ON THE STRUCTURE OF FROBENIUS GROUPS

WALTER FEIT

1. Introduction. Let G be a group which has a faithful representation as a transitive permutation group on m letters in which no permutation other than the identity leaves two letters unaltered, and there is at least one permutation leaving exactly one letter fixed. It is easily seen that if G has order mh, a necessary and sufficient condition for G to have such a representation is that G contains a subgroup H of order h which is its own normalizer in G and is disjoint[1] from all its conjugates. Such a group G is called a Frobenius group of type (h, m).

Some immediate consequences of the definition are that in a Frobenius group G of type (h,m), h divides $m - 1$, every element other than the identity whose order divides h is contained in exactly one subgroup of order h, and any two subgroups of order h are conjugate. A fundamental property of a Frobenius group G of type (h,m) is that G contains exactly m elements whose order divides m and these form (**2**, p. 334) a normal subgroup M of G. This subgroup M will be called the regular subgroup of G, since the above mentioned permutation representation of G when restricted to M is just the regular representation of M.

Burnside has shown that the regular subgroup of a Frobenius group of type (h, m), with even h, is abelian of odd order (**2**, p. 172). Conversely it is not hard to show that an abelian group of odd order m can be imbedded in a Frobenius group G of type $(2, m)$ as the regular subgroup of G. In general, the regular subgroup of a Frobenius group need not be abelian (see **4** for a counter example), however it has been conjectured that it must always be nilpotent.[2] The main result proved below is that, under certain conditions, the regular subgroup of a Frobenius group is nilpotent. If it can be shown that no exceptional groups exist (in the sense of §4), then the nilpotency would be proved in general. The result can also be restated in a different form using the language of automorphisms[3] as is done in the Corollary in §4.

2. Some properties of Frobenius groups

LEMMA 2.1. *Let G be a group of order hm, where h and m are relatively prime and unequal to* 1. *Then G is a Frobenius group of type (h, m) if and only if G*

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[1]Two subgroups of a group are said to be disjoint if their intersection consists of only the identity element.

[2]I am indebted to Professor Marshall Hall for telling me about this conjecture.

[3]After I had written up this paper I was informed by Professor Herstein that he and Professor Wielandt had proved a result essentially equivalent to the Corollary in §4, though their methods were somewhat different from those used here.

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contains a normal subgroup M of order m, and the order of every element divides either h or m.

Proof. If G is a Frobenius group of type (h, m), then its regular subgroup M is normal in G and has order m. Since h and m are relatively prime, every element x in G can be written as a product $x = x_1x_2$, where x_1 and x_2 commute and $x_1^h = 1 = x_2^m$. If $x_1 \neq 1$, then it lies in some subgroup H of order h, hence

$$x_2^{-1}x_1x_2 = x_1 \in H \cap x_2^{-1}Hx_2.$$

Since distinct subgroups of order h are disjoint, x_2 must lie in the normalizer of H, and as H is its own normalizer, x_2 must lie in H; therefore $x_2 = 1$. Consequently either $x_1 = 1$ or $x_2 = 1$, and the order of every element in G must divide either h or m.

Conversely, as the index of M in G is relatively prime to m, G contains a subgroup H of order h (**5**, p. 132, Theorem 25) and every element whose order divides h lies in a subgroup conjugate to H (**3**, p. 184, Lemma 6.1). Let N be the normalizer of H, $M \cap N$ is a normal subgroup of N, hence N is the direct product of H and $M \cap N$, thus if $M \cap N \neq \{1\}$, N contains elements where order divides neither h nor m which is impossible, therefore $N = H$, and G contains m subgroups of order h conjugate to H. Each of these subgroups contains $h - 1$ elements other than the identity, if some element $x \neq 1$ is contained in two of these subgroups, then the total number of elements unequal to the identity whose order divides h is strictly less than $m(h - 1) = mh - m$. All the elements whose order divides m lie in M and thus there are exactly m of them, hence the number of elements in G, other than the identity, whose order divides h must equal $mh - m$. Therefore the assumption that G contains two subgroups of order h which are not disjoint is untenable and the proof is complete.

LEMMA 2.2. *Let G be a Frobenius group of type (h, m). If G_1 is a subgroup of order h_1m_1, where h_1 divides h,m_1 divides m, and $h_1 \neq 1 \neq m_1$, then G_1 is a Frobenius group of type (h_1,m_1).*

Proof. By Lemma 2.1, every element of G_1 has order dividing h_1 or m_1. If M is the regular subgroup of G, then $M \cap G_1$ is a normal subgroup of G_1 whose order is m_1, hence Lemma 2.1 implies the result.

LEMMA 2.3. *Let G be a Frobenius group of type (h,m). If $\bar{G}$ is a homomorphic image of G whose order is hm_1, with $m_1 > 1$, then $\bar{G}$ is a Frobenius group of type (h, m_1).*

Proof. Let K of order k be the kernel of the homomorphism, then $K \in M$, and the image $\bar{M}$ of M is a normal subgroup of $\bar{G}$ whose order is $m/k = m_1$.

If $\bar{x}$ is an element of $\bar{G}$ and x is some element of G which is mapped onto $\bar{x}$, then $x^n = 1$ implies that $\bar{x}^n = 1$. Since the order of x divides either h or m, this is also the case for $\bar{x}$. If the order of $\bar{x}$ divides m, then it must divide the

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greatest common divisor m_1 of m and the order m_1h of $\bar{G}$. Hence the order of every element in $\bar{G}$ divides either h or m_1 and Lemma 2.1 now implies that $\bar{G}$ is a Frobenius group of type (h_1, m_1).

