X)$ are the Hermite polynomials.

Recall that a sequence $\{a_n\}_{n\geq 0}$ is called to satisfy the Turán inequality, if

$$T(a_n) :\, = {a_n}^2 - a_{n-1}a_{n+1} \geq 0.$$

We say that $\{a_n\}_{n\geq 0}$ satisfies the double Turán inequality, if

$$T(T(a_n)) :\, = ({a_n}^2 - a_{n-1}a_{n+1})^2 - ({a_{n-1}}^2 - a_{n-2}a_n)({a_{n+1}}^2 - a_n a_{n+2}) \geq 0.$$

A sequence $\{a_n\}_{n\geq 1}$ is called to satisfy the Laguerre inequality of order m [26], if

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k} a_{n+k}a_{n+2m-k} \geq 0. \tag{1.4}$$

The Laguerre inequalities have profound connections to the Riemann hypothesis and the Laguerre-Pólya class, see [9], [11], [15], [21] and [22]. Recently, the Laguerre inequalities for the Maclaurin coefficients of the Riemann Ξ-function, the partition function and some celebrated sequences have been extensively studied, see [1], [7], [10], [25], [26] and [27]. Denote the sum of the terms with positive (negative, respectively) coefficients of $L_m(a_n)$ by $L_m^+(a_n)$ ($L_m^-(a_n)$, respectively). One can see that (1.4) equals to $\frac{L_m^+(a_n)}{L_m^-(a_n)} \geq 1$. It is also interested to consider the companion Laguerre inequalities, i.e., give an upper bound for $\frac{L_m^+(a_n)}{L_m^-(a_n)}$. For more recent work, see [1], [7] and [10].

In this paper, we will investigate these inequalities for $c(n)$. This paper is organized as follows. In Section 2, we shall prove that as $n \to \infty$, $c(n)$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

satisfies the Turán inequality of any order and the double Turán inequality. In Section 3, we will show $c(n)$ satisfies the companion Turán inequality for sufficiently large n. In Section 4, we will show $c(n)$ asymptotically satisfies the Laguerre inequality of any order and its companion version. In Section 5, as applications, we will show the numbers of commuting ℓ-tuples in S_n, the partition without sequence, the plane partition, the partition into k-gonal numbers, the finite-dimensional representations of groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$ and the coefficients of infinite product generating functions asymptotically satisfy these inequalities.

2 Turán type inequalities

In this section, we will prove that as $n \to \infty$, $c(n)$ meets the requires of the Turán inequality of any order and the double Turán inequality.

To prove that $c(n)$ fulfills the Turán inequality of any order, we just need to verify $c(n)$ conform to the conditions of Theorem 1.1.

Theorem 2.1. *Assume that $c(n)$ is a sequence of the form* (1.1)*, let $\mathcal{S} := \{\lambda_1, \ldots, \lambda_s\}$ where $1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ and $\mathcal{T} := \{\mu_1, \ldots, \mu_t\}$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t \geq 0$, κ, C, A_λ, $\beta_\mu \in \mathbb{R}$ with $\beta_0 = 1$ if $0 \in \mathcal{T}$. If $A_{\lambda_1} > 0$, then for sufficiently large n, $c(n)$ fulfills the Turán inequality of any order.*

Proof. By the Taylor expansion of $\log x$, we have that

$$\log(x) = x - 1 - \frac{(x-1)^2}{2} + \cdots + (-1)^{r-1}\frac{(x-1)^r}{r} + o(x^r), \ x \to 1. \tag{2.1}$$

From (1.1) we get

$$\log \frac{c(n+j)}{c(n)} \sim \sum_{\lambda \in \mathcal{S}} A_\lambda (n+j)^\lambda - \sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda + \log \frac{n^\kappa}{(n+j)^\kappa} + \log \frac{\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu}}. \tag{2.2}$$

We proceed to estimate $\sum_{\lambda \in \mathcal{S}} A_\lambda \left((n+j)^\lambda - n^\lambda\right)$, $\log \frac{n^\kappa}{(n+j)^\kappa}$ and $\log \frac{\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu}}$ respectively.

First, we estimate $\sum_{\lambda \in \mathcal{S}} A_\lambda \left((n+j)^\lambda - n^\lambda\right)$. Since

$$(n+j)^\lambda - n^\lambda = \sum_{i=1}^{N} \binom{\lambda}{i} n^{-i+\lambda} j^i + o(n^{-N+\lambda}) \qquad (N \geq r). \tag{2.3}$$

Thus

$$\sum_{\lambda \in \mathcal{S}} A_\lambda \left((n+j)^\lambda - n^\lambda\right) = \sum_{\lambda \in \mathcal{S}} A_\lambda \sum_{i=1}^{N} \binom{\lambda}{i} n^{-i+\lambda} j^i + o(n^{-N+\lambda}) \qquad (N \geq r). \tag{2.4}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Next we turn to estimating $\log \frac{n^\kappa}{(n+j)^\kappa}$. Since

$$\frac{n^\kappa}{(n+j)^\kappa} = \sum_{i=0}^{N} \binom{-\kappa}{i} n^{-i} j^i + O(n^{-N-1}), \tag{2.5}$$

by (2.1) we obtain

$$\log \frac{n^\kappa}{(n+j)^\kappa} = \sum_{i=1}^{N} f_i(\kappa) n^{-i} j^i + O(n^{-N-1}), \tag{2.6}$$

where $f_i(\kappa) = \frac{(-1)^i \kappa}{i}$.

Finally, we only need to estimate $\log \frac{\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu}}$. Since

$$\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu} = \sum_{i=1}^{t} \beta_{\mu_i} n^{-\mu_i}, \quad \sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{(n+j)^\mu} = \sum_{i=1}^{t} \beta_{\mu_i} (n+j)^{-\mu_i}, \tag{2.7}$$

from (2.1) we deduce that

$$\log \frac{\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{(n+j)^\mu}}{\sum_{\mu\in\mathcal{T}} \frac{\beta_\mu}{n^\mu}} = \sum_{i=1}^{N} G_i(n) j^i + o(G_N(n)) \tag{2.8}$$

where $G_i(n)$ are certain polynomials of n with degree no more than $-i$ and $|G_i(n)| \le \frac{1}{i}\mu_1^i n^{-i}$.

Combining (2.4), (2.6) and (2.8), as $n \to \infty$ we get that

$$\begin{aligned}\log \frac{c(n+j)}{c(n)} &\sim \sum_{\lambda\in\mathcal{S}} A_\lambda \sum_{i=1}^{N} \binom{\lambda}{i} n^{-i+\lambda} j^i + \sum_{i=1}^{N} f_i(\kappa) n^{-i} j^i + \sum_{i=1}^{N} G_i(n) j^i + o(n^{-N+\lambda}) \\ &=: \sum_{i=1}^{N} \left(\sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n) \right) j^i + o(n^{-N+\lambda}),\end{aligned} \tag{2.9}$$

where $B_{\lambda,i}(n) = A_\lambda \binom{\lambda}{i} n^{-i+\lambda}$, $F_i(n) = f_i(\kappa) n^{-i} + G_i(n)$. Employing (2.6) and (2.8), we get $|F_i(n)| \le \frac{1}{i}(\mu_1^i + \kappa) n^{-i}$.

