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GENERALISED SECOND-ORDER DERIVATIVES OF CONVEX FUNCTIONS IN REFLEXIVE BANACH SPACES

JAMES LOUIS NDOUTOUME AND MICHEL THÉRA

Generalised second-order derivatives introduced by Rockafellar in the finite dimensional setting are extended to convex functions defined on reflexive Banach spaces. Our approach is based on the characterisation of convex generalised quadratic forms defined in reflexive Banach spaces, from the graph of the associated subdifferentials. The main result which is obtained is the exhibition of a particular generalised Hessian when the function admits a generalised second derivative. Some properties of the generalised second derivative are pointed out along with further justifications of the concept.

1. INTRODUCTION

Generalised second-order derivatives of extended real-valued functions have been extensively studied and used from many points of view in the last decade in nonsmooth analysis as the following nonexhaustive list of works developed by Borwein and Noll, Cominetti and Correa, Do, Hiriart-Urruty, Loewen and Zheng, Ndoutoume, Noll, Penot, Rockafellar, Seeger,... shows. The desire to obtain second order expansions (Taylor's expansion for example) for nonsmooth functions, by means of differential tools based on epiconvergence of second order differential quotients led Rockafellar in [**13**] to introduce "generalised" quadratic functionals (see the precise definition later), which may be extended real-valued, as giving second derivatives of convex functions at certain points of nonsmoothness. More precisely, given an arbitrary lower semicontinuous extended real-valued proper (that is, not identically equal to $+\infty$) convex function $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, a point x where f is finite and a point $x^* \in \partial f(x)$ (the convex subdifferential of f at x), according to Rockafellar [**13**] (see also Borwein and Noll [**5**]), f is said to have a generalised second derivative (or to be second differentiable in the generalised sense) at x relative to x^*, if the second order difference quotients $\left(\Delta_t^{[2]} f\right)_{x,x^*}(\cdot) := 1/t^2\{f(x+t\cdot) - f(x) - t\langle x^*, \cdot\rangle\}$ epiconverge to a convex generalised quadratic form denoted by f''_{x,x^*} (as $t \downarrow 0^+$). The linear and symmetric operator T_{x,x^*} such that

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$f''_{x,x^*}(h) = 1/2\langle T_{x,x^*}(h), h\rangle$ for every $h \in \operatorname{Dom} f''_{x,x^*}$ (the effective domain of f''_{x,x^*}) is then called the *generalised Hessian* of f at x relative to x^*.

The main ingredient of this approach is the characterisation of convex generalised quadratic forms defined on $\mathbb{R}^n$, which are exactly convex functions $q\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for which the graph of the subdifferential mapping ∂q is a vector subspace. In [**13**] this property results from geometric considerations, namely from the fact that the graph of a maximal monotone operator defined on $\mathbb{R}^n$, is a finite dimensional Lipschitzian manifold; it means that it can be viewed locally as the graph of a Lipschitzian mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

An infinite dimensional version of this approach for second derivatives of convex functions in terms of generalised quadratic functionals and Clarke's tangent cone to the graph of the subdifferential mapping of ∂f, has been provided recently by Ndoutoume [**11**]. The main tool used in this context is the so-called Moreau-Yosida approximation. More precisely, it has been proved in [**11**] that a lower semicontinuous proper convex function q defined on a Hilbert space, is a generalised quadratic form if and only if for each $\epsilon > 0$, q_ϵ (the Moreau-Yosida approximate of index ϵ of q) is a convex quadratic form. An immediate consequence of this result is that when a lower semicontinuous proper convex function f defined on a Hilbert space admits a generalised second derivative at x relative to $x^* \in \partial f(x)$, then the mapping which to $h \in \operatorname{Dom} f''_{x,x^*}$ assigns $A^0(h)$ (here $A^0(h)$ stands for the element of minimal norm in the closed convex subset $\partial f''_{x,x^*}(h)$) is a generalised Hessian of f at x relative to x^*.

Another significant contribution to the characterisation of convex generalised quadratic forms, within the framework of Hilbert spaces, has been provided by Borwein and Noll in [5] and has led to comparable results to those given in [**13**].

It is purpose of the present paper to examine the corresponding question for nonsmooth convex functions defined on reflexive Banach spaces. We assume the reader is familiar with elementary definitions and techniques from convex analysis (see [**16**] for instance). We limit ourselves to the presentation of basic definitions and properties pertaining to the generalised second order derivative of convex functions defined on a reflexive Banach space, mentioning at the end further related developments. The paper is organised as follows:

2. Convex generalised quadratic forms;
3. Generalised second derivatives of convex functions;
4. Application to an optimal control problem.

In the sequel, unless we specify the contrary, X will be a reflexive Banach space and X^* will represent its continuous dual. We recall that a function $q\colon X \to \mathbb{R}$ is called a *quadratic form* when it can be written as

$$q(h) = \frac{1}{2}\langle L(h), h\rangle \quad \text{for all } h \in X, \tag{1.1}$$

where $L: X \to X^*$ is a symmetric linear continuous operator. It is well-known that quadratic forms on X are exactly the functions $q: X \to \mathbb{R}$ for which the gradient $\nabla q: X \to X^*$ (in the Gâteaux sense) is linear and continuous (see [10] for details).

Throughout the following, as usual, for an extended real-valued lower semicontinuous proper convex function $q: X \to \mathbb{R} \cup \{+\infty\}$, we denote by

$$\operatorname{Dom} q := \{x \in X \mid q(x) < +\infty\},$$

the *domain* of q;

$$\operatorname{epi} q := \{(x, r) \in X \times \mathbb{R} \mid r \geqslant q(x)\}$$

the *epigraph* of q;

$$\partial q(u) := \{z \in X^* \mid q(v) \geqslant q(u) + \langle z, v - u\rangle, \quad \text{for all } v \in X\}$$

the *convex subdifferential* of q at u,

$$R(\partial q) := \bigcup_{x \in X} \partial q(x)$$

the *range* of ∂q;

$$D(\partial q) := \{x \in X \mid \partial q(x) \neq \emptyset\}$$

the *domain* of ∂q; and finally,

$$\operatorname{Graph} \partial q := \{(x, x^*) \in X \times X^* \mid x^* \in \partial q(x)\}$$

the *graph* of ∂q.

