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TWO-WEIGHT NORM INEQUALITY AND CARLESON MEASURE IN WEIGHTED HARDY SPACES

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ABSTRACT. Let $(\mathbf{X}, \nu, d)$ be a homogeneous space and let Ω be a doubling measure on $\mathbf{X}$. We study the characterization of measures μ on $\mathbf{X}^+ = \mathbf{X} \times \mathbf{R}^+$ such that the inequality $\|H_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$, where $q < p$, holds for the maximal operator $H_\nu f$ studied by Hörmander. The solution utilizes the concept of the "balayée" of the measure μ.

1. **Introduction.** In [3], Carleson characterized those finite positive measures μ on the unit ball $\mathbf{B}$ in $\mathbf{C}^1$ such that

$$\Big(\int_{\mathbf{B}} |U(z)|^p \, d\mu\Big)^{1/p} \leq C\|f\|_{H^p}$$

for every function f in the Hardy space H^p $(0 < p < \infty)$, where $U(z)$ is the Poisson integral of f. He showed that the above inequality holds if and only if $\mu(S) \leq Ch$ for every set of the form

$$S = \{re^{i\theta} : 1 - h \leq r < 1, \theta_0 \leq \theta \leq \theta_0 + h\}.$$

Such a measure μ is now often called a Carleson measure. Hörmander [5] obtained a more general result in $\mathbf{C}^N$ using a maximal function, Marcinkiewicz interpolation theorem, and a simple covering argument. Using some of Hörmander's ideas, Duren [4] proved the following: for $0 < p \leq q < \infty$

$$\Big(\int_{\mathbf{B}} |U(z)|^q \, d\mu\Big)^{1/q} \leq C\|f\|_{H^p}$$

for every f in H^p, if and only if $\mu(S) \leq Ch^\alpha$, where $1 \leq \alpha = q/p$. Such a measure is called an *α-Carleson measure*. Videnskii [8] generalized the Duren's theorem to the case $q < p$. He proved that the space of measures whose "balayées" belong to a certain $L^{\frac{1}{1-\alpha}}$ space is the replacement of the space of α-Carleson measures when $q < p$. For the higher dimension case , see Luecking [6].

In $\mathbf{R}^N$, the maximal function used by Hörmander is defined by

$$Hf(x,t) = \sup \frac{1}{|Q|} \int_Q |f| \, dx \quad x \in \mathbf{R}^N, t > 0$$

where the supremum is taken over the cubes Q in $\mathbf{R}^N$ centered at x with sides parallel to the axes and has side length at least t.

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Given two measures μ and Ω, the question of determining if H is a bounded map from $L^p(\Omega)$ into $L^q(\mu)$ is referred as a two-weight norm problem. In [7] a characterization of the two-weight norm inequality for H with $p = q$ was obtained by Francisco J. Ruiz and José L. Torrea.

The following question arises: Can we solve the two-weight norm problem when $q < p$?

Using Hörmander's method, it will be shown that H is bounded from $L^p(\Omega)$ into $L^q(\mu)$ (when $q \geq p > 1$) if and only if μ is an α-Carleson measure with $\alpha = q/p$. Using the "balayée" of a measure μ as employed by E. Amar and A. Bonami [1], we are able to prove that if Ω satisfies Muckenhoupt's condition, then H is bounded from $L^p(\Omega)$ into $L^q(\mu)$ with $q < p$ if and only if the "balayée" of μ belongs to $L^{\frac{1}{1-q/p}}(\Omega)$. We also generalize Duren's theorem and Videnskii's result to the weighted Hardy spaces with weights satisfying Muckenhoupt's condition.

We shall present the main results in Section 2. In Section 3 we collect the results for Hörmander operator with $\alpha \geq 1$. We shall prove our main result for Hörmander operator in Section 4 and prove the results for weighted Hardy spaces in Section 5. In the last section , we shall give another characterization of the spaces of "balayées".

I would like to express my sincere appreciation to professor William T. Sledd for his guidance, encouragement, and, in particular, for his careful review and invaluable discussion of this paper.

2. **Definitions and main results.** Let $\mathbf{X}$ be a topological space with a positive Borel measure ν. Let d be a real-valued function in $\mathbf{X} \times \mathbf{X}$. We shall call the triple $(\mathbf{X}, \nu, d)$ a homogeneous space if $(\mathbf{X}, \nu, d)$ satisfies the following properties:

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x) > 0$ if $x \neq y$;
3. there is a constant C_d such that $d(x, z) \leq C_d[d(x, y) + d(y, z)]$ for all x, y and z;
4. given a neighborhood N of a point x there exists a $r > 0$, such that the sphere $B(x, r) = \{y \mid d(x, y) < r\}$ with center at x is contained in N;
5. the spheres $B(x, r) = \{y \mid d(x, y) < r\}$ are measurable and there is a constant C_ν, $C_\nu > 1$, such that

$$0 < \nu\big(B(x, 2r)\big) \leq C_\nu \nu\big(B(x, r)\big) < \infty$$

for all r and x.

A measure satisfying condition 5 is called a *doubling measure*.

In this paper, we shall also assume that the class of compactly supported continuous functions is dense in the space of integrable functions $L^1(\nu)$.