Let $A(n) = \sum_{\lambda\in\mathcal{S}} B_{\lambda,1}(n) + F_1(n)$, $\delta(n) = \sqrt{-\sum_{\lambda\in\mathcal{S}} B_{\lambda,2}(n) - F_2(n)}$, $g_i(n) = \sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)$ $(i \ge 3)$. Then $i \ge 3$, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{g_i(n)}{\delta(n)^i} &= \lim_{n\to\infty} \frac{\sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)}{\left(-\sum_{\lambda\in\mathcal{S}} B_{\lambda,2}(n) - F_2(n)\right)^{\frac{i}{2}}} \\ &= \lim_{n\to\infty} \frac{\sum_{\lambda\in\mathcal{S}} A_\lambda \binom{\lambda}{i} n^{-i+\lambda}}{(-\sum_{\lambda\in\mathcal{S}} A_\lambda \binom{\lambda}{2} n^{-2+\lambda})^{\frac{i}{2}}} \\ &= \lim_{n\to\infty} \frac{\sum_{\lambda\in\mathcal{S}} A_\lambda \binom{\lambda}{i}}{(-\sum_{\lambda\in\mathcal{S}} A_\lambda \binom{\lambda}{2})^{\frac{i}{2}}} n^{\lambda(1-\frac{i}{2})} \\ &= 0.\end{aligned} \tag{2.10}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

In summary, we have that $c(n)$ satisfies the conditions of Theorem 1.1. Thus, for sufficiently large n, $c(n)$ satisfies the Turán inequality of any order. $\square$

Now we turn to proving the double Turán inequality for $c(n)$.

Theorem 2.2. *For sufficiently large n, $c(n)$ satisfies the double Turán inequality, i.e.,*

$$\begin{aligned} T(T(c(n))) &= (c(n)^2 - c(n-1)c(n+1))^2 \\ &\quad - (c(n-1)^2 - c(n-2)c(n))(c(n+1)^2 - c(n)c(n+2)) \geq 0. \end{aligned} \tag{2.11}$$

Proof. Following the framework in the proof of Theorem 1.3 in [5], we claim that for all $\lambda \in \mathcal{S}$,

$$\exp\left(A_\lambda \left(2n^\lambda - (n+1)^\lambda - (n-1)^\lambda\right)\right) = 1 - \frac{\gamma_{\lambda,1}}{n^{2-\lambda}} + \frac{\gamma_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)$$

where $\gamma_{\lambda,1} = 2A_\lambda \binom{\lambda}{2}$,

$$\exp\left(A_\lambda \left(n^\lambda + (n+2)^\lambda - 2(n+1)^\lambda\right)\right) = 1 + \frac{\mu_{\lambda,1}}{n^{2-\lambda}} + \frac{\mu_{\lambda,2}}{n^{3-\lambda}} + \frac{\mu_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)$$

where $\mu_{\lambda,1} = 2A_\lambda \binom{\lambda}{2}$ and $\mu_{\lambda,2} = 6A_\lambda \binom{\lambda}{3}$,

$$\exp\left(A_\lambda \left(n^\lambda + (n-2)^\lambda - 2(n-1)^\lambda\right)\right) = 1 + \frac{\omega_{\lambda,1}}{n^{2-\lambda}} + \frac{\omega_{\lambda,2}}{n^{3-\lambda}} + \frac{\omega_{\lambda,1}^2}{2n^{4-2\lambda}} + o\left(n^{-4+2\lambda}\right)$$

where $\omega_{\lambda,1} = 2A_\lambda \binom{\lambda}{2}$ and $\omega_{\lambda,2} = -6A_\lambda \binom{\lambda}{3}$.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Then, we get that

$$
\begin{aligned}
&\exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda\left(2n^\lambda-(n-1)^\lambda-(n+1)^\lambda\right)\right)\\
&=\prod_{\lambda\in\mathcal{S}}\left(1-\frac{\gamma_{\lambda,1}}{n^{2-\lambda}}+\frac{\gamma_{\lambda,1}^2}{2n^{4-2\lambda}}+o\left(n^{-4+2\lambda}\right)\right)\\
&=1-\frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}}-\cdots-\frac{\gamma_{\lambda_2,1}}{n^{2-\lambda_2}}+\frac{\gamma_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right),\\
&\exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda\left(n^\lambda+(n+2)^\lambda-2(n+1)^\lambda\right)\right)\\
&=\prod_{\lambda\in\mathcal{S}}\left(1+\frac{\mu_{\lambda,1}}{n^{2-\lambda}}+\frac{\mu_{\lambda,2}}{n^{3-\lambda}}+\frac{\mu_{\lambda,1}^2}{2n^{4-2\lambda}}+o\left(n^{-4+2\lambda}\right)\right)\\
&=1+\frac{\mu_{\lambda_1,1}}{n^{2-\lambda_1}}+\cdots+\frac{\mu_{\lambda_s,1}}{n^{2-\lambda_s}}+\frac{\mu_{\lambda_1,2}}{n^{3-\lambda_1}}+\cdots+\frac{\mu_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}}+\frac{\mu_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right),\\
&\exp\left(\sum_{\lambda\in\mathcal{S}} A_\lambda\left(n^\lambda+(n-2)^\lambda-2(n-1)^\lambda\right)\right)\\
&=\prod_{\lambda\in\mathcal{S}}\left(1+\frac{\omega_{\lambda,1}}{n^{2-\lambda}}+\frac{\omega_{\lambda,2}}{n^{3-\lambda}}+\frac{\omega_{\lambda,1}^2}{2n^{4-2\lambda}}+o\left(n^{-4+2\lambda}\right)\right)\\
&=1+\frac{\omega_{\lambda_1,1}}{n^{2-\lambda_1}}+\cdots+\frac{\omega_{\lambda_s,1}}{n^{2-\lambda_s}}+\frac{\omega_{\lambda_1,2}}{n^{3-\lambda_1}}+\cdots+\frac{\omega_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}}+\frac{\omega_{\lambda_1,1}^2}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right).
\end{aligned}
\tag{2.12}
$$

Here $\lambda_{s'} > 2\lambda_1 - 1$.

A direct calculation gives that for $n > 1$,

$$
\begin{aligned}
\frac{1}{n^{2\kappa}} &= \frac{1}{(n-1)^\kappa(n+1)^\kappa}\left(1+\sum_{j\ge 2}\frac{a_j}{n^j}\right),\\
\frac{1}{n^\kappa(n+2)^\kappa} &= (n+1)^{-2\kappa}\left(1+\sum_{j\ge 2}\frac{b_j}{n^j}\right),\\
\frac{1}{n^\kappa(n-2)^\kappa} &= (n-1)^{-2\kappa}\left(1+\sum_{j\ge 2}\frac{c_j}{n^j}\right).
\end{aligned}
\tag{2.13}
$$

According to (2.12) and (2.13), we have

$$
T(T(c(n))) \sim C^4\frac{\exp\left(2\sum_{\lambda\in\mathcal{S}}A_\lambda((n-1)^\lambda+(n+1)^\lambda)\right)}{(n-1)^{2\kappa}(n+1)^{2\kappa}}\left(C'^2-A'B'\right),
$$

where

$$
A' = \left(\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+1)^\mu}\right)^2
$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

$$
-\left(1+\frac{\mu_{\lambda_1,1}}{n^{2-\lambda_1}}+\cdots+\frac{\mu_{\lambda_s,1}}{n^{2-\lambda_s}}+\frac{\mu_{\lambda_1,2}}{n^{3-\lambda_1}}+\cdots+\frac{\mu_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}}+\frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right)\right)
$$

$$
\left(1+O\left(n^{-2}\right)\right)\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}\sum_{\nu\in\mathcal{T}}\frac{\beta_\nu}{(n+2)^\nu},
$$

$$
B'=\left(\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n-1)^\mu}\right)^2
$$

$$
-\left(1+\frac{\omega_{\lambda_1,1}}{n^{2-\lambda_1}}+\cdots+\frac{\omega_{\lambda_s,1}}{n^{2-\lambda_s}}+\frac{\omega_{\lambda_1,2}}{n^{3-\lambda_1}}+\cdots+\frac{\omega_{\lambda_{s'},2}}{n^{3-\lambda_{s'}}}+\frac{\omega^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right)\right)
$$

$$
\left(1+O\left(n^{-2}\right)\right)\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}\sum_{\nu\in\mathcal{T}}\frac{\beta_\nu}{(n-2)^\nu},
$$

$$
C'=\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n-1)^\mu}\sum_{\nu\in\mathcal{T}}\frac{\beta_\nu}{(n+1)^\nu}
$$

$$
-\left(1-\frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}}-\cdots-\frac{\gamma_{\lambda_s,1}}{n^{2-\lambda_s}}-\cdots+\frac{\gamma^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}+o\left(n^{-4+2\lambda_1}\right)\right)\left(1+O\left(n^{-2}\right)\right)\left(\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}\right)^2.
$$