2. Convex generalised quadratic forms

We begin with:

Definition 2.1: A function $q: X \to \mathbb{R} \cup \{+\infty\}$ is called a generalised quadratic form on X, if $\operatorname{Dom} q$ is a linear subspace of X and if there exists a linear symmetric operator L from $\operatorname{Dom} q$ to X^*, such that q may be represented as

$$q(h) = \frac{1}{2}\langle L(h), h\rangle, \quad \text{for all } h \in \operatorname{Dom} q. \tag{2.1}$$

If furthermore the operator L appearing in (2.1) is closed, then q is referred to as a purely quadratic form (see [**5**] for further details).

Among the generalised quadratic forms on X, we shall only be interested here in those that are convex and which may be characterised as follows:

PROPOSITION 2.1. *Given a generalised quadratic form $q\colon X \to \mathbb{R} \cup \{+\infty\}$, the following statements are equivalent:*

(i) *q is convex;*
(ii) *q is non negative (that is $q(x) \geqslant 0$ for every $x \in \operatorname{Dom} q$).*

PROOF: It suffices to remark that when q is a generalised quadratic form on X, then for all x, $y \in \operatorname{Dom} q$, and for all $\lambda \in [0, 1]$, one has:

$$\lambda(\lambda - 1)q(x - y) = q(\lambda x + (1 - \lambda)y) - \lambda q(x) - (1 - \lambda)q(y). \tag{2.2}$$

and the proof follows. □

Otherwise, the following elementary fact may be observed:

PROPOSITION 2.2. *Let q be an extended real-valued function defined on X. The following statements are equivalent:*

(i) *q is a convex generalised quadratic form on X.*
(ii) *$\operatorname{Dom} q$ is a vector subspace of X, q is positively homogeneous of degree 2 and non negative on $\operatorname{Dom} q$. Moreover, for every $v \in \operatorname{Dom} q$, the mapping defined by $u \in \operatorname{Dom} \mapsto q(u + v) - q(u) - q(v)$ is linear and symmetric.*

PROOF: Trivial. □

We now turn to our main concern in this section; namely the characterisation of convex generalised quadratic forms defined in reflexive Banach spaces, from the graph of the associated subdifferentials. For that purpose, we need to recall the concept of complementary subspaces:

Two subspaces M and N of X are declared to be *algebraic complements* in X when: (i) $M + N = X$ and (ii) $M \cap N = \{0\}$.

Algebraic complements are said to be *topological complements*, if the projection onto M along N, that is, the mapping defined by $P(m + n) = m$ is continuous. This obviously forces M and N to be closed subspaces of X. In the setting of Banach spaces, every subspace admits by Zorn's Lemma an algebraic complement. Moreover, closed algebraic complements are always topological complements. It is no longer true that a closed subspace of X admits a topological complement. For instance, the space C_0 of all real sequences converging to zero has no topological complement in the space of bounded real sequences. However, in a Banach space, any finite dimensional subspace, any closed subspace of finite codimension (that is, which admits an algebraic complement of finite dimension) is topological complemented. Finally, the fact that every closed subspace admits a topological complement characterises Hilbert spaces. For further details the reader is referred to [**9**], [**6**] and [**8**].

The following elementary fact may be observed:

LEMMA 2.3. *Let q be a convex lower semicontinuous generalised quadratic form on X, with the following representation:*

$$q(h) = \frac{1}{2}\langle T(h), h\rangle, \text{ for all } h \in \operatorname{Dom} q. \tag{2.3}$$

Then

(i) $R(\partial q) = T(\operatorname{Dom} q) + (\operatorname{Dom} q)^{\perp}$;
(ii) $D(\partial q) = \operatorname{Dom} q$;
(iii) $\operatorname{Graph} \partial q$ *is a linear subspace of* $X \times X^*$;
(iv) ∂q *satisfies the symmetry property, that is,*

$$(u, u^*), (v, v^*) \in \operatorname{Graph} \partial q \Longrightarrow \langle u, v^*\rangle = \langle v, u^*\rangle.$$

PROOF: Let us prove that for all $x \in \operatorname{Dom} q$, one has

$$\partial q(x) = T(x) + (\operatorname{Dom} q)^{\perp}. \tag{2.4}$$

Indeed, consider $y^* \in X^*$ such that $y^* - T(x) \in (\operatorname{Dom} q)^{\perp}$; since $\operatorname{Dom} q$ is a linear subspace, for all $u \in \operatorname{Dom} q$, we then have:

$$\langle y^*, u - x\rangle = \langle T(x), u - x\rangle. \tag{2.5}$$

Furthermore, since

$$q(u) - q(x) = \frac{1}{2}\langle T(x), u - x\rangle + \frac{1}{2}\langle u - x, T(u - x)\rangle + \frac{1}{2}\langle x, T(u - x)\rangle, \tag{2.6}$$

using the fact that

$$\frac{1}{2}\langle u - x, T(u - x)\rangle \geqslant 0 \tag{2.7}$$

it follows from (2.6) that

$$q(u) - q(x) \geqslant \frac{1}{2}\langle T(x), u - x\rangle + \frac{1}{2}\langle T(x), u - x\rangle. \tag{2.8}$$

From (2.5), using the symmetry of T, we get

$$q(u) - q(x) \geqslant \langle y^*, u - x\rangle \text{ for all } u \in \operatorname{Dom} q. \tag{2.9}$$

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That means that $y^* \in \partial q(x)$.