Let $\mathbf{X}^+ = \mathbf{X} \times \mathbf{R}^+$ with the product topology. Denote

$$T\big(B(x, t)\big) = \{(y, s) \in \mathbf{X}^+ \mid B(y, s) \subset B(x, t)\}.$$

Following the notation of E. Amar and A. Bonami [1], for $0 \leq \alpha < \infty$, we shall call a Borel measure μ on $\mathbf{X}^+$ an α-Carleson measure relative to Ω if

$$|\mu|\Big(T\big(B(x, t)\big)\Big) \leq C\Big(\Omega\big(B(x, t)\big)\Big)^\alpha.$$

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Let

$$S_\Omega(x,y,t)=\frac{1}{\Omega\big(B(x,t)\big)}\chi_{B(x,t)}(y).$$

For $f\geq 0$, define

$$S_\Omega f(x,t)=\int_{\mathbf{X}} S_\Omega(x,y,t)f(y)\,d\Omega(y),$$

$$H_\Omega f(x,t)=\operatorname{Sup}\frac{1}{\Omega\big(B(y,s)\big)}\int_{B(y,s)} f(u)\,d\Omega(u),$$

where the supremum is taken over all balls $B(y,s)\supset B(x,t)$.

Define

$$M_\Omega f(x)=\sup_{r>0}\frac{1}{\Omega\big(B(x,r)\big)}\int_{B(x,r)} f(u)\,d\Omega(u)$$

and

$$S^*_\Omega\mu(y)=\int_{\mathbf{X}^+} S_\Omega(x,y,t)\,d\mu(x,t).$$

The nontangential maximal operator is defined by

$$N(u)(x)=\sup\big\{|u(y,t)|:d(x,y)\leq t\big\}=\sup\big\{|u(y,t)|:(y,t)\in\Gamma(x)\big\},$$

where u is a function in $\mathbf{X}^+$ and $\Gamma(x)=\big\{(y,t):d(y,x)\leq t\big\}$.

DEFINITION. Let $0\leq\alpha<\infty$ and let μ be a Borel measure on $\mathbf{X}^+$. Define

$$V^\alpha_\Omega=\Big\{\mu:|\mu|T\big(B(x,t)\big)\leq C\Big(\Omega\big(B(x,t)\big)\Big)^\alpha\Big\},$$

$$W^\alpha_\Omega=\{\mu:S^*_\Omega|\mu|\in L^{\frac{1}{1-\alpha}}(\Omega)\}.$$

For $0<\alpha<1$, W^α_Ω is the complex interpolation space $(V^0_\Omega,V^1_\Omega)_\alpha$ (see [1]).

We say that ω satisfies Muckenhoupt's A_p condition if for any ball B

$$\int_B\omega d\nu\Big[\int_B\omega^{-\frac{1}{p-1}}\,d\nu\Big]^{p-1}\leq C_\omega[\nu(B)]^p,\quad 1<p<\infty$$

$$\int_B\omega\,d\nu\leq C_\omega\nu(B)\operatorname{essinf}_{x\in B}\omega(x),\quad p=1.$$

Note that by Hölder's inequality, $\omega\in A_p$ ($p>1$) implies that Ω is a doubling measure.

The main results of this paper are the following:

THEOREM 2.1. *Let $0<\alpha<1$, and let $q>0$, $p>1$, $q/p=\alpha$. Let μ be a positive measure on $\mathbf{X}^+$. Suppose $\omega\in A_p$ and set $d\Omega=\omega\,d\nu$. If $\mu\in W^\alpha_\Omega$ then there is a constant C such that*

$$\|H_\nu f\|_{L^q(\mu)}\leq C\|f\|_{L^p(\Omega)}$$

for every $f\in L^p(\Omega)$.

Conversely, let $0<q<p<\infty$ and let $\alpha=q/p$. Suppose that Ω is a doubling measure on $\mathbf{X}$. If

$$\|S_\nu f\|_{L^q(\mu)}\leq C\|f\|_{L^p(\Omega)}$$

for every $f\in L^p(\Omega)$, then $\mu\in W^\alpha_\Omega$.

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THEOREM 2.2. *Let $\alpha < 1$ and let $q/p = \alpha$. Then*

$$||u(x,t)||_{L^q(\mu)} \le C||Nu||_{L^p(\Omega)}$$

for all $u(x,t)$ satisfying $Nu \in L^p(\Omega)$ if and only if $\mu \in W^\alpha_\Omega$.

In particular, if $\mathbf{X} = R^N$ and $d\Omega = \omega dm$, where m denotes the Lebesgue measure, then

(1) $\mu \in W^\alpha_\Omega$ implies $||u(x,t)||_{L^q(\mu)} \le C||Nu||_{L^p(\Omega)}$;

(2) Suppose $\omega \in A_p$. If $p > 1$ and $||u(x,t)||_{L^q(\mu)} \le C||Nu||_{L^p(\Omega)}$ for all harmonic functions $u(x,t)$ satisfying $Nu \in L^p(\Omega)$, then $\mu \in W^\alpha_\Omega$;

(3) Suppose $\omega \in A_r$ for some $r \ge 1$. If $p \le 1$ and $||u(x,t)||_{L^q(\mu)} \le C||Nu||_{L^p(\Omega)}$ for all subharmonic functions satisfying $Nu \in L^p(\Omega)$, then $\mu \in W^\alpha_\Omega$.

3. **Results for $\alpha \ge 1$.** In this section, we always assume μ is a positive measure.