As $\beta_0=1$, we take A mod $(o(n^{-4+2\lambda_1}))$,

$$
\sum_{\substack{\mu,\nu\in\mathcal{T}\\0\le\mu+\nu\le4-2\lambda_1}}\frac{\beta_\mu\beta_\nu}{(n+1)^{\mu+\nu}}-\left(\sum_{\substack{\mu,\nu\in\mathcal{T}\\0\le\mu+\nu\le4-2\lambda_1}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}+\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}\sum_{\substack{\mu,\nu\in\mathcal{T}\\0\le\mu+\nu\le2-2\lambda_1+\lambda_i}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}\right.
$$

$$
\left.+\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}+\frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right)
$$

$$
=2\sum_{\substack{\mu\in\mathcal{T}\\0\le\mu\le2-\lambda_1}}\beta_\mu^2\left(\frac{1}{(n+1)^{2\mu}}-\frac{1}{n^\mu(n+2)^\mu}\right)+\sum_{\substack{\mu\in\mathcal{T}\\0\le\mu<\nu\\1\le\mu+\nu\le4-2\lambda_1}}\beta_\mu\beta_\nu\left(\frac{2}{(n+1)^{\mu+\nu}}-\frac{1}{n^\mu(n+2)^\nu}-\frac{1}{n^\nu(n+2)^\mu}\right)
$$

$$
-\left(\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}+\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}\sum_{\substack{\mu,\nu\in\mathcal{T}\\1\le\mu+\nu\le2-2\lambda_1+\lambda_i}}\frac{\beta_\mu\beta_\nu}{n^\mu(n+2)^\nu}+\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}+\frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right)
$$

$$
=\begin{cases}-\dfrac{2\beta_1}{n^3}-\displaystyle\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}-2\beta_1\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}}-\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}-\frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & 0<\lambda_1\le\dfrac{1}{2},\\ -\displaystyle\sum_{i=1}^{s}\frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}}-2\beta_1\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}}-\sum_{i=1}^{s'}\frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}}-\frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & \dfrac{1}{2}<\lambda_1<1.\end{cases}\tag{2.14}
$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Similarly,

$$B'\left(\mod (o(n^{-4+2\lambda_1}))\right) = \begin{cases} -\dfrac{2\beta_1}{n^3} - \displaystyle\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & 0 < \lambda_1 \le \dfrac{1}{2}, \\ -\displaystyle\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & \dfrac{1}{2} < \lambda_1 < 1. \end{cases} \tag{2.15}$$

and

$$C'\left(\mod (o(n^{-4+2\lambda_1}))\right) = \begin{cases} \dfrac{2\beta_1}{n^3} + \displaystyle\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & 0 < \lambda_1 \le \dfrac{1}{2}, \\ \displaystyle\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}, & \dfrac{1}{2} < \lambda_1 < 1. \end{cases} \tag{2.16}$$

Combining (2.14), (2.15) and (2.16), we get that for $0 < \lambda_1 \le \frac{1}{2}$, $C'^2 - A'B'$ can be simplified to

$$\begin{aligned} &\left(\frac{2\beta_1}{n^3} + \sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right)^2 - \\ &\left(-\frac{2\beta_1}{n^3} - \sum_{i=1}^{s} \frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right) \\ &\left(-\frac{2\beta_1}{n^3} - \sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right) \\ &= -\frac{16A^3_{\lambda_1}\binom{\lambda_1}{2}^3}{n^{-6+3\lambda_1}} + o(n^{-6+3\lambda_1}). \end{aligned}$$

For $\frac{1}{2} < \lambda_1 < 1$, $C'^2 - A'B'$ can be simplified to

$$\begin{aligned} &\left(\sum_{i=1}^{s} \frac{\gamma_{\lambda_i,1}}{n^{2-\lambda_i}} + 2\beta_1 \sum_{i=1}^{s'} \frac{\gamma_{\lambda_i,1}}{n^{3-\lambda_i}} - \frac{\gamma^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right)^2 - \\ &\left(-\sum_{i=1}^{s} \frac{\mu_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\mu_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\mu^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right) \\ &\left(-\sum_{i=1}^{s} \frac{\omega_{\lambda_i,1}}{n^{2-\lambda_i}} - 2\beta_1 \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,1}}{n^{3-\lambda_i}} - \sum_{i=1}^{s'} \frac{\omega_{\lambda_i,2}}{n^{3-\lambda_i}} - \frac{\omega^2_{\lambda_1,1}}{2n^{4-2\lambda_1}}\right) \\ &= -\frac{16A^3_{\lambda_1}\binom{\lambda_1}{2}^3}{n^{-6+3\lambda_1}} + o(n^{-6+3\lambda_1}). \end{aligned}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Since $A_{\lambda_1} > 0$ and $0 < \lambda_1 < 1$, we conclude that

$$\operatorname{sgn}\left(-16A_{\lambda_1}^3 \binom{\lambda_1}{2}^3\right) = 1. \tag{2.17}$$

This completes the proof. $\square$

3 The companion Turán inequality

In this section, we will show $c(n)$ adheres the companion Turán inequality for sufficiently large n.

Theorem 3.1. *Assume that $c(n)$ satisfies the form (1.1), then $c(n)$ asymptotically satisfies the companion Turán inequality, i.e.,*

$$c(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) c(n+1)c(n-1) \le 0, \tag{3.1}$$

where $\gamma_{\lambda_1,1}$ is defined as in Theorem 2.2 and $\overline{\gamma}$ is any real number larger than $-\gamma_{\lambda_1,1}$.