Conversely, if $y^* \in \partial q(x)$, for all $t > 0$ and for all $u \in \operatorname{Dom} q$, one has:

$$\langle y^*, tu\rangle \leqslant q(x+tu) - q(x) = t\langle T(x), u\rangle + \frac{t^2}{2}\langle T(u), u\rangle. \tag{2.10}$$

Dividing (2.10) by $t > 0$ we get

$$\langle T(x), u\rangle + \frac{t}{2}\langle T(u), u\rangle \geqslant \langle y^*, u\rangle. \tag{2.11}$$

Hence, by passing to the limit as $t \to 0^+$, we obtain

$$\langle y^* - T(x), u\rangle \leqslant 0, \quad \text{for all } u \in \operatorname{Dom} q. \tag{2.12}$$

As $\operatorname{Dom} q$ is a linear subspace, this yields, $\langle y^* - T(x), u\rangle = 0$ for all $u \in \operatorname{Dom} q$, and therefore $y^* - T(x) \in (\operatorname{Dom} q)^\perp$, as desired.

The proof of assertion (ii) is trivial since it suffices to observe that by virtue of (2.4), we have $T(x) \in \partial q(x)$ for all $x \in \operatorname{Dom} q$. Assertions (iii) and (iv) are immediate consequences of (2.4). ∎

REMARK 2.1. Lemma 2.3 specifies that a necessary condition for a lower semicontinuous proper convex function to be a generalised quadratic form, is that its effective domain coincides with the domain of the associated subdifferential mapping. Hence, without loss of generality, we shall restrict hereafter our attention to the convex functions which satisfy this property.

LEMMA 2.4. *Let $q\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function such that* $\operatorname{Graph} \partial q$ *is a linear subspace of $X \times X^*$. Assume that $q(0) = 0$ and* $\operatorname{Dom} q = D(\partial q)$. *Then,*

(i) *q is positively homogeneous of degree 2 and non negative on* $\operatorname{Dom} q$;
(ii) $\operatorname{Dom} q$ *is a linear subspace of X;*
(iii) $(\operatorname{Dom} q)^\perp = \partial q(0)$.

PROOF: Let us prove assertion (ii) via Rockafellar's integration method (see [**17**] for details), according to which for every $x \in X$, we have

$$q(x) = \sup_{c(0,x)} \{q(0) + \sum_{p=1}^{n} \langle \partial q(x_{i-1}), x_i - x_{i-1}\rangle\}. \tag{2.13}$$

The supremum is taken over all finite chains $c(0, x)$ (that is $x_0 = 0$, $x_1 \cdots x_{n-1}$, $x_n = x$ such that $x_0, x_1, x_2, \cdots x_{n-1}$ belongs to $D(\partial q)$. For any finite chain $c(0, x)$ and for all $t \in \mathbb{R}_+$, since $\partial q(tu) = t\partial q(u)$ *for every* $u \in \operatorname{Dom} q$, one has

$$t^2 \sum_{p=1}^{n} \langle \partial q(x_{i-1}), x_i - x_{i-1}\rangle = \langle \partial q(tx_{i-1}), t(x_i - x_{i-1})\rangle, \tag{2.14}$$

from which we derive

$$(2.15)\qquad \begin{aligned} t^2 \sup_{c(0,x)} \{q(0) + \sum_{p=1}^{p=n} \langle \partial q(x_{i-1}),\, x_i - x_{i-1}\rangle\} \\ = \sup_{c(0,tx)} \{q(0) + \sum_{p=1}^{p=n} \langle \partial q(y_{i-1}),\, y_i - y_{i-1}\rangle\}. \end{aligned}$$

This means: $t^2 q(x) = q(tx)$, that is q is positively homogeneous of degree 2. Let us now verify that $\operatorname{Dom} q$ is a linear subspace of X. Consider $(t, h) \in \mathbb{R} \times \operatorname{Dom} q$; since $q(th) = t^2 q(h)$, necessarily $th \in \operatorname{Dom} q$. On the other hand, consider $h_1, h_2 \in \operatorname{Dom} q$. By using the fact that q is convex and positively homogeneous of degree 2, one has

$$(2.16)\qquad q(h_1 + h_2) = 4q\left(\frac{h_1 + h_2}{2}\right) \leqslant 2\,(q(h_1) + q(h_2)) < +\infty,$$

and therefore $h_1 + h_2 \in \operatorname{Dom} q$.

Let us now prove that $(\operatorname{Dom} q)^{\perp} = \partial q(0)$. Consider $z^* \in (\operatorname{Dom} q)^{\perp}$; then for all $u \in \operatorname{Dom} q$ one has $\langle z^*, u\rangle = 0$; consequently

$$(2.17)\qquad q(u) \geqslant q(0) + \langle z^*, u - 0\rangle, \quad \text{for all } u \in \operatorname{Dom} q,$$

that means $z^* \in \partial q(0)$. Conversely, pick $z^* \in \partial q(0)$; then for all $u \in \operatorname{Dom} q$, and for all $t > 0$, one has $q(tu) \geqslant \langle z^*, tu\rangle$. Since $q(tu) = t^2 q(u)$, we get $tq(u) \geqslant \langle z^*, u\rangle$ for all $t > 0$. Letting t go to zero, we obtain $0 \geqslant \langle z^*, u\rangle$ for all $u \in \operatorname{Dom} q$. Using the fact that $\operatorname{Dom} q$ is a linear subspace, it follows: $\langle z^*, u\rangle = 0$ for all $u \in \operatorname{Dom} q$. That means $z^* \in (\operatorname{Dom} q)^{\perp}$. ∎

We are finally able to characterise convex generalised quadratic forms from the graphs of the associated subdifferentials.