LEMMA 3.1. *Let F be a family of $\{B(x,r)\}$ of balls with bounded radii in $\mathbf{X}$. Then there is a countable subfamily $\{B(x_i, r_i)\}$ consisting of pairwise disjoint balls such that each ball in F is contained in one of the balls $B(x_i, br_i)$, where $b = 3C_d^2$ and C_d is the constant in condition 3 .*

For the proof, see [2].

The following theorem is essentially due to Hörmander [6], which gives a relation between an α-Carleson measure and the L^q-norm of the operator H_Ω.

THEOREM 3.2. *Let $\alpha \ge 1$, $p > 1$. Suppose that Ω is a doubling measure on $\mathbf{X}$. Then $\mu \in V^\alpha_\Omega$ if and only if*

$$||H_\Omega f||_{L^q(\mu)} \le C||f||_{L^p(\Omega)}, \quad f \in L^p(\Omega)$$

where $q/p = \alpha$.

PROOF. That $||H_\Omega f||_{L^q(\mu)} \le C||f||_{L^p(\Omega)}$ implies $\mu \in V^\alpha_\Omega$ follows from the standard argument by taking $f = \chi_{B(x,t)}(y)$.

For each n, $n > 0$, we define

$$(H^n_\Omega f)(x,t) = \sup_{s \le n, B(y,s) \supset B(x,t)} \frac{1}{\Omega\big(B(y,s)\big)} \int_{B(y,s)} |f(u)|\, d\Omega(u)$$

and we shall show that the inequality above holds with H_Ω replaced by H^n_Ω with C independent of n. Once this is established, the theorem will follow by letting n tend to infinity.

It is clear that H^n_Ω is of type (∞, ∞). If we can show that H^n_Ω is also of weak type $(1, \alpha)$, the conclusion will follow from Marcinkiewicz interpolation theorem.

Let $\lambda > 0$ and let $E = \{(x,t) \in \mathbf{X}^+ : H^n_\Omega f(x,t) > \lambda\}$. For each $(x,t) \in E$, there is a ball $B(y,r)$ containing x such that $n \ge r \ge t$ and

$$\frac{1}{\Omega\big(B(y,r)\big)} \int_{B(y,r)} |f(u)|\, d\Omega(u) > \lambda.$$

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Let **B** be the collection of all such balls and let $\{B(y_i, r_i)\}$ be the countable subfamily of pairwise disjoint balls of **B** as in Lemma 3.1 . Then $\bigcup_{\mathbf{B}} B(y, r) = \bigcup B(y_i, br_i)$ and that each $B \in \mathbf{B}$ is contained in one of $B(y_i, br_i)$.

It is clear that $E \subset \bigcup T\big(B(y_i, br_i)\big)$. Therefore

$$\begin{aligned}
\mu(E) &\le \mu\Big(\bigcup T\big(B(y_i, br_i)\big)\Big)\\
&\le \sum \mu\Big(T\big(B(y_i, br_i)\big)\Big)\\
&\le C\sum\Big(\Omega\big(B(y_i, br_i)\big)\Big)^{\alpha}\\
&\le C\sum\Big(\Omega\big((B(y_i, r_i)\big)\Big)^{\alpha}\\
&\le \frac{C}{\lambda^{\alpha}}\sum\Big(\int_{B(y_i,r_i)} |f|\,d\Omega\Big)^{\alpha}\\
&\le \frac{C}{\lambda^{\alpha}}\Big(\int |f|\,d\Omega\Big)^{\alpha}.
\end{aligned}$$

That is H^n_Ω is of weak type $(1, \alpha)$. The conclusion follows.

Next we give a similar estimate to the operator H_ν.

Let $\gamma > 1$. If $\omega \in A_\gamma$, by Hölder's inequality, it is easy to show that

$$H_\nu f(x, t) \le C[H_\Omega(|f|^\gamma)]^{1/\gamma}$$

with C only depends on A_γ condition. Thus we have:

THEOREM 3.3. *Let $\alpha \ge 1$. If $\omega \in A_p$ and let $d\Omega = \omega d\nu$, then $\mu \in V^\alpha_\Omega$ if and only if*

$$\|H_\nu f\|_{L^q(\mu)} \le C\|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega)$$

for any $p > 1$, $q > 0$, such that $q/p = \alpha$.

PROOF. That $\|H_\nu f\|_{L^q(\mu)} \le C\|f\|_{L^p(\Omega)}$ implies $\mu \in V^\alpha_\Omega$ follows from the standard argument by taking $f = \chi_{B(x,t)}(y)$.

Now suppose $\mu \in V^\alpha_\Omega$. Since $p > 1$, there is a $\gamma > 1$, $\gamma < p$ such that $\omega \in A_\gamma$ [2]. Then

$$\begin{aligned}
\int_{\mathbf{X}^+} |H_\nu f|^q\,d\mu &\le C\int_{\mathbf{X}^+} |H_\Omega |f|^\gamma|^{\frac{q}{\gamma}}\,d\mu\\
&\le C\Big[\int_{\mathbf{X}^+} |f|^p\,d\Omega\Big]^{q/p}.
\end{aligned}$$

The last inequality follows from Theorem 3.2, since Ω is a doubling measure, $\frac{\frac{q}{\gamma}}{\frac{p}{\gamma}} = q/p = \alpha$ and $\frac{p}{\gamma} > 1$. The proof is complete.