Proof. We have

$$\begin{aligned}
&c\left(n\right)^2 - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) c\left(n+1\right) c\left(n-1\right) \\
&\sim C^2 \frac{\exp\left(2\sum_{\lambda\in S} A_\lambda n^\lambda\right)}{n^{2\kappa}} \left(\left(\sum_{\mu\in T} \frac{\beta_\mu}{n^\mu}\right)^2 \right. \\
&\qquad \left. - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right)\left(1 + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + o\left(n^{-2+\lambda_1}\right)\right)\left(1 + O\left(n^{-2}\right)\right) \sum_{\mu\in T} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu\in T} \frac{\beta_\nu}{(n-1)^\nu} \right).
\end{aligned}$$

The sign of this is dictated by

$$\sum_{\mu,\nu\in T} \frac{\beta_\mu\beta_\nu}{n^{\mu+\nu}} - \left(1 + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right)\left(1 + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + o\left(n^{-2+\lambda_1}\right)\right)\left(1 + O\left(n^{-2}\right)\right) \sum_{\mu\in T} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu\in T} \frac{\beta_\nu}{(n-1)^\nu}.$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

As $\beta_0 = 1$, the above formula $\mod \left(o(n^{-2+\lambda_1})\right)$ equals,

$$\begin{aligned}
&\sum_{\mu,\nu\in\mathcal{T}} \frac{\beta_\mu\beta_\nu}{n^{\mu+\nu}} - \left(\sum_{\mu,\nu\in\mathcal{T}} \frac{\beta_\mu\beta_\nu}{(n+1)^\mu(n-1)^\nu} + \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} + \frac{\overline{\gamma}}{n^{2-\lambda_1}}\right) \\
&= 2\sum_{\substack{\mu\in\mathcal{T}\\ 0\le\mu\le 1-\frac{\lambda_1}{2}}} \beta_\mu^2\left(\frac{1}{n^{2\mu}} - \frac{1}{(n+1)^\mu(n-1)^\mu}\right) \\
&\quad + \sum_{\substack{\mu,\nu\in\mathcal{T}\\ 0\le\mu<\nu\\ 1\le\mu+\nu\le 2-\lambda_1}} \beta_\mu\beta_\nu\left(\frac{2}{n^{\mu+\nu}} - \frac{1}{(n+1)^\mu(n-1)^\nu} - \frac{1}{(n+1)^\nu(n+1)^\mu}\right) \qquad (3.2)\\
&\quad - \frac{\gamma_{\lambda_1,1}}{n^{2-\lambda_1}} - \frac{\overline{\gamma}}{n^{2-\lambda_1}}.
\end{aligned}$$

For $n > 1$,

$$\frac{1}{(n+1)^\nu(n+1)^\mu} = n^{-2\mu} + O\left(n^{-2\mu-2}\right)$$

and

$$\frac{1}{(n+1)^\mu(n-1)^\nu} - \frac{1}{(n+1)^\nu(n+1)^\mu} = 2n^{-\mu-\nu}\left(1 + \sum_{r\ge1}\frac{p_r}{n^{2r}}\right).$$

Thus (3.2) becomes

$$O(n^{-2+\lambda_1}) - \frac{\gamma_{\lambda_1,1} + \overline{\gamma}}{n^{2-\lambda_1}}.$$

When $\overline{\gamma} > -\gamma_{\lambda_1,1}$, it is obvious that (3.1) holds for sufficiently large n. □

4 Laguerre inequalities

In this section, we demonstrate $c(n)$ fulfills the Laguerre inequality and their companion inequality of any order. Recently, Wang and Yang [27] proved that for any r and sufficiently large n, one has $L_r(\alpha(n)) > 0$.

Theorem 4.1 (Wang and Yang). *Let $\{\alpha(n)\}, \{\delta(n)\}, \{A_i(n)\}$ be sequences with $\alpha(n)$ positive, $\delta(n) \to 0^+$, $A_2(n) < 0$, $A_2(n) = \Theta(\delta(n)^t)$ and $A_{2i}(n) = o(\delta(n)^{it})$ $(i \ge 2)$ for some positive t, and for $-r < j < r$,*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = \sum_{i=1}^{2r} A_i(n)j^i + o\left(\delta(n)^{tr}\right), \qquad (4.1)$$

then for sufficiently large n,

$$L_r(\alpha(n)) = \frac{1}{2}\sum_{k=0}^{2r}(-1)^{k+r}\binom{2r}{k}\alpha(n+k)\alpha(n+2r-k) > 0. \qquad (4.2)$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Since the conditions of Theorem 1.1 are stonger than those of Theorem 4.1, we get $c(n)$ meets the requirements of the above Theorem from the proof of Theorem 2.1. Thus $c(n)$ fulfills the Laguerre inequality of any order for sufficiently large n. Next we prove $c(n)$ satisfies the companion Laguerre inequalities. First, let us recall two combinatorial identities which will be used in our proofs, see [10] and [27].

Lemma 4.2. *For positive integers m and $t \le 2m$,*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(m-k)^t=\begin{cases}0, & t<2m,\\ (2m)!, & t=2m.\end{cases} \tag{4.3}$$

Lemma 4.3. *For positive integers r,*

$$\sum_{k=0}^{2r}\binom{2r}{k}(r-k)^2=2^{2r-1}r. \tag{4.4}$$

Now we are in a position to establish the companion Laguerre inequalities for $c(n)$.

Theorem 4.4. *For any fixed $r \ge 2$ and sufficiently large n, $c(n)$ satisfies the companion Laguerre inequality of order r, i.e.,*

$$\left(1+\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\right)L_r^-(c(n-r))>L_r^+(c(n-r)). \tag{4.5}$$

Proof. The inequality (4.5) can be rewritten as

$$\begin{aligned}&\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\frac{L_r^-(c(n-r))}{c(n)^2}\\ >&\frac{L_r^+(c(n-r))}{c(n)^2}-\frac{L_r^-(c(n-r))}{c(n)^2}\\ =&\frac{L_r(c(n-r))}{c(n)^2}.\end{aligned} \tag{4.6}$$

The right-hand side of the above inequality can be expressed as

$$\frac{L_r(c(n-r))}{c(n)^2}=\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\frac{c(n-r+j)c(n+r-j)}{c(n)^2}.$$

From (2.9) we get

$$\frac{c(n+r-j)c(n-r+j)}{c(n)^2}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

$$
\begin{aligned}
&= \exp\left(\log\frac{c(n+r-j)}{c(n)} + \log\frac{c(n-r+j)}{c(n)}\right)\\
&\sim \exp\Bigg(\sum_{\lambda\in\mathcal{S}} A_\lambda\left((n+r-j)^\lambda - n^\lambda\right) + \log\frac{n^\kappa}{(n+r-j)^\kappa} + \log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n+r-j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}\\
&\quad + \sum_{\lambda\in\mathcal{S}} A_\lambda((n-r+j)^\lambda - n^\lambda) + \log\frac{n^\kappa}{(n-r+j)^\kappa} + \log\frac{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{(n-r+j)^\mu}}{\sum_{\mu\in\mathcal{T}}\frac{\beta_\mu}{n^\mu}}\Bigg)\\
&= \exp\left(\sum_{i=1}^{N}\left(\sum_{\lambda\in\mathcal{S}} B_{\lambda,i}(n) + F_i(n)\right)\left((r-j)^i + (r-j)^i(-1)^i\right) + o(n^{-N+\lambda})\right)\\
&= \exp\left(2\sum_{i=1}^{N}\left(\sum_{\lambda\in\mathcal{S}} B_{\lambda,2i}(n) + F_{2i}(n)\right)(r-j)^{2i} + o(n^{-N+\lambda})\right)\\
&=: \exp\left(2\sum_{i=1}^{N} B_{2i}(n)(r-j)^{2i} + o(B_{2N}(n))\right),
\end{aligned}
$$

where $B_{2i}(n)$ are certain polynomials of n with degree no more than $-2i+\lambda_1$ and the coefficient of $n^{-2i+\lambda_1}$ is $A_{\lambda_1}\binom{\lambda_1}{2i}$.