Theorem 2.5. *Let $q: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function such that $\operatorname{Dom} q = D(\partial q)$ and $q(0) = 0$. Then the following statements are equivalent:*

(i) *$\operatorname{Graph} \partial q$ is a linear subspace of $X \times X^*$;*
(ii) *q is a convex generalised quadratic form on X.*

Proof: Let us prove the implication (i) $\Rightarrow$ (ii). We denote by $S(\partial q(0))$ an algebraic complement of $\partial q(0)$ in $R(\partial q)$. Consider $x \in D(\partial q)$. We first claim that for all $y, y' \in \partial q(x)$, one has

$$(2.18)\qquad \operatorname{proj}_{S(\partial q(0))}(y) = \operatorname{proj}_{S(\partial q(0))}(y'),$$

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that is, the projection of y onto $S(\partial q(0))$ coincides with the projection of y' onto $S(\partial q(0))$. Indeed, set $y = y_1 + y_2$ and $y' = y'_1 + y'_2$ with $y_1, y'_1 \in S(\partial q(0))$ and $y_2, y'_2 \in \partial w(0)$. Since the graph of ∂q is a linear subspace and $y, y' \in \partial q(x)$, it follows:

$$y - y' = y_1 - y'_1 - (y_2 - y'_2) \in \partial q(0), \tag{2.19}$$

that is, $y_1 - y'_1 \in S(\partial q(0)) \cap \partial q(0)$. Hence $y_1 - y'_1 = 0$ and (2.18) holds true. Let us now set for all $y \in \partial q(x)$

$$T(x) := \operatorname{proj}_{S(\partial q(0))}(y). \tag{2.20}$$

Since $\partial q(0)$ and $S(\partial q(0))$ are algebraic complements in $R(\partial q)$, then T is obviously linear on $\operatorname{Dom} q$. In summary, we have proved that for each $x \in \operatorname{Dom} q = D(\partial q)$, one has

$$\partial q(x) = T(x) + \partial q(0), \tag{2.21}$$

where $T(x)$ denotes the unique vector defined in (2.21). From here, the rest of the proof is similar to that given in [5] within the framework of Hilbert spaces. Indeed, observing that $T(x) \in \partial q(x)$ for all $x \in \operatorname{Dom} q$, one has for all $t \in \mathbb{R}$

$$q(x + tx) - q(x) \geqslant \langle T(x), tx \rangle, \tag{2.22}$$

that is,

$$(2t + t^2) q(x) \geqslant \langle T(x), tx \rangle. \tag{2.23}$$

Dividing (2.23) by positive (respectively negative) t, and letting t go to zero, we obtain:

$$q(x) = \frac{1}{2} \langle T(x), x \rangle, \quad \text{for every } x \in \operatorname{Dom} q.$$

On the other hand, since for all $x, y \in \operatorname{Dom} q$,

$$\langle T(x), y \rangle \leqslant \frac{1}{t}(q(x + ty) - q(x)) = \frac{1}{2}(\langle T(x), y \rangle + \langle T(y), x \rangle + (t \langle T(y), y \rangle),$$

letting t go to zero, we deduce that T is symmetric on $\operatorname{Dom} q$. This ends the proof of the implication (i) $\Rightarrow$ (ii). The reverse implication (ii) $\Rightarrow$ (i) is an immediate consequence of Lemma 2.3. ∎

We are interested now in the duality point of view. For that purpose, we recall that the Legendre-Fenchel conjugate of q is the functional q^* defined by:

$$q^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - q(x)\}.$$

Theorem 2.6. (Conjugacy) *Let $q\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function such that* $\operatorname{Dom} q = D(\partial q)$ *and* $q(0) = 0$. *Then q is a convex generalised quadratic form on X if and only if q^* is a convex generalised quadratic form on X^*.*

Proof: It is an immediate consequence of Theorem 2.5 and the fact that ∂q and ∂q^* are inverse to each other. ∎

Recall that for a convex function $q\colon X \to \mathbb{R} \cup \{+\infty\}$, the Moreau-Yosida approximate of q of index $\lambda > 0$ is denoted by q_λ and is defined by

$$q_\lambda(h) = \inf_{x \in X} \left\{ q(x) + \frac{1}{2\lambda} \|x - h\|^2 \right\}, \quad \text{for all } h \in X. \tag{2.24}$$

In the sequel we shall suppose as in [**2**, Theorem 1.20] that X is a reflexive Banach space such that X and X^* are equipped with strictly convex and Kadec norms. We shall denote by $(\mathcal{C})$ this class of Banach spaces which includes the class of Banach spaces which are uniformly convex along with their duals (for instance $L^p(T, \mathcal{B}, \mu)$, with μ σ-finite and $1 < p < +\infty$, $W^{k,p}(\Omega)$, with Ω bounded, $k \in]1, +\infty[$, $k \in \mathbb{N}$). In the class $(\mathcal{C})$, the infimum in (2.24) is attained at a unique point denoted by $J_\lambda^q h$. It should also be observed that in the class $(\mathcal{C})$, the metric projection on any convex closed set is single-valued and continuous. Throughout the paper we shall denote by $A^0(x)$ the element of minimal norm in the closed convex set $\partial q(x) := \min_{u \in \partial q(x)} \|x\|$.

Theorem 2.7. *Let X be of class $(\mathcal{C})$, and let $q\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function such that* $\operatorname{Dom} q = D(\partial q)$ *and* $q(0) = 0$. *The following statements are equiavlent:*

(i) $\operatorname{Graph} \partial q$ *is a linear subspace of* $X \times X^*$;

(ii) *The mapping $x \in \operatorname{Dom} q \mapsto A^0(x)$ is linear and symmetric. Moreover, q admits the representation:*

$$q(h) = \frac{1}{2} \langle A^0(h), h \rangle, \quad \text{for all } h \in \operatorname{Dom} q.$$

Proof: Let us firstly prove that under assumption (i), equality: $q(h) = 1/2 \langle A^0(h), h \rangle$ holds true for all $h \in \operatorname{Dom} q$. In reality, we are going to prove that this equality holds true whenever q is positively homogeneous of degree 2. In fact, for all $x \in \operatorname{Dom} q = D(\partial q)$, and for all $t \in \mathbb{R}$, we have

$$q(th) + \frac{1}{2\lambda} \|tx - th\|^2 = t^2 \left(q(x) + \frac{1}{2\lambda} \|x - h\|^2 \right), \ \lambda, h \in X. \tag{2.25}$$

Hence,

$$\min_{u \in X} \left\{ q(u) + \frac{1}{2\lambda} \|u - th\|^2 \right\} = t^2 \min_{x \in X} \left\{ q(x) + \frac{1}{2\lambda} \|x - h\|^2 \right\}. \tag{2.26}$$