The next lemma is due to E. Amar and A. Bonami [1].

LEMMA 3.4. *Let μ be a positive measure on $\mathbf{X}^+$. Let*

$$g(y) = \int_{\mathbf{X}^+} S_\Omega(x, y, t)\,d\mu(x, t).$$

Then the measure

$$\{S_\Omega(1/g)(x, t)\}\mu \in V^1_\Omega.$$

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PROOF. We need to show that for any ball B

$$\int_{T(B)} S_\Omega(1/g)(x,t)\,d\mu(x,t) \le C\Omega(B).$$

By definition

$$\begin{aligned} I &= \int_{T(B)} S_\Omega(1/g)(x,t)\,d\mu(x,t) \\ &= \int_{\mathbf{X}^+} \chi_{T(B)}(x,t)\Big[\int_{\mathbf{X}} S_\Omega(x,y,t)\frac{1}{g(y)}\,d\Omega(y)\Big]\,d\mu(x,t) \\ &= \int_{\mathbf{X}} \frac{1}{g(y)}\Big[\int_{\mathbf{X}^+} \frac{\chi_{T(B)}(x,t)\chi_{B(x,t)}(y)}{\Omega\big(B(x,t)\big)}\,d\mu(x,t)\Big]\,d\Omega(y). \end{aligned}$$

Since $(x,t) \in T(B)$ and $y \in B(x,t)$ imply that $B(x,t) \subset B$ and $y \in B$, then

$$\chi_{T(B)}(x,t)\chi_{B(x,t)}(y) \le \chi_B(y)\chi_{B(x,t)}(y).$$

Thus

$$\begin{aligned} I &\le \int_B \frac{1}{g(y)}\Big[\int_{\mathbf{X}^+} \frac{\chi_{B(x,t)}(y)}{\Omega\big(B(x,t)\big)}\,d\mu(x,t)\Big]\,d\Omega(y) \\ &= \int_B d\Omega(y) \\ &= \Omega(B). \end{aligned}$$

The proof is complete.

The last theorem of this section is due to Calderón in [2].

THEOREM 3.5. *If* $1 < p < \infty$, $d\Omega = \omega d\nu$ *with* $\omega \in A_p$, *then*

$$\Big[\int |M_\nu f|^p\,d\Omega\Big]^{1/p} \le C\Big[\int |f|^p\,d\Omega\Big]^{1/p}$$

for $f \in L^p(\Omega)$.

4. **Proof of main results for H_ν.** We shall need the following lemma:

LEMMA 4.1. *Let* $(\mathbf{X},\Omega,d)$ *be a homogeneous space and let* $a > 0$. *Then there is a constant* $C > 0$, *such that if* $B(x,r) \cap B(y,r') \ne \phi$, *and* $r \le ar'$, *then* $B(x,r) \subset B(y,Cr')$.

For the proof, see [2].

PROOF OF THEOREM 2.2. Suppose $\mu \in W^\alpha_\Omega$ and suppose $q/p = \alpha, p > 1$. Let

$$g(y) = \int_{\mathbf{X}^+} S_\Omega(x,y,t)\,d\mu(x,t).$$

Then $\mu \in W^\alpha_\Omega$ implies $g \in L^{\frac{1}{1-\alpha}}(\Omega)$.

Note that by Hölder's inequality

$$[S_\Omega(1/g)(x,t)]^{-1} \le S_\Omega g(x,t).$$

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If $f \in L^p(\Omega)$, then

$$\begin{aligned}\int_{\mathbf{X}^+} |H_\nu f|^q \, d\mu &= \int_{\mathbf{X}^+} |H_\nu f|^q [S_\Omega(1/g)(x,t)]^{-1} S_\Omega(1/g)(x,t)\, d\mu(x,t)\\ &\leq \int_{\mathbf{X}^+} |H_\nu f|^q S_\Omega g(x,t) S_\Omega(1/g)(x,t)\, d\mu(x,t)\\ &\leq \Big[\int_{\mathbf{X}^+} |H_\nu f|^p S_\Omega(1/g)(x,t)\, d\mu(x,t)\Big]^{q/p}\\ &\qquad \times \Big[\int_{\mathbf{X}^+} |S_\Omega g(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)\, d\mu(x,t)\Big]^{1-q/p}\\ &\leq [\int_{\mathbf{X}^+} |H_\nu f|^p S_\Omega(1/g)(x,t)\, d\mu(x,t)]^{q/p}\\ &\qquad \times \Big[\int_{\mathbf{X}^+} |H_\Omega g(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)\, d\mu(x,t)\Big]^{1-q/p}\\ &= A \times B.\end{aligned}$$

By Lemma 3.4, $S_\Omega(1/g)(x,t)\mu \in V^1_\Omega$. It follows from Theorem 3.3 that

$$A \leq C\Big[\int |f|^p \, d\Omega\Big]^{q/p}$$

and from Theorem 3.2 that

$$B \leq C\Big[\int_{\mathbf{X}} |g|^{\frac{1}{1-\alpha}} \, d\Omega\Big]^{1-q/p}.$$

Therefore

$$\begin{aligned}\int_{\mathbf{X}^+} |H_\nu f|^q \, d\mu &\leq C\Big[\int_{\mathbf{X}} |f|^p \, d\Omega\Big]^{q/p}\Big[\int_{\mathbf{X}} |g|^{\frac{1}{1-\alpha}} \, d\Omega\Big]^{1-q/p}\\ &\leq C\|f\|^q_{L^p(\Omega)}.\end{aligned}$$

For the converse, suppose that Ω is a doubling measure on $\mathbf{X}$, and that

$$\|S_\nu f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\Omega)}$$

for every $f \in L^p(\Omega)$. From the definition of W^α_Ω, we need to show $g \in L^{\frac{1}{1-\alpha}}(\Omega)$.