Let

$$
\phi_r(x) := \sum_{i=1}^{2r}\frac{x^i}{i!} = \exp(x) + o\left(x^{2r}\right). \tag{4.7}
$$

Then as $n\to\infty$, we get

$$
\begin{aligned}
&\frac{L_r(c(n-r))}{c(n)^2}\\
=&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\exp\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i} + o(B_{2N}(n)\right)\\
=&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\exp\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right)\exp\left(o(B_{2N}(n))\right)\\
=&\left(\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) + o\left(B_{2N}(n)^{2Nr}\right)\right)\exp\left(o(B_{2N}(n))\right)\\
=&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N} B_{2i}(n)2(r-j)^{2i}\right) + o\left(B_{2N}(n)\right).
\end{aligned}
$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Now we focus on the expression

$$\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N}B_{2i}(n)2(r-j)^{2i}\right). \tag{4.8}$$

The ϕ_r-terms can be viewed as a polynomial in $(r-j)^2$, i.e., there exist coefficients $B'_{2i}(n)$ $(0\le i\le 2rN)$ such that

$$\phi_r\left(\sum_{i=1}^{N}B_{2i}(n)2(r-j)^{2i}\right)=\sum_{i=0}^{2rN}B'_{2i}(n)(r-j)^{2i}. \tag{4.9}$$

where $B'_{2i}(n)$ are certain polynomials of n with degree no more than $(-2+\lambda_1)i$ and the coefficient of $n^{(-2+\lambda_1)i}$ is $\frac{2^iA^i_{\lambda_1}\binom{\lambda_1}{2}^i}{i!}$.

By Lemma 4.2, one can see that

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N}B_{2i}(n)2(r-j)^{2i}\right)\\
=&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\sum_{i=0}^{2rN}B'_{2i}(n)(r-j)^{2i}\\
=&\frac{1}{2}\sum_{i=0}^{2rN}B'_{2i}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2i}\\
=&\frac{1}{2}\sum_{i=r}^{2rN}B'_{2i}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2i}.
\end{aligned} \tag{4.10}$$

(The sum of terms with $i<r$ vanishes). It implies that as $n\to\infty$, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}\phi_r\left(\sum_{i=1}^{N}B_{2i}(n)2(r-j)^{2i}\right)\\
=&\frac{1}{2}B'_{2r}(n)\sum_{j=0}^{2r}(-1)^{j+r}\binom{2r}{j}(r-j)^{2r}+o\left(B'_{2r}(n)\right)\\
=&\frac{(2r)!}{r!}2^{r-1}(-1)^rA^r_{\lambda_1}\binom{\lambda_1}{2}^r n^{(-2+\lambda_1)r}+o\left(n^{(-2+\lambda_1)r}\right).
\end{aligned} \tag{4.11}$$

On the other hand, we obtain

$$\begin{aligned}
&\frac{c(n+2j+1)c(n-2j-1)}{c(n)^2}\sim\exp\left(2\sum_{i=1}^{n}B_{2i}(n)(2j+1)^{2i}+o(B_{2N}(n))\right)\\
&=1+2A_{\lambda_1}\binom{\lambda_1}{2}(2j+1)^2n^{-2+\lambda_1}+o\left(n^{-2+\lambda_1}\right).
\end{aligned} \tag{4.12}$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Thus, by Lemma 4.3 we get

$$\begin{aligned}
\frac{L_r^-(c(n-r))}{c(n)^2} &= \frac{1}{2}\sum_{j=-[r-1/2]}^{[r-1/2]} \binom{2r}{r+2j+1} \frac{c(n+2j+1)c(n-2j-1)}{c(n)^2} \\
=& 2^{2r-2} + \frac{1}{2}\sum_{j=-[r-1/2]}^{[r-1/2]} \binom{2r}{r+2j+1} \left(2A_{\lambda_1}\binom{\lambda_1}{2}(2j+1)^2 n^{-2+\lambda_1} + o\left(n^{-2+\lambda_1}\right)\right) \\
=& 2^{2r-2} + \frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2} n^{-2+\lambda_1} \left(2\sum_{j=-[r-1/2]}^{[r-1/2]} \binom{2r}{r+2j+1}(2j+1)^2\right) + o\left(n^{-2+\lambda_1}\right) \\
=& 2^{2r-2} + \frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2} n^{-2+\lambda_1} \left(\sum_{j=-[r-1/2]}^{[r-1/2]} \binom{2r}{r+2j+1}(2j+1)^2 + \sum_{j=-[r/2]}^{[r/2]} \binom{2r}{r+2j}(2j)^2\right) + o\left(n^{-2+\lambda_1}\right) \quad (4.13) \\
=& 2^{2r-2} + \frac{1}{2}A_{\lambda_1}\binom{\lambda_1}{2} n^{-2+\lambda_1} \sum_{j=-r}^{r} \binom{2r}{r+j} j^2 + o\left(n^{-2+\lambda_1}\right) \\
=& 2^{2r-2} + 2^{2r-2} r A_{\lambda_1}\binom{\lambda_1}{2} n^{-2+\lambda_1} + o\left(n^{-2+\lambda_1}\right).
\end{aligned}$$

Combining (4.11) and (4.13) gives

$$\begin{aligned}
&\left(1+\frac{(2r)!}{r!}2^{1-r}(-A_{\lambda_1})^r \binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)}\right)\frac{L_r^-(c(n-r))}{c(n)^2} - \frac{L_r^+(c(n-r))}{c(n)^2} \\
=& \frac{(2r)!}{r!}2^{r-1}(-A_{\lambda_1})^r \binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)} + \frac{(2r)!}{(r-1)!}2^{r-1}(-1)^r (A_{\lambda_1})^{r+1}\binom{\lambda_1}{2}^{r+1} n^{(-2+\lambda_1)r} \\
&- \frac{(2r)!}{r!}2^{r-1}(-1)^r A_{\lambda_1}^r \binom{\lambda_1}{2}^r n^{(-2+\lambda_1)r} + o(n^{(-2+\lambda_1)r}) \qquad (4.14) \\
=& \frac{(2r)!}{r!}2^{r-1}A_{\lambda_1}^r(-1)^r \binom{\lambda_1}{2}^r n^{(-2+\lambda_1)(r-1)} + o(n^{(-2+\lambda_1)(r-1)}).
\end{aligned}$$

In view of $\lambda_1 \in (0,1)$, we get $(-1)^r\binom{\lambda_1}{2}^r > 0$. Hence, we conclude the positivity of (4.14) as $n \to \infty$. □

5 Applications

In this section, we will apply Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 4.4 to show that some sequences of the form $c(n)$ satisfy the companion Turán inequality, the double Turán inequality, the Turán inequality of any order, the Laguerre inequality of any order r and its companion version.

The number of commuting ℓ-tuples in S_n. Denote by $N_\ell(n)$ the number of ℓ-tuples of elements in the symmetric group S_n with commuting components, normalized by the order of S_n. Notice that $N_2(n) = p(n)$, where $p(n)$

denotes the number of partitions of n. Bringmann, Franke and Heim [5] gave the asymptotical formulas for $N_\ell(n)$ for arbitrary ℓ as follows.