From (2.26), we get $q_\lambda(th) = t^2 q_\lambda(h)$. Consequently, since q_λ is Fréchet differentiable, then one obtains: $\nabla q_\lambda(th) = t\nabla q_\lambda(h)$. As a result

$$\frac{d}{dt} q_\lambda(th) = \langle \nabla q_\lambda(th),\, h\rangle = t\langle \nabla q_\lambda(h),\, h\rangle. \tag{2.27}$$

By integrating (2.27) on $[0, 1]$, we get:

$$q_\lambda(h) - q_\lambda(0) = \int_0^1 \langle \nabla q_\lambda(h),\, h\rangle t\, dt, \tag{2.28}$$

from which we derive

$$q_\lambda(h) = \frac{1}{2}\langle \nabla q_\lambda(h),\, h\rangle, \quad \text{for all } h \in X. \tag{2.29}$$

Letting λ go to zero and using the fact that $\nabla q_\lambda(h) \to A^0(h)$ for all $h \in D(\partial q)$ (see [1] for instance), we get

$$q(h) = \frac{1}{2}\langle A^0(h),\, h\rangle, \quad \text{for all } h \in \operatorname{Dom} q. \tag{2.30}$$

Let us prove now that A^0 is linear on $\operatorname{Dom} q$. Indeed, let $x \in \operatorname{Dom} q = D(\partial q)$. Since $A^0(x)$ is the element of least norm in $\partial q(x)$, we have

$$\left\|A^0(x)\right\| = \min_{z \in \partial q(x)} \|z\|. \tag{2.31}$$

Using the fact that $\partial q(x) = T(x) + \partial q(0)$ (see the proof of Theorem 2.5), we deduce from (2.31) that:

$$\left\|A^0(x)\right\| = \|w_0 - T(x)\| = \min_{w \in \partial q(0)} \|w - T(x)\|,$$

where w_0 denotes the projection of $T(x)$ on $\partial q(0)$. That means that

$$A^0(x) = T(x) - \operatorname{proj}_{\partial q(0)} T(x), \quad \text{for all } x \in \operatorname{Dom} q.$$

The linearity of T implies that of A^0. Moreover, the symmetry property of A^0 is easily obtained from the relation:

$$\langle A^0(u),\, v\rangle = \langle T(u),\, v\rangle \text{ for all } u,\, v \in \operatorname{Dom} q.$$

The implication (ii) $\Rightarrow$ (i) is an immediate consequence of Theorem 2.5. ∎

3. Generalised second derivatives of convex functions

Generalised second derivatives of extended real-valued functions, as defined by Rockafellar [**13**] within the framework of finite dimensional spaces, requires in the present setting the concept of Mosco-convergence of sets (see Attouch's book [**1**] or the survey [**3**]). Let (X, τ) be a first countable topological space. Given a sequence $\{C_n \subseteq X \mid n \in \mathbb{N}\}$ of subsets of X, the τ-lower limit of the sequence $\{C_n \mid n \in \mathbb{N}\}$, denoted by $\tau\text{-}\liminf_{n\to\infty} C_n$ is the closed subset of X defined by

$$\tau\text{-}\liminf_{n\to\infty} C_n := \left\{x \in X \mid \exists (x_n)_{n\in\mathbb{N}} \xrightarrow{\tau} x \text{ such that } x_n \in C_n (\forall n \in \mathbb{N})\right\}.$$

The τ-upper limit of the sequence $\{C_n \mid n \in \mathbb{N}\}$, denoted by $\tau\text{-}\limsup_{n\to\infty} C_n$ is the closed subset of X defined by

$$\tau\text{-}\limsup_{n\to\infty} C_n := \left\{x \in X \mid \exists (n_k)_{k\in\mathbb{N}}, \exists \{x_k\}_{k\in\mathbb{N}} \xrightarrow{\tau} x, \text{ such that } x_k \in C_{n_k} (\forall k \in N)\right\}.$$

Definition 3.1: The sequence $\{C_n \mid n \in \mathbb{N}\}$ is declared *Kuratowski-Painlevé* convergent for the topology τ, or briefly τ-convergent, if the following equality holds:

$$\tau\text{-}\limsup_{n\to\infty} C_n \tau\text{-}\liminf_{n\to\infty} C_n.$$

Its limit, denoted by $C = \tau\text{-}\lim C_n$, is the closed subset of X equal to this common value

$$C = \tau\text{-}\liminf_{n\to\infty} C_n = \tau\text{-}\limsup_{n\to\infty} C_n = \tau\text{-}\lim_{n\to\infty} C_n.$$

When X is a normed linear space, the sequential weak upper limit of a sequence $\{C_n \mid n \in \mathbb{N}\}$ of subsets of X is defined by

$$\text{seq -}w\text{-}\limsup_{n\to\infty} C_n := \Big\{x \in X \mid \exists (n_k)_{k\in\mathbb{N}}, \exists (x_k)_{k\in\mathbb{N}} x_k \xrightarrow{w} x, \text{ such that } x_k \in C_{n_k} (\forall k \in \mathbb{N})\Big\}.$$

Definition 3.2: A sequence $\{C_n \mid n \in \mathbb{N}\}$ of subsets of a normed linear space X is said to Mosco converge to a set C, and we write $C = M\text{-}\lim_{n\to\infty} C_n$, if:

$$\text{seq -}w\text{-}\limsup_{n\to\infty} C_n \subseteq C \subseteq \liminf_{n\to\infty} C_n.$$

Equivalently, $C = M\text{-}\lim_{n\to\infty} C_n$, if and only if both of the following conditions hold:

(i) for each $x \in C$, there exists a sequence $\{x_n \mid n \in \mathbf{N}\}$ norm converging to x such that $x_n \in C_n$ for each $n \in \mathbf{N}$;

(ii) for each subsequence $\{n_k \mid k \in \mathbf{N}\}$ and $\{x_k \mid k \in \mathbf{N}\}$ such that $x_k \in C_{n_k}$, the weak convergence of $\{x_k \mid k \in \mathbf{N}\}$ to $x \in C$ forces x to belong to C.

It is an immediate consequence of these definitions that Mosco convergence implies Kuratowski-Painlevé convergence and that the two notions coincide whenever X is finite dimensional. It turns out that Mosco-convergence is a basic concept when considering sequences of convex sets in reflexive Banach spaces.