Let f be in $L^{p/q}(\Omega)$ which is the dual of $L^{\frac{1}{1-\alpha}}(\Omega)$. For any $y \in B(x,t)$, by Lemma 4.1 and the fact that Ω is a doubling measure, we have

$$S_\Omega f(x,t) \leq CM_\Omega f(y),$$

hence

$$\begin{aligned}|S_\Omega f(x,t)|^{1/q} &\leq \frac{C}{\nu\big(B(x,t)\big)} \int_{B(x,t)} |M_\Omega f(y)|^{1/q} \, d\nu(y)\\ &= CS_\nu(|M_\Omega f|^{1/q})(x,t).\end{aligned}$$

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Therefore

$$\begin{aligned}\left|\int_{\mathbf{X}} g(y)f(y)\,d\Omega\right| &\le \int_{\mathbf{X}^+} |S_\Omega f(x,t)|\,d\mu(x,t)\\ &= \int_{\mathbf{X}^+} |S_\Omega f(x,t)|^{(1/q)q}\,d\mu(x,t)\\ &\le C\int_{\mathbf{X}^+}[S_\nu(|M_\Omega f|^{1/q})]^q\,d\mu(x,t)\\ &\le C\Big[\int_{\mathbf{X}}(M_\Omega f)^{p/q}\,d\Omega\Big]^{q/p} \text{ (by the hypothesis)}\\ &\le C\Big[\int_{\mathbf{X}}|f|^{p/q}\,d\Omega\Big]^{q/p} < \infty.\end{aligned}$$

Since $p/q > 1$, the last inequality follows from a similar argument used in the proof of Theorem 3.2, we leave the details to the reader. Therefore $g \in L^{\frac{1}{1-\alpha}}(\Omega)$. The proof is complete.

COROLLARY 4.2. *Let $0 < q < p$, $1 < p < \infty$ such that $\alpha = q/p$. Let $f \in L^p(\mathbf{R}^N)$ and let U(x,t) denote the Poisson integral of f. Let μ be a positive measure and let m denote the Lebesgue measure on $\mathbf{R}^N$. Then $\mu \in W_m^\alpha$ if and only if there is a constant C such that*

$$\Big(\int_{\mathbf{R}_+^{N+1}} |U(x,t)|^q\,d\mu\Big)^{1/q} \le C\Big(\int_{\mathbf{R}^N} |f|^p\,dm\Big)^{1/p}.$$

PROOF. It suffices to prove the theorem for positive functions $f \ge 0$.

Let m denote the Lebesgue measure on $\mathbf{R}^N$ and let

$$P(x,t) = \frac{C_N t}{(|x|^2 + t^2)^{\frac{N+1}{2}}}$$

be the Poisson kernel in $\mathbf{R}_+^{N+1}$. Let $U(x,t)$ be the Poisson integral of f. Then there exist C_1, C_2 such that

$$C_1 S_m f(x,t) \le U(x,t) \le C_2 H_m f(x,t)$$

for all (x,t).

Therefore the conclusion follows immediately from Theorem 2.2 .

REMARK. 1. Corollary 4.2 is still true when $\mathbf{R}_+^{N+1}$ is replaced by the unit ball of $\mathbf{C}^1$. We leave the details to the reader.

2. In Corollary 4.2, the space $(\mathbf{R}^N, m)$ can be replaced by the homogeneous space $(\mathbf{R}^N, \omega dm, d)$ under the assumptions of Theorem 2.2.

5. **Results for weighted Hardy spaces.** On $\mathbf{R}^N$, let Ω be a doubling measure such that $d\Omega = \omega\,dm$, where m denotes the Lebesgue measure. The weighted Hardy space is defined by

$$H^p(\Omega) = \{u : u \text{ is harmonic in } \mathbf{R}_+^{\mathbf{N+1}}, N(u)(x) \in L^p(\Omega)\}$$

with $||u||_{H^p(\Omega)} = ||N(u)||_{L^p(\Omega)}$.

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LEMMA 5.1. *Let*

$$\Gamma(x) = \{(y,t) : d(x,y) < t\}.$$

Then if $(y,t) \in \Gamma(x)$, *for any function* f *defined on* $\mathbf{X}$, *we have*

$$H_\Omega f(y,t) \leq CM_\Omega f(x).$$

Hence $N(H_\Omega f)(x) \leq CM_\Omega f(x)$.