Theorem 5.1 (Bringmann, Franke and Heim)**.** *We have, as $n \to \infty$,*

$$N_3(n) \sim \frac{e^{-\frac{\zeta'(-1)}{2} - \frac{\pi^2}{288\zeta(3)}} \zeta(3)^{\frac{11}{72}}}{2^{\frac{11}{24}} \cdot 3^{\frac{47}{72}} \cdot \pi^{\frac{11}{72}} \cdot n^{\frac{47}{72}}} \exp\left(\frac{(3\pi)^{\frac{2}{3}} \zeta(3)^{\frac{1}{3}}}{2} n^{\frac{2}{3}} - \frac{\pi^{\frac{4}{3}}}{4 \cdot 3^{\frac{2}{3}} \cdot \zeta(3)^{\frac{1}{3}}} n^{\frac{1}{3}} \right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{3,j}}{n^{\frac{j}{3}}} \right)$$

for certain numbers $B_{3,j}$,

$$\begin{aligned} N_4(n) &\sim \frac{e^{\frac{\zeta'(-2)}{24}} \pi^{\frac{1}{4}} \zeta(3)^{\frac{1}{8}}}{2^{\frac{13}{8}} \cdot 3^{\frac{1}{4}} \cdot 5^{\frac{1}{8}} \cdot n^{\frac{5}{8}}} \\ &\quad \times \exp\left(\frac{2^{\frac{7}{4}} \cdot \pi^{\frac{3}{2}} \cdot \zeta(3)^{\frac{1}{4}}}{3^{\frac{3}{2}} \cdot 5^{\frac{1}{4}}} n^{\frac{3}{4}} + A_{4,2} n^{\frac{1}{2}} + A_{4,3} n^{\frac{1}{4}} + A_{4,4} \right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{4,j}}{n^{\frac{j}{4}}} \right), \\ N_5(n) &\sim \frac{e^{\frac{\zeta'(-2)}{2880}} (\pi\zeta(3)\zeta(5))^{\frac{1}{10}}}{2^{\frac{2}{5}} \cdot 3^{\frac{1}{5}} \cdot 5^{\frac{3}{5}} \cdot n^{\frac{3}{5}}} \\ &\quad \times \exp\left(\frac{5^{\frac{4}{5}} \cdot \pi^{\frac{6}{5}} \cdot (\zeta(3)\zeta(5))^{\frac{1}{5}}}{2^{\frac{9}{5}} \cdot 3^{\frac{2}{5}}} n^{\frac{4}{5}} + A_{5,2} n^{\frac{3}{5}} + A_{5,3} n^{\frac{2}{5}} + A_{5,4} n^{\frac{1}{5}} + A_{5,5} \right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{5,j}}{n^{\frac{j}{5}}} \right), \end{aligned}$$

with computable constants $A_{4,j}$ $(2 \le j \le 4)$ and $A_{5,j}$ $(2 \le j \le 5)$ and certain $B_{4,j}$ and $B_{5,j}$, and for $\ell \ge 6$,

$$N_\ell(n) \sim \frac{(\ell-1)!^{\frac{1}{2\ell}} \sqrt{Z_\ell}}{\sqrt{2\pi\ell} n^{\frac{\ell+1}{2\ell}}} \exp\left(\frac{\ell \Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell - 1} n^{\frac{\ell-1}{\ell}} + \sum_{k=2}^{\ell} A_{\ell,k} n^{\frac{\ell-k}{\ell}} \right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{\ell,j}}{n^{\frac{j}{\ell}}} \right),$$

where $Z_\ell := (\zeta(2) \cdot \zeta(3) \cdots \zeta(\ell))^{\frac{1}{\ell}}$ for certain $A_{\ell,k}$ and $B_{\ell,j}$.

Theorem 5.1 suggests the asymptotical formulas for $N_\ell(n)$ satisfies the form of $c(n)$. Combining Theorem 2.1 and Theorem 2.2, we get that for sufficiently large n, $N_\ell(n)$ satisfies the Turán inequality of any order and the double Turán inequality.

Theorem 5.2. *For $\ell \ge 2$ and sufficiently large n, $N_\ell(n)$ satisfies the Turán inequality of any order and the double Turán inequality.*

Set $\lambda_1 = \frac{\ell-1}{\ell}$ and $A_{\lambda_1} = \frac{\ell\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell-1}$. Theorem 3.1 gives that $N_\ell(n)$ satisfies the following companion Turán inequality.

Theorem 5.3. *For $\ell \ge 2$ and sufficiently large n, we have $N_\ell(n)$ satisfies the companion Turán inequality, i.e.,*

$$N_\ell(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{1+\frac{1}{\ell}}} \right) N_\ell(n+1) N_\ell(n-1) \le 0,$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

where $\overline{\gamma}$ is any real number larger than $\frac{\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell}$.

Since $1+\frac{1}{\ell}>1$, from Theorem 5.3 we deduce that the log-convexity of $|C_{\ell,n}| = n! N_\ell(n)$, which give an affirmative answer to the open question (3) proposed by Bringmann, Franke and Heim in [5].

Corollary 5.4. *For $\ell \geq 3$ and sufficiently large n, we have $|C_{\ell,n}|$ is log-convex.*

For Laguerre inequalities, we get that $N_\ell(n)$ satisfies the Laguerre inequality of any order r and its companion version from Theorem 4.4.

Theorem 5.5. *For $\ell \geq 2$, $r \geq 2$ and sufficiently large n, we have*

$$1 < \frac{L_r^+(N_\ell(n-r))}{L_r^-(N_\ell(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-\frac{\ell \Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell - 1} \right)^r \binom{\frac{\ell-1}{\ell}}{2}^r n^{(-1-\frac{1}{\ell})(r-1)}.$$

The following sequences have also the asymptotical behavior of the form $c(n)$, then we can obtain results similar to $N_\ell(n)$.

Partitions without sequences. The partition without sequence is the partition of n that do not contain any consecutive integers as parts. Let $p_2(n)$ denote the number of partition without sequence of n. Bringmann and Mahlburg [6] first succeeded in proving an asymptotic formula for $p_2(n)$. Mauth [16] gave the following formula and proved the log-concavity of $p_2(n)$ for $n \geq 482$,

$$p_2(n) = \frac{e^{\frac{2\pi\sqrt{n}}{3}}}{n^{\frac{3}{4}}} \left(\sum_{i=0}^{9} \frac{\beta_i}{n^{\frac{i}{4}}} + O\left(\frac{1}{n^{\frac{5}{2}}} \right) \right),$$

where β_i are constants. Note that $p_2(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{2}$ and $A_{\lambda_1} = \frac{2\pi}{3}$. Hence, by the theorems above, we deduce the following result.

Theorem 5.6. *For any $r \geq 2$ and sufficiently large n, $p_2(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the following companion Turán inequality*

$$p_2(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{3}{2}}} \right) p_2(n+1) p_2(n-1) \leq 0,$$

where $\overline{\gamma}$ is any real number larger than $\frac{\pi}{6}$, the Laguerre inequality of any order r and its companion version as follow

$$1 < \frac{L_r^+(p_2(n-r))}{L_r^-(p_2(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-\frac{2\pi}{3} \right)^r \binom{\frac{1}{2}}{2}^r n^{-\frac{3}{2}(r-1)}.$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Plane partition. A plane partition of size n is a two-dimensional array of non-negative integers $\pi_{j,k}$, for which $\sum_{j,k}\pi_{j,k}=n$, such that $\pi_{j,k}\geq\pi_{j,k+1}$ and $\pi_{j,k}\geq\pi_{j+1,k}$ for any $j,k\in N$, denote the number of plane partitions of size n by $\mathrm{PL}(n)$. Wright [28] proved that

$$\mathrm{PL}(n)=\frac{C}{n^{\frac{25}{36}}}e^{A_1n^{\frac{2}{3}}}\left(1+\sum_{j=2}^{N+1}\frac{B_j}{n^{\frac{2(j-1)}{3}}}+O\left(n^{-\frac{2(N+1)}{3}}\right)\right),$$

where $A_1>0$, C and B_j are constants.

Heim, Neuhauser and Tröger [14] conjectured plane partition is log-concave and proved the conjecture for almost all n. Ono, Pujahari and Rolen [19] derived the following explicit asymptotic formula and proved that $J_{\mathrm{PL}}^{d,n}(x)$ is hyperbolic for sufficiently large n.