DEFINITION 3.3: Let X be a normed space and $\{f, f_n \mid n \in \mathbb{N}\}$ be a sequence of functions from X into $\mathbb{R} \cup \{+\infty\}$. We say that f is the Mosco-epi-limit of the sequence $\{f_n \mid n \in \mathbb{N}\}$, and we write $f = M-\operatorname{epi}\lim_{n\to\infty} f_n$, if the sequence $\{\operatorname{epi} f_n \mid n \in \mathbb{N}\}$ Mosco converges to $\operatorname{epi} f$.

This is equivalent to saying that, for any $x \in X$, the two following statements hold:

$$\text{(i)} \quad \text{for any } \{x_n\} \xrightarrow{w} x, \text{ then } f(x) \leqslant \liminf_{n\to\infty} f_n(x_n)$$

and

$$\text{(ii)} \quad \text{for each } x \in X, \text{ there exists } \{x_n\} \xrightarrow{s} x \text{ such that } \limsup_{n\to\infty} f_n(x_n) \leqslant f(x).$$

This notion of convergence can be characterised in many other equivalent ways, but we do not need to go into the details here; see [**1**] or [**3**] for additional description and references.

Consider now a lower semicontinuous proper convex function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in X$ where f is finite. Given a point $x^* \in \partial f(x)$, we introduce the second-order difference quotient as follows:

$$\left(\Delta_t^{[2]} f\right)_{x,x^*}(h) := \frac{1}{t^2}\{f(x + th) - f(x) - t\langle x^*, h\rangle\}. \tag{3.1}$$

Different types of second-order directional derivatives can be introduced if one considers different types of convergence for the family $\left(\Delta_t^{[2]} f\right)_{x,x^*}$ as $(t \downarrow 0)$. The pointwise version has been studied in a detailed way in Seeger [**18**]. The epigraphical version has been investigated in Rockafellar [**14**]. What is looked for there as a generalised second order derivative of f at x relative to x^*, is some generalised quadratic form as a substitute for $\langle \nabla^2 f(x)u, v\rangle$. This leads in the present setting to consider the following definition: f will be said to have a *generalised second order derivative* at x relative to x^* if the functions $\left(\Delta_t^{[2]} f\right)_{x,x^*}$ Mosco converge to a convex generalised quadratic form, denoted by f''_{x,x^*} (as $t \downarrow 0$). In this case the linear symmetric operator T_{x,x^*} such that

$$f''_{x,x^*} = \langle T_{x,x^*}(h), h\rangle \quad \text{for all } h \in \operatorname{Dom} f''_{x,x^*} \tag{3.2}$$

is called the *generalised Hessian* of f at x relative to x^*. The next theorem explains our desire already expressed in the first part of this work, to obtain a characterisation of convex generalised quadratic forms from the graph of the associated subdifferential operator.

THEOREM 3.1. *Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous proper convex function. Suppose $x \in \operatorname{Dom} f$ and consider $x^* \in \partial f(x)$. Then f has a generalised second derivative at x relative to x^* if and only if the family of sets*

$$\left[\frac{\operatorname{Graph}\partial f - (x, x^*)}{t}\right]_{t>0}$$

converges in the Kuratowski-Painlevé sense (as $t \downarrow 0$) to a linear subspace.

PROOF: It suffices to observe that $\operatorname{Graph}\partial\left(\Delta_t^{[2]}f\right)_{x,x^*}$ coincides with $[(\operatorname{Graph}\partial f - (x, x^*))/t]$ and then to combine Attouch's subdifferential convergence Theorem [1, Theorem 3.66] with Theorem 2.5. ∎

COROLLARY 3.2. *Let X be of class $(\mathcal{C})$. Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous proper convex function. Suppose $x \in \operatorname{Dom} f$ and consider $x^* \in \partial f(x)$. Assume that f admits a generalised second derivative at x relative to x^* denoted by f''_{x,x^*}. Then the mapping $h \in \operatorname{Dom} q \mapsto A^0(h)$, (where $A^0(h)$ denotes the element of minimal norm in the closed convex set $\partial f''_{x,x^*}(h)$) is a generalised Hessian of f at x relative to x^*.*

PROOF: It is an immediate consequence of Theorem 2.6. ∎

THEOREM 3.3. (Conjugacy) *Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous proper convex function. Suppose $x \in \operatorname{Dom} f$ and consider $x^* \in \partial f(x)$. Then f admits a generalised second derivative at x relative to x^* if and only if f^* admits a generalised second derivative at x^* relative to x. Under these conditions, we have:*

$$\left(f''_{x,x^*}\right)^* = \left(f_{x^*,x}\right)''.$$

PROOF: Observing that $\left(\left(\Delta_t^{[2]}f\right)_{x,x^*}\right)^* = \left(\left(\Delta_t^{[2]}f^*\right)_{x^*,x}\right)$, the result follows from the continuity of the Legendre-Fenchel transformation with respect to Mosco convergence (see [1]). ∎

Let us recall that the eqisum (or infimal convolution) of two convex lower semicontinuous proper functions f and g is defined by $f \underset{e}{+} g(x) := \inf_{y\in X}\{f(x) + g(y - x)\}$.

COROLLARY 3.4. *Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous proper convex function. Let φ be a function constructed from f in either of the following ways.*

(i) *$\varphi := f + g$ where g is a convex function of class C^2 on X;*

(ii) *$\varphi = f \underset{e}{+} g$, where g is the conjugate of a convex function h of class C^2 on X.*

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Then, for a given pair $(x, x^*) \in \operatorname{Graph} \partial f$, f *admits a generalised second derivative at* x *relative to* x^* *if and only if* φ *has a generalised second derivative at* w *relative to* w^*, *where* $(w, w^*) := (x, x^* + \nabla g(x))$ *(respectively* $(w, w^*) := (x + \nabla h(x^*), x^*)$ *in case (ii)).*

PROOF: (i) is trivial. (ii) follows from the combination of Theorem 3.3 and the formula $\left(f \underset{e}{+} g\right)^* = f^* + h$. ∎

COROLLARY 3.5. *Suppose that* X *is a Hilbert space and let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous proper convex function and let* f_λ *be its Moreau-Yosida approximate of index* $\lambda > 0$. *Then, for a given* $(x, z) \in \operatorname{Graph} \partial f$, f *admits a generalised second derivative at* $u = (I + \lambda \partial f)^{-1}(x)$ *relative to* z *if and only if* f_λ *admits a second generalised derivative at* x *relative to* z.