PROOF.

$$H_\Omega f(y,t) = \sup_{B(z,s)\supset B(y,t)} \frac{1}{\Omega\big(B(z,s)\big)} \int_{B(z,s)} f(u)\, d\Omega(u).$$

Since for any $(y,t) \in \Gamma(x)$ and $B(z,s) \supset B(y,t)$, we have $x \in B(y,t) \subset B(z,s)$. Therefore, by Lemma 4.1 there are constants $C > C_1 > 0$ independent of x, y, z, s and t such that $B(z,s) \subset B(x, C_1 s) \subset B(z, Cs)$. Since Ω is a doubling measure, there is a constant A such that

$$\Omega\big(B(z,s)\big) \geq A\Omega\big(B(z,Cs)\big) \geq A\Omega\big(B(x,C_1 s)\big).$$

Therefore

$$\begin{aligned}\frac{1}{\Omega\big(B(z,s)\big)} \int_{B(z,s)} f(u)\, d\Omega(u) &\leq C\frac{1}{\Omega\big(B(x,C_1 s)\big)} \int_{B(x,C_1 s)} f(u)\, d\Omega(u)\\ &\leq CM_\Omega f(x).\end{aligned}$$

The conclusion follows from the above inequality.

THEOREM 5.2. *Let* $\alpha \geq 1$. *Let* Ω *be a doubling measure on* $\mathbf{X}$. *Then* $\mu \in V_\Omega^\alpha$ *if and only if*

$$\|u(x,t)\|_{L^q(\mu)} \leq C\|Nu\|_{L^p(\Omega)}$$

for all measurable functions u *satisfying* $Nu(x) \in L^p(\Omega)$ *with* $q/p = \alpha$.

In particular, if $\mathbf{X} = \mathbf{R}^N$ *and* $d\Omega = \omega\, dm$, *then*

(1) Suppose $\omega \in A_p$. *If* $p > 1$ *and* $\|u(x,t)\|_{L^q(\mu)} \leq C\|Nu\|_{L^p(\Omega)}$ *for all harmonic functions* $u(x,t)$ *satisfying* $Nu \in L^p(\Omega)$, *then* $\mu \in V_\Omega^\alpha$;

(2) Suppose $\omega \in A_r$ *for some* $r \geq 1$. *If* $p \leq 1$ *and* $\|u(x,t)\|_{L^q(\mu)} \leq C\|Nu\|_{L^p(\Omega)}$ *for all subharmonic functions satisfying* $Nu \in L^p(\Omega)$, *then* $\mu \in V_\Omega^\alpha$.

PROOF. Suppose $p > 1$ and suppose $\mu \in V_\Omega^\alpha$. If $y \in B(x,t)$, then

$$|u(x,t)| \leq Nu(y).$$

Thus

$$\begin{aligned}H_\Omega(Nu)(x,t) &\geq \frac{1}{\Omega\big(B(x,t)\big)} \int_{B(x,t)} Nu(y)\, d\Omega(y)\\ &\geq |u(x,t)|.\end{aligned}$$

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Therefore

$$||u(x,t)||_{L^q(\mu)} \leq C||H_\Omega(Nu)||_{L^q(\mu)} \leq C||Nu||_{L^p(\Omega)}.$$

The last inequality follows from Theorem 3.2 .

For $p \leq 1$, take $r > 0$ such that $p/r > 1$. Let $G(x,t) = |u(x,t)|^r$, then $NG(x) = |Nu(x)|^r \in L^{p/r}(\Omega)$. Then the conclusion follows from the case $p > 1$.

The other direction follows by letting $u(y,s) = \chi_{T(B(x,t))}(y,s)$.

We now prove the particular case.

(1) Let $\chi_{B(y,s)}$ be the characteristic function of $B(y,s)$. Let $U(x,t)$ be the Poisson integral of $\chi_{B(y,s)}$. Then there are C_1, $C_2 > 0$ such that

$$C_1 H_m(x,t) \geq U(x,t) \geq C_2 S_m(\chi_{B(y,s)})(x,t)$$

for all (x,t). Thus if $(x,t) \in TB(y,s)$, then $U(x,t) \geq C_2 S_m(\chi_{B(y,s)})(x,t) \geq C_2$. Hence

$$\Big(\mu\big(TB(y,s)\big)\Big)^{1/q} \leq C||U||_{L^q(\mu)}.$$

By Lemma 5.1, $N(H_m\chi_{B(y,s)})(x) \leq CM_m(\chi_{B(y,s)})$, therefore

$$\begin{aligned}\Big(\mu\big(TB(y,s)\big)\Big)^{1/q} &\leq C||U||_{L^q(\mu)}\\ &\leq C||NU||_{L^p(\Omega)}\\ &\leq C||N(H_m\chi_{B(y,s)})||_{L^p(\Omega)}\\ &\leq C||M_m\chi_{B(y,s)}||_{L^p(\Omega)} \text{ (Lemma 5.1)}\\ &\leq C||\chi_{B(y,s)}||_{L^p(\Omega)}\\ &= C\Big(\Omega\big(B(y,s)\big)\Big)^{1/p}.\end{aligned}$$

The last inequality follows from Theorem 3.5 .

(2) Suppose $p \leq 1$, $\omega \in A_r$ for some $r \geq 1$ and suppose $||u(x,t)||_{L^q(\mu)} \leq C||Nu||_{L^p(\Omega)}$ for all subharmonic functions with $Nu \in L^p(\Omega)$. Let $l \geq r$. For any harmonic function u with $Nu \in L^l(\Omega)$, take $k \geq 1$ such that $l/k = p$. Then $G(x,t) = |u(x,t)|^k$ is subharmonic and $N(G) = |Nu|^k \in L^p(\Omega)$. Thus

$$\begin{aligned}||u||_{L^{l\alpha}(\mu)} &= ||G||^{1/k}_{L^{\alpha l/k}(\mu)}\\ &= ||G||^{1/k}_{L^{\alpha p}(\mu)}\\ &\leq C||NG||^{1/k}_{L^p(\Omega)}\\ &= C||Nu||_{L^l(\Omega)}.\end{aligned}$$

The conclusion follows from the case $p > 1$.