Theorem 5.7 (Ono, Pujahari and Rolen)**.** *If $r\in\mathbb{Z}^+$, then for every integer $n\geq\max(n_r,\ell_r,87)$, we have*

$$\mathrm{PL}(n)=\frac{e^{c+3AN_n^2}}{2\pi}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_sb_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}}N_n^{2s+2m+\frac{25}{12}}}+E_r^{\mathrm{maj}}(n)+E^{\mathrm{min}}(n),$$

where $|E_r^{\mathrm{maj}}(n)|\leq\widehat{E}_r^{\mathrm{maj}}(n)$, $N_n:=(\frac{n}{2A})^{\frac{1}{3}}$ and

$$|E^{\mathrm{min}}(n)|\leq\exp\left(\left(3A-\frac{2}{5}\right)n^2/(2A)^{\frac{2}{3}}\right).$$

Pandey [20] found a lower bound such that $J_{\mathrm{PL}}^{d,n}(x)$ has all real roots for all $n\geq N_{\mathrm{PL}}(d)$, where

$$N_{\mathrm{PL}}(d)\leq 279928\times d(d-1)\left(6d^3(22.2)^{\frac{3(d-1)}{2}}\right)^{2d}e^{\frac{\Gamma(2d^2)}{(2\pi)^{2d+2}}}.$$

Obviously, $\mathrm{PL}(n)$ have the form (1.1) with $\lambda_1=\frac{2}{3}$ and $A_{\lambda_1}=A_1$. Thus, employing the theorems above, we get the following result.

Theorem 5.8. *For any $r\geq 2$ and sufficiently large n, $\mathrm{PL}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the following companion Turán inequality*

$$\mathrm{PL}(n)^2-\left(1+\frac{\overline{\gamma}}{n^{\frac{4}{3}}}\right)\mathrm{PL}(n+1)\mathrm{PL}(n-1)\leq 0,$$

where $\overline{\gamma}$ is any real number larger than $\frac{2A_1}{9}$, the Laguerre inequality of any order r and its companion version as follow

$$1<\frac{L_r^+(\mathrm{PL}(n-r))}{L_r^-(\mathrm{PL}(n-r))}<1+\frac{(2r)!}{r!}2^{1-r}\left(-A_1\right)^r\binom{\frac{2}{3}}{2}^rn^{-\frac{4}{3}(r-1)}.$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

The coefficients of infinite product generating functions. $\mathrm{p}_d(n)$ is defined by the generating function [13]

$$\sum_{n=0}^{\infty} \mathrm{p}_d(n) q^n :\ = \prod_{n=1}^{\infty} (1-q^n)^{-n^{d-1}}.$$

Note that $\mathrm{p}_1(n) = p(n)$ and $\mathrm{p}_2(n) = \mathrm{PL}(n)$. Bringmann, Franke and Heim [5] provided that

$$\mathrm{p}_d(n) \sim \frac{C}{n^b} e^{A_1 n^{\frac{d+1}{d+2}}} \left(1 + \sum_{j\geq 1} \frac{E_{d,j}}{n^{\frac{j}{d+2}}}\right),$$

where C, A_1, b and $E_{d,j}$ are constants. One can see that $\mathrm{p}_d(n)$ have the form (1.1) with $\lambda_1 = \frac{d+1}{d+2}$ and $A_{\lambda_1} = A_1$. Thus, we can get the following result by the theorems above.

Theorem 5.9. *For any $r \geq 2$ and sufficiently large n, $\mathrm{p}_d(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the following companion Turán inequality*

$$\mathrm{p}_d(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{1+\frac{1}{d+2}}}\right) \mathrm{p}_d(n+1)\mathrm{p}_d(n-1) \leq 0,$$

where $\overline{\gamma}$ is any real number larger than $\frac{A_1(d+1)}{(d+2)^2}$, the Laguerre inequality of any order r and its companion version as follow

$$1 < \frac{L_r^+(\mathrm{p}_d(n-r))}{L_r^-(\mathrm{p}_d(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} (-A_1)^r \binom{\frac{d+1}{d+2}}{2}^r n^{(-1-\frac{1}{d+2})(r-1)}.$$

Partitions into k-gonal numbers. We denote the number of partitions of n into k-gonal numbers by $p_k(n)$. We have generating function

$$\sum_{n\geq 0} p_k(n) q^n = \prod_{n\geq 1} \frac{1}{1-q^{P_k(n)}},$$

where

$$P_k(n) := \frac{1}{2}\left((k-2)n^2 + (4-k)n\right)$$

is the n-th k-gonal number. Bridges, Brindle, Bringmann and Franke [3] provided that

$$p_k(n) = \frac{C(k) e^{A(k) n^{\frac{1}{3}}}}{n^{\frac{5k-6}{6(k-2)}}} \left(1 + \sum_{j=1}^{N} \frac{B_{j,k}}{n^{\frac{j}{3}}} + O_N\left(n^{-\frac{N+1}{3}}\right)\right),$$

where $C(k)$, $A(k)$ and $B_{j,k}$ are constants.

Note that $p_k(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{3}$ and $A_{\lambda_1} = A(k)$. Then, we can get the following result by the theorems above.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

Theorem 5.10. *For any $r \geq 2$ and sufficiently large n, $p_k(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the companion Turán inequality*

$$p_k(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{5}{3}}}\right) p_k(n+1)p_k(n-1) \leq 0,$$

where $\overline{\gamma}$ is any real number larger than $\frac{2A(k)}{9}$, the Laguerre inequality of any order r and its companion version as follow

$$1 < \frac{L_r^+(p_k(n-r))}{L_r^-(p_k(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-A(k)\right)^r \binom{\frac{1}{3}}{2}^r n^{-\frac{5}{3}(r-1)}.$$

The finite-dimensional representations of groups $\mathfrak{su}(3)$. The unitary group $\mathfrak{su}(3)$, whose irreducible representations $W_{j,k}$ indexed by pairs of positive integers. The numbers $r_{\mathfrak{su}(3)}(n)$ of n-dimensional representations, have the generating function

$$\sum_{n \geq 0} r_{\mathfrak{su}(3)}(n) q^n = \prod_{j,k \geq 1} \frac{1}{1 - q^{\frac{jk(j+k)}{2}}},$$

with $r_{\mathfrak{su}(3)}(0) := 1$. In [24], Romik proved that, as $n \to \infty$,

$$r_{\mathfrak{su}(3)}(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right),$$

with explicit constants C_0, A_1, A_2, A_3, A_4 expressible in terms of zeta and gamma values. Romik asked for lower order terms in the asymptotical expansion of $r_{\mathfrak{su}(3)}(n)$. Bringmann and Franke [4] showed the following form of $r_{\mathfrak{su}(3)}(n)$.

Theorem 5.11 (Bringmann and Franke)**.** *As $n \to \infty$, for any $N \in \mathbb{N}$,*

$$r_{\mathfrak{su}(3)}(n) = \frac{C_0}{n^{\frac{3}{5}}} e^{A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}} \left(1 + \sum_{j=1}^{N} \frac{C_j}{n^{\frac{j}{10}}} + O_N\left(n^{-\frac{N}{10} - \frac{3}{80}}\right)\right),$$

where the constants C_j do not depend on N and n and can be calculated explicitly.

Note that $r_{\mathfrak{su}(3)}(n)$ have the form (1.1) with $\lambda_1 = \frac{2}{5}$ and $A_{\lambda_1} = A_1$. Hence, we can deduce the following result by the theorems above.