PROOF: It suffices to observe that $f_\lambda = f \underset{e}{+} \widetilde{g}$ with $\widetilde{g}(x) = (1/2\lambda) \|x\|^2$ and $\widetilde{g}^*(p) = (\lambda/2) \|p\|^2$. ∎

We now give some elementary examples in order to clarify the meaning of f''_{x,x^*} (the generalised second derivative of f at x relative to x^*).

EXAMPLE 3.1. Let $f\colon X \to \mathbb{R}$ be a Fréchet differentiable convex function in a neighbourhood of $x \in X$. If f is twice Fréchet differentiable at x, then f admits a generalised second derivative at x relative to $x^* = \nabla f(x)$, with $\nabla^2 f(x)$ as a generalised Hessian. In other words

$$f''_{x,x^*}(h) = \frac{1}{2} \langle \nabla^2 f(x) h, h \rangle \qquad \forall h \in X.$$

The reader will note that the converse is true when X is finite-dimensional (see [7, Proposition 4.1] for a proof).

EXAMPLE 3.2. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a convex generalised quadratic form. Then, for all $(x, x^*) \in \operatorname{Graph} \partial f$, it is readily seen that

$$\lim_{t \downarrow 0} \frac{1}{t} [\operatorname{Graph} \partial f - (x, x^*)]$$

exist and coincides with $\operatorname{Graph} \partial f$. That means f admits a generalised second derivative at x relative to x^* and f''_{x,x^*} coincides with f.

EXAMPLE 3.3. Let X be of class $(\mathcal{C})$ and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function. Given $(x, x^*) \in \operatorname{Graph} \partial f$, assume that the second order difference quotient functions

$$\left(\Delta_t^{[2]} f\right)_{x,x^*}(h) = \frac{1}{t^2} \{ f(x + th) - f(x) - t \langle x^*, h \rangle \} \qquad \forall h \in X$$

Mosco epiconverge (as $t \downarrow 0$) to some function Φ such that $\Phi(0) \neq -\infty$. It is well-known in this case (see [7] for example) that Φ is a lower semicontinuous proper convex and positively homogeneous of degree 2 function. If we assume in addition that $\operatorname{Dom}\Phi$ coincides with $D(\partial\Phi)$, then we can prove as in Theorem 2.6 that Φ may be represented as:

$$\Phi(h) = \frac{1}{2}\langle A^0(h), h\rangle \qquad \forall h \in \operatorname{Dom}\Phi.$$

Here $A^0(h)$ stands for the element of minimal norm of the closed convex subset $\partial\Phi(h)$. Hence, under the above assumptions, it is readily seen that f admits a generalised second derivative at x relative to x^*, if and only if the mapping $A^0\colon \operatorname{Dom}\Phi \to X^*$ is linear.

EXAMPLE 3.4. Consider a convex integrand $f\colon \Omega \times \mathbb{R}^d \cup \{+\infty\}$ with the corresponding integral $I_f(x) = \int_\Omega f(\omega, x(\omega))d\omega$ on $x \in L^p(\Omega)$ $(1 < p < +\infty)$, the space of p-integrable measurable functions x from a measurable space Ω to $\mathbb{R}^d$. We recall the well-known fact that when I_f is proper, one has that: an element x^* of $L^q(\Omega)$ belongs to $\partial I_f(x)$ if and only if $x^*(\omega)$ belongs to $\partial f_\omega(x(\omega))$ for almost all $\omega \in \Omega$, where $f_\omega(\cdot) = f(\omega, \cdot)$ (with $1/p + 1/q = 1$) (see [15] for details). Assume that $f_\omega(\cdot)$ admits a generalised second derivative at $x(\omega)$ relative to $x^*(\omega)$ for almost all $\omega \in \Omega$. It follows that I_f also admits a generalised second derivative at x relative to x^*, and

$$(I_f)''_{x,x^*}(h) = \int_\Omega f''_{(x(\omega),x^*(\omega))}(\omega, h(\omega))d\omega \qquad \forall h \in L^p(\Omega).$$

For the proof, we first recall the well-known fact (see [7, Theorem 5.5] for instance) that the existence of the Mosco-epi-limit of the second order difference quotients

$$\left(\Delta_t^{[2]} f\right)_{x(\omega),x^*(\omega)}(u) = \frac{1}{t^2}\{f(\omega, x(\omega) + tu) - f(\omega, x(w)) - t\langle x^*(\omega), u\rangle\}$$

(as $t \downarrow 0$) for almost all $\omega \in \Omega$, implies the existence of the Mosco-epi-limit of the second order difference quotient functions

$$\left(\Delta_t^{[2]} I_f\right)_{x,x^*}(h) = \frac{1}{t^2}\{I_f(x + th) - I_f(x) - t\langle x^*, h\rangle\}$$

(as $t \downarrow 0$). Moreover, if we set $q =$ Mosco-epi- $\lim\limits_{t\to 0+} \left(\Delta_t^{[2]} I_f\right)_{x,x^*}$, it follows that

$$q(h) = \int_\Omega f''_{x(\omega),x^*(\omega)}(\omega, h(\omega))d\omega \qquad \forall h \in L^p(\Omega).$$

Then, using the fact that for almost all $\omega \in \Omega$, $f''_{x(\omega),x^*(\omega)}$ is a convex generalised quadratic form, we can deduce from Proposition 2.2 that q is also a convex generalised quadratic form.