We now turn to the proof of Theorem 2.3 .

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PROOF. We only prove the special case. The proof for the general case is similar.

(1) Suppose $\mu \in W^{\alpha}_{\Omega}$. Let g be the balayée of μ w.r.t. Ω as in Lemma 3.4. Note that by Hölder's inequality

$$[S_\Omega(1/g)(x,t)]^{-1} \le S_\Omega g(x,t).$$

Then

$$\begin{aligned}
\int_{\mathbf{X}^+} |u(x,t)|^q \, d\mu &= \int_{\mathbf{X}^+} |u(x,t)|^q [S_\Omega(1/g)(x,t)]^{-1} S_\Omega(1/g)(x,t)\, d\mu(x,t) \\
&\le \int_{\mathbf{X}^+} |u(x,t)|^q S_\Omega g(x,t) S_\Omega(1/g)(x,t)\, d\mu(x,t) \\
&\le \Big[\int_{\mathbf{X}^+} |u(x,t)|^p S_\Omega(1/g)(x,t)\, d\mu(x,t)\Big]^{q/p} \\
&\qquad \times \Big[\int_{\mathbf{X}^+} |S_\Omega g(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)\, d\mu(x,t)\Big]^{1-q/p} \\
&\le \Big(\int_{\mathbf{X}^+} |u(x,t)|^p S_\Omega(1/g)(x,t)\, d\mu\Big)^{q/p} \\
&\qquad \times \Big(\int_{\mathbf{X}^+} |H_\Omega(g)(x,t)|^{\frac{1}{1-\alpha}} S_\Omega(1/g)(x,t)\, d\mu\Big)^{1-q/p} \\
&\le C\Big(\int_{\mathbf{X}} |Nu|^p \, d\Omega\Big)^{q/p},
\end{aligned}$$

The last inequality follows from Theorem 5.2 and Theorem 3.3 since by Lemma 3.4 $S_\Omega(1/g)(x,t)\mu \in V^1_\Omega$.

(2) Suppose for all harmonic functions $u(x,t)$ with $Nu \in L^p(\Omega)$, $\|u(x,t)\|_{L^q(\mu)} \le C\|Nu\|_{L^p(\Omega)}$. Suppose $p > 1$ and that g is as above. Note that similar to the proof of Theorem 2.2, for any $y \in B(x,t)$, by Lemma 4.1 and the fact that Ω is a doubling measure, we have

$$S_\Omega f(x,t) \le C M_\Omega f(y).$$

Hence

$$\begin{aligned}
|S_\Omega f(x,t)|^{1/q} &\le \frac{C}{m\big(B(x,t)\big)} \int_{B(x,t)} |M_\Omega f(y)|^{1/q} \, dm(y) \\
&= C S_m(|M_\Omega f|^{1/q})(x,t).
\end{aligned}$$

Let $f \in L^{p/q}(\Omega)$. Then

$$\begin{aligned}
\Big|\int_{\mathbf{X}} g(y) f(y)\, d\Omega(y)\Big| &\le \int_{\mathbf{X}^+} [S_\Omega |f|(x,t)]\, d\mu \\
&\le \int_{\mathbf{X}^+} [(S_\Omega |f|)^{1/q}]^q \, d\mu \\
&\le C \int_{\mathbf{X}^+} \Big[S_m\big((M_\Omega |f|)^{1/q}\big)\Big]^q \, d\mu \\
&\le C \int_{\mathbf{X}^+} \Big|U\big((M_\Omega |f|)^{1/q}\big)\Big|^q \, d\mu,
\end{aligned}$$

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where $U\big((M_\Omega|f|)^{1/q}\big)$ denotes the Poisson integral of $(M_\Omega|f|)^{1/q}$. Then by the hypothesis,

$$\begin{aligned}\Big|\int_{\mathbf{X}} g(y)f(y)\,d\Omega(y)\Big| &\leq C\Big(\int_{\mathbf{X}}\big|N\big[U\big((M_\Omega|f|)^{1/q}\big)\big]\big|^p\,d\Omega\Big)^{q/p}\\ &\leq C\Big(\int_{\mathbf{X}}|M_m[(M_\Omega|f|)^{1/q}]|^p\,d\Omega\Big)^{q/p} \text{ (by Lemma 2)}\\ &\leq C\Big(\int_{\mathbf{X}}(M_\Omega|f|)^{p/q}\,d\Omega\Big)^{q/p}\\ &\leq C\Big(\int_{\mathbf{X}}|f|^{p/q}\,d\Omega\Big)^{q/p} \leq \infty.\end{aligned}$$

The last two inequalities follow from Theorem 3.5 since $p > 1$, $p/q > 1$ and $\omega \in A_p$. Therefore $g \in L^{\frac{1}{1-\alpha}}(\Omega)$, that is, $\mu \in W_\Omega^\alpha$.