Theorem 5.12. *For any $r \geq 2$ and sufficiently large n, $r_{\mathfrak{su}(3)}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the companion Turán inequality*

$$r_{\mathfrak{su}(3)}(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{8}{5}}}\right) r_{\mathfrak{su}(3)}(n+1) r_{\mathfrak{su}(3)}(n-1) \leq 0,$$

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

where $\overline{\gamma}$ is any real number larger than $\frac{6A_1}{25}$, the Laguerre inequality of any order r and its companion version

$$1 < \frac{L_r^+(r_{\mathfrak{su}(3)}(n-r))}{L_r^-(r_{\mathfrak{su}(3)}(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-A_1\right)^r \binom{\frac{2}{5}}{2}^r n^{-\frac{8}{5}(r-1)}.$$

The finite-dimensional representations of groups $\mathfrak{so}(5)$. This framework generalizes to other groups. For example, one can investigate the *Witten zeta function* for $\mathfrak{so}(5)$, which is (for more background to this function, see [17] and [18])

$$\zeta_{\mathfrak{so}(5)}(s) :\ = \sum_{\varphi} \frac{1}{\dim(\varphi)^s} = 6^s \sum_{n,m \geq 1} \frac{1}{m^s n^s (m+n)^s (m+2n)^s},$$

where the φ are running through the finite-dimensional irreducible representations of $\mathfrak{so}(5)$. Bridges, Brindle, Bringmann and Franke [3] showed the following form of $r_{\mathfrak{so}(5)}(n)$.

Theorem 5.13 (Bridges, Brindle, Bringmann and Franke)**.** *As $n \to \infty$, for any $N \in \mathbb{N}$,*

$$r_{\mathfrak{so}(5)}(n) = \frac{C}{n^{\frac{7}{12}}} e^{A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4} \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right)$$

where C, A_1, A_2, A_3, A_4 and B_j are constants.

Bringmann, Franke and Heim [5] have proved that $r_{\mathfrak{su}(3)}(n)$ and $r_{\mathfrak{so}(5)}(n)$ asymptotically satisfy the Turán inequality. Note that $r_{\mathfrak{so}(5)}(n)$ have the form (1.1) with $\lambda_1 = \frac{1}{3}$ and $A_{\lambda_1} = A_1$. Thus, we can get the following result by the theorems above.

Theorem 5.14. *For any $r \geq 2$ and sufficiently large n, $r_{\mathfrak{so}(5)}(n)$ satisfies the double Turán inequality, the Turán inequality of any order, the following companion Turán inequality*

$$r_{\mathfrak{so}(5)}(n)^2 - \left(1 + \frac{\overline{\gamma}}{n^{\frac{5}{3}}}\right) r_{\mathfrak{so}(5)}(n+1) r_{\mathfrak{so}(5)}(n-1) \leq 0,$$

where $\overline{\gamma}$ is any real number larger than $\frac{2A_1}{9}$, the Laguerre inequality of any order r and its companion version

$$1 < \frac{L_r^+(r_{\mathfrak{so}(5)}(n-r))}{L_r^-(r_{\mathfrak{so}(5)}(n-r))} < 1 + \frac{(2r)!}{r!} 2^{1-r} \left(-A_1\right)^r \binom{\frac{1}{3}}{2}^r n^{-\frac{5}{3}(r-1)}.$$

Acknowledgments. This work was supported by the National Science Foundation of China (grant number 12171254).

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

References

[1] K. Banerjee, Invariants of the quartic binary form and proofs of Chen's conjectures on inequalities for the partition function and the Andrews' spt function, Res. Math. Sci., 12 (1) (2025).

[2] W. Bridges and K. Bringmann, Log-concavity for unimodal sequences, Res. Number Theory, 10 (6) (2023).

[3] W. Bridges, B. Brindle, K. Bringmann and J. Franke, Asymptotic expansions for partitions generated by infinite products, Math. Ann., 390 (2024), 2593–2632.

[4] K. Bringmann and J. Franke, An asymptotic formula for the number of n-dimensional representations of $\mathfrak{su}(3)$, Rev. Mat. Ibero., 39 (5) (2023), 1599–1640.

[5] K. Bringmann, J. Franke and B. Heim, Asymptotics of commuting l-tuples in symmetric groups and log-concavity, Res. Number Theory, 10 (83) (2024).

[6] K. Bringmann and K. Mahlburg, An extension of the Hardy-Ramanujan Circle Method and applications to partitions without sequences, Amer. J. Math., 133 (4) (2011), 1157–1178.

[7] W.Y.C. Chen, The spt-function of Andrews. Surveys in combinatorics 2017, 141–203, London Math. Soc. Lecture Note Ser., 440, Cambridge Univ. Press, Cambridge, 2017.

[8] G. Csordas, T.S. Norfolk and R.S. Varga, The Riemann hypothesis and the Turán inequalities, Trans. Amer. Math. Soc., 296 (2) (1986), 521–541.

[9] G. Csordas and A. Vishnyakova, The generalized Laguerre inequalities and functions in the Laguerre-Pólya class, Cent. Eur. J. Math., 11 (9) (2013), 1643–1650.

[10] L.-M. Dou, L.X.W. Wang and N.N.Y.Yang, Unified coefficients for the reversed inequalities associated with $\mathcal{LP}$ class, submitted.

[11] G. Gasper, Positivity and special functions, Proceedings of an Advanced Seminar Sponsored by the Mathematics Research Center, the University of Wisconsin-Madison, March 31-April 2, (1975), 375–433.

[12] M. Griffin, K. Ono, L. Rolen and D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, Proc. Nat. Acad. Sci. USA, 116 (23) (2019), 11103–11110.

https://doi.org/10.4153/S0008414X25101661 Published online by Cambridge University Press

[13] B. Heim and M. Neuhauser, Log-concavity of infinite product generating functions, Res. Number Theory, 8 (53) (2022), 1–14.

[14] B. Heim, M. Neuhauser and R. Tröger, Inequalities for plane partitions, Ann. Comb., 27(1) (2023), 87–108.

[15] E. Laguerre, Oeuvres, vol.1, (Paris: Gaauthier-Villars, 1989).

[16] L. Mauth, Log-concavity for partitions without sequences, arXiv:2306.07459.

[17] K. Matsumoto, On Mordell–Tornheim and other multiple zeta-functions, in Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften, 360 (2003), 17.

[18] K. Matsumoto and H. Tsumura, On Witten multiple Zeta-functions associated with semisimple Lie algebras I, Ann. Inst. Fourier, Grenoble 56 (5) (2006), 1457–1504.

[19] K. Ono, S. Pujahari and L. Rolen, Turán inequalities for the plane partition function. Adv. Math., 409 (2022), 108692.

[20] B.V. Pandey, Higher Turán inequalities for the plane partition function, Res. Number Theory, 10 (69) (2024).

[21] M.L. Patrick, Some inequalities concerning Jacobi polynomials, SIAM J. Math. Anal., 2 (2) (1971), 213–220.

[22] M.L. Patrick, Extensions of inequalities of the Laguerre and Turán type. Pacific J. Math., 44 (2) (1973), 675–682.

[23] R. Rhoades, Asymptotics for the number of strongly unimodal sequences, Int. Math. Res. Notices., 3 (2014), 700–719.

[24] D. Romik, On the number of n-dimensional representations of $\mathfrak{su}(3)$, the Bernoulli numbers, and the Witten zeta function, Acta Arith., 180(2), (2017), 111–159.

[25] I. Wagner, On a new class of Lagueree-Pólya type functions with applications in number theory, Pacific J. Math., 320 (2022), 177–192.

[26] L.X.W. Wang and E.Y.Y. Yang, Laguerre inequalities for discrete sequences, Adv. Appl. Math., 139 (2022), 120357.

[27] L.X.W. Wang and N.N.Y. Yang, Laguerre inequalities and complete monotonicity for the Riemann Xi-function and the partition function, Trans. Amer. Math. Soc., 377 (2024), 4703–4725.

[28] E.M. Wright, Asymptotic partition formulae. I. Plane partitions. Q. J. Math., Oxf. Ser., 2 (1931), 177–189.