We conclude this section with the following comment:

COMMENT 3.1. We observe that when the function f is twice Fréchet differentiable at x, then $\partial f''_{x,x^*}$ (with $x^* = \nabla f(x)$) coincides with $\nabla^2 f(x)$ (the second Fréchet derivative of f at x). In this case, the symmetry property (see (iv) Lemma 2.3) of $\partial f''_{x,x^*}$ can be rewritten as follows:

$$\langle \nabla^2 f(x)u, v\rangle = \langle u, \nabla^2 f(x)v\rangle \qquad \forall u, n \in X.$$

This means that the linear mapping $\nabla^2 f(x)\colon X \to X^*$ is self-adjoint. We are all familar with this property of the second order Fréchet derivative. In a nondifferentiable setting where $\partial f''_{x,x^*}$ can be considered as a substitute for $\nabla^2 f(x)$, the symmetry property of $\partial f''_{x,x^*}$ plays a role parallel to the one played by the well-known symmetry property of the Hessian of a function at a given point. For more background on the usefulness of the symmetry property of $\partial f''_{x,x^*}$ within the framework of Hilbert spaces, we refer the reader to [**12**]. In this case, the symmetry property of $\partial f''_{x,x^*}$ was proved by using the Moreau-Yosida approximation (see [**12,** Theorem 2.1]).

4. Application to an optimal control problem

We consider the following nonsmooth control problem:

$$(\mathcal{P}) \qquad \begin{cases} \text{minimise the function } g(y) + h(u) \\ \text{over all } (y, u) \text{ subject to the state system (variational inequality)} \\ Ay + \partial F(y) \ni Bu + f \end{cases}$$

where A is a linear operator from the state space X (which is a reflexive Banach space) to X^* (its topological dual), ∂F is a subgradient operator (that is, the subdifferential of a lower semicontinuous proper convex function F on X), and B is a linear continuous operator from the space of controls U (which is also a reflexive Banach space) to X^*. g is of class C^1 on X and h is a lower semicontinuous proper convex function on U.

We observe the following:

THEOREM 4.1. *(First-order optimality conditions). Let $(\overline{y}, \overline{u})$ be any optimal solution for problem $(\mathcal{P})$, and set $\overline{r} := B\overline{u} - A\overline{y} + f$. Assume that F has a generalised second derivative at $\overline{y}$ relative to $\overline{r}$ and the linear mapping A is weak to norm continuous, that is, for every sequence $\{x_n \mid n \in \mathbb{N}\}$ such that $x_n \xrightarrow{w} x$ in X, then $A(x_n) \xrightarrow{\tau} A(x)$ in X^*. Then*

$$-Z^*(\nabla g(\overline{y})) \in \partial h(\overline{u}),$$

where Z^* denotes the adjoint of the linear Lipschitz mapping $Z\colon X \to X^*$ defined by

$$Z(v) = \left(A + \partial F''_{\overline{y},\overline{r}}\right)^{-1} \circ B(v), \quad \text{for all } u \in U.$$

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PROOF: The proof relies strongly on the fact that Graph $\partial F''_{\overline{y},\overline{r}}$ is a linear subspace. Indeed, let us first recall that for all $u \in U$, the variational inequality

$$A(y) + \partial F(y) \ni Bu + f$$

admits an unique solution denoted by $y(u)$. Moreover, the mapping $u \mapsto y(u)$ is Lipschitz (see [4, Theorem 2.1] for details). Then, for every $v \in U$ and $\lambda > 0$, we have

$$g(\overline{y}(\overline{u} + \lambda v)) + h(\overline{u} + \lambda v) \geqslant g(\overline{y}(\overline{u})) + h(\overline{u}). \tag{4.1}$$

Set $Z_\lambda(v) := (\overline{y}(\overline{u} + \lambda v) - \overline{y}(\overline{u}))/\lambda$. Then, using the fact that g is of class C^1, we get

$$o(\lambda) - \langle \nabla g(\overline{y}(\overline{u})),\, Z_\lambda(v)\rangle \leqslant \frac{h(\overline{u} + \lambda v) - h(\overline{u})}{\lambda}. \tag{4.2}$$

Since the mapping $u \mapsto y(u)$ is Lipschitz, the family $\{Z_\lambda \mid \lambda > 0\}$ is bounded. Hence, there exists $Z(v)$ such that $Z_{\lambda_n}(v) \overset{w}{\to} Z(v)$ for some $\lambda_n \downarrow 0$. Noticing that

$$B(\overline{u}) - A(\overline{y}(\overline{u})) + f + \lambda_n[B(v) - A(Z_{\lambda_n}(v))] \in \partial F(\overline{y}(\overline{u}) + \lambda_n Z_{\lambda_n}(v)) \tag{4.3}$$

holds for all $n \in \mathbb{N}$, it follows that

$$B(v) = A(Z(v)) \in \partial F''_{\overline{y},\overline{r}}(Z(v)). \tag{4.4}$$

Then,

$$Z(v) = \left(A + \partial F''_{\overline{y},\overline{r}}\right)^{-1} \circ B(v). \tag{4.5}$$

Since $\partial F_{\overline{y},\overline{r}}$ has a closed linear graph, then $\left(A + \partial F''_{\overline{y},\overline{r}}\right)^{-1}$ is obviously a linear Lipschitz mapping. This implies that Z is linear and Lipschitian. Hence, by virtue of (4.2), we get:

$$\langle -\nabla g(\overline{y}),\, Z(v)\rangle \leqslant h'(u,\, v), \tag{4.6}$$

where

$$h'(u,\, v) = \lim_{\lambda \downarrow 0} \frac{h(u + \lambda v) - h(v)}{\lambda}.$$

Then,

$$\langle -Z^*(\nabla g(\overline{y})),\, v\rangle \leqslant h'(u,\, v). \tag{4.7}$$

(4.7) being true for every $v \in U$, this yields

$$-Z^*(\nabla g(\overline{y})) \in \partial h(\overline{u}),$$

and the proof is complete. ∎

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Institut Africain d'Informatique B.P. 2263
Libreville
Gabon
and
LACO URA-CNRS 1586
Université de Limoges
F-87060 Limoges, Cedex
France

LACO URA-CNRS 1586
Université de Limoges
F-87060 Limoges, Cedex
France
e-mail: thera@cix.cict.fr