(3) Similar to the proof of particular case (2) of Theorem 5.2 .

6. Another characterization of W_Ω^α.

Let $K_\mu(x) = \sup_{r>0} \frac{|\mu|(TB(x,r))}{\Omega(B(x,r))}$. Let $d\Omega = \omega d\nu$.

THEOREM 6.1. *Let $\alpha < 1$ and $\omega \in A_\gamma$ for some $\gamma \geq 1$. Then*

$$W_\Omega^\alpha = \{\mu : K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)\}.$$

PROOF. Suppose $\mu \in W_\Omega^\alpha$. We may assume that μ is positive. Then for any $y \in \mathbf{X}$ and $r > 0$,

$$\begin{aligned}\frac{1}{\Omega\big(B(y,r)\big)}\int_{B(y,r)} S_\Omega^*|\mu|(s)\,d\Omega(s) &= \frac{1}{\Omega\big(B(y,r)\big)}\int_{\mathbf{X}^+}\int_{\mathbf{X}} \frac{\chi_{B(y,r)}(s)\chi_{B(x,t)}(s)}{\Omega\big(B(x,t)\big)}\,d\Omega(s)\,d\mu(x,t)\\ &= \frac{1}{\Omega\big(B(y,r)\big)}\int_{\mathbf{X}^+}\int_{\mathbf{X}} \frac{\chi_{B(y,r)\bigcap B(x,t)}(s)}{\Omega\big(B(x,t)\big)}\,d\Omega(s)\,d\mu(x,t)\\ &= \frac{1}{\Omega\big(B(y,r)\big)}\int_{\mathbf{X}^+} \frac{\Omega\big(B(y,r)\bigcap B(x,t)\big)}{\Omega\big(B(x,t)\big)}\,d\mu(x,t)\\ &\geq \frac{1}{\Omega\big(B(y,r)\big)}\int_{TB(y,r)} \frac{\Omega\big(B(y,r)\bigcap B(x,t)\big)}{\Omega\big(B(x,t)\big)}\,d\mu(x,t).\end{aligned}$$

Since if $(x,t) \in TB(y,r)$, then $B(x,t) \subset B(y,r)$. Thus

$$\begin{aligned}\frac{1}{\Omega\big(B(y,r)\big)}\int_{B(y,r)} S_\Omega^*|\mu|(s)\,d\Omega(s) &\geq \frac{1}{\Omega\big(B(y,r)\big)}\int_{TB(y,r)} d\mu(x,t)\\ &= \frac{\mu\big(TB(y,r)\big)}{\Omega\big(B(y,r)\big)}.\end{aligned}$$

Therefore $M_\Omega(S_\Omega^*|\mu|)(y) \geq K_\mu(y)$. By Theorem 3.5, $M_\Omega(S_\Omega^*|\mu|) \in L^{\frac{1}{1-\alpha}}(\Omega)$ if $S_\Omega^*|\mu| \in$

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$L^{\frac{1}{1-\alpha}}(\Omega)$. Hence $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$.

Conversely, suppose $K_\mu \in L^{\frac{1}{1-\alpha}}(\Omega)$. We first prove the following:

LEMMA 6.2. $\{S_\Omega(\frac{1}{K_\mu})(x,t)\}\mu \in V_\Omega^1$.

PROOF. Given any $B(y,r)$, we need to prove that

$$\int_{TB(y,r)} S_\Omega\Big(\frac{1}{K_\mu}\Big)(x,t)\,d\mu(x,t) \le C\Omega\big(B(y,r)\big)$$

with C independent of y and r.

Note that if $s \in B(x,t)$ and $(x,t) \in TB(y,r)$, then $s \in B(y,r)$. By Lemma 4.1 , there are $C_1, C_2 > 0$ independent of s, y and r, such that $B(y,r) \subset B(s, C_1 r) \subset B(y, C_2 r)$. Since Ω is a doubling measure, we have

$$\begin{aligned}\frac{1}{K_\mu(s)} &\le \frac{\Omega\big(B(s,C_1 r)\big)}{\mu\big(TB(s,C_1 r)\big)}\\ &\le \frac{\Omega\big(B(y,C_2 r)\big)}{\mu\big(TB(y,r)\big)}\\ &\le C\frac{\Omega\big(B(y,r)\big)}{\mu\big(TB(y,r)\big)}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{TB(y,r)} S_\Omega\Big(\frac{1}{K_\mu}\Big)(x,t)\,d\mu(x,t) &= \int_{TB(y,r)} \frac{1}{\Omega\big(B(x,t)\big)}\int_{B(x,t)}\frac{d\Omega(s)}{K_\mu(s)}\,d\mu(x,t)\\ &\le \int_{TB(y,r)} C\frac{\Omega\big(B(y,r)\big)}{\mu\big(TB(y,r)\big)}\,d\mu(x,t)\\ &= C\Omega\big(B(y,r)\big).\end{aligned}$$

Now, similar to the proof of the first part of Theorem 2.2 (with g replaced by K_μ), for any $f \in L^\gamma(\Omega)$, take $q < \gamma$ such that $\frac{q}{\gamma} = \alpha$, we have $\|H_\nu f\|_{L^q(\mu)} \le C\|f\|_{L^\gamma(\Omega)}$. Then the second part of Theorem 2.2 implies that $\mu \in W_\Omega^\alpha$. The proof is complete.

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