LEMMA 2.4. *Suppose G is a Frobenius group of type (h, m), let r be a prime dividing h, then G contains a subgroup G_1 which is a Frobenius group of type (r, m). The regular subgroup of G is also the regular subgroup of G_1.*

Proof. Let M be the regular subgroup of G. G contains a subgroup R of order r. Since M is normal in G, $G_1 = MR$ is a group of order mr which contains M as a normal subgroup and in which the order of every element divides either m or r, hence Lemma 2.1 yields the desired result.

LEMMA 2.5. *Suppose that G is a Frobenius group of type (h, m). Let p be a prime dividing m and let T be a subgroup of some Sylow p-group P of G which is normal in the normalizer $N(P)$ of P. If the normalizer $N(T)$ of T has order n, then h divides n and $N(T)$ is a Frobenius group of type $(h, n/h)$.*

Proof. As the regular subgroup M of G is normal in G and its order m is divisible by the highest power of p which divides the order of G, all Sylow p-groups lie in M and hence are conjugate in M. Therefore the number of Sylow p-groups (in G as well as in M) is a divisor of m, then the index of the normalizer $N(P)$ of P in G divides m, hence h divides the order of $N(P)$. By assumption T is normal in $N(P)$, therefore h divides the order n of $N(T)$. By Lemma 2.2 $N(T)$ is a Frobenius group of the desired type.

LEMMA 2.6. *Let G be a Frobenius group of type (h,m). Suppose that A is a subgroup of G of order q^2 or qr, where q and r are primes dividing h, then A is cyclic.*

Proof. Let M be the regular subgroup of G. As M is normal in G, MA is a group and hence by Lemma 2.2 a Frobenius group whose regular subgroup is M. Let p be a prime dividing m and let P be a Sylow p-group of M. Let C be the subgroup of P consisting of all elements in the center of P whose pth power is the identity. Clearly C is a characteristic subgroup of P hence normal in $N(P)$, therefore by Lemma 2.5 $N(C)$ is a Frobenius group. The group $N(C)$ contains a Frobenius group $G_1 = A_1C$ whose regular subgroup is C, and which contains a subgroup A_1 conjugate to A.

Each element x in A_1 defines an automorphism of C which sends y into $x^{-1}yx$ for y in C, hence A_1 can be considered to be a group of automorphisms of C. As A_1C is a Frobenius group with regular subgroup C, no element of A_1 commutes with any element of C. In other words, no automorphism of A_1 leaves any element of C other than the identity fixed. An argument of Burnside[4] can now be applied which shows that A_1 is cyclic, as A is conjugate to A_1, it too must be cyclic.

[4]Burnside deduced a false theorem from a correct argument; this can be found in (**2**, pp. 334–335). For a statement and proof of the result, in the form needed above, see (**6**, p. 196).

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3. The structure of a special class of groups. This section is devoted to investigating groups satisfying certain assumptions. In order to prevent repetition, the basic hypothesis will be stated separately. The symbols $\otimes$ and $\oplus$ will stand for direct product and direct sum respectively. For any subset S of G, the centralizer of S in G will be denoted by $C(S)$.

Hypothesis I. *The order of G is $p^a q^b$, where p and q are distinct primes and $a,b > 0$. The Sylow p-group P of G is the direct product of groups of order p and is normal in G. A Sylow q-group of G is the direct product of groups of order q.*

LEMMA 3.1. *Suppose that G satisfies hypothesis* I. *There is a one to one mapping from P onto an a-dimensional vector space V over the field F of p elements with the property that $\overline{y_1 y_2} = \bar{y}_1 + \bar{y}_2$ for y_1,y_2 in P, where $\bar{y}$ denotes the image of y in V. Let Q be a Sylow q group of G, for x in Q define the linear transformation $A(x)$ acting on V by $A(x)\bar{y} = \overline{xyx^{-1}}$ for all y in P. The mapping of Q into the group of linear transformations on V defined in this way is a completely reducible representation Γ of Q. A subgroup of P is normal in G if and only if the corresponding subspace $\bar{P}$ of V is invariant under the representation Γ.*

Proof. Every statement of the Lemma can easily be checked. The complete reducibility of Γ follows from the fact that the characteristic p of F does not divide the order q^b of Q.

LEMMA 3.2. *If G satisfies hypothesis* I, *and if P contains a subgroup T which is normal in G, then G contains a normal subgroup T_1 with the property that $P = T \otimes T_1$.*

Proof. Let $\bar{T}$ be the subspace of V corresponding to T under the mapping defined in Lemma 3.1. Since T is normal in G, $\bar{T}$ is invariant under the representation Γ. The complete reducibility of Γ implies the existence of an invariant subspace $\bar{T}_1$ such that $V = \bar{T} \oplus \bar{T}_1$. Hence by Lemma 3.1, T_1 is a normal subgroup of G and $P = T \otimes T_1$.

LEMMA 3.3. *Suppose that G satisfies hypothesis* I, *let Q be a Sylow q-group of G. If x is in Q, then $C(x) \cap P$ is a normal subgroup of G. If P_0 is a minimal normal subgroup of G which is contained in P, then q^{b-1} divides the order of $C(P_0)$.*

Proof. Since P is a normal subgroup of G, $C(x) \cap P$ is a normal subgroup of $C(x)$. Therefore the normalizer of $C(x) \cap P$ contains $C(x)$ which contains Q, since x lies in the abelian group Q. As P is abelian, it lies in the normalizer of every subgroup. Hence $G = PQ$ is contained in the normalizer of $C(x) \cap P$.

The group $C(x) \cap P_0$ is a normal subgroup of G since it is the intersection of the normal subgroups $C(x) \cap P$ and P_0. Therefore since P_0 is a minimal normal subgroup of G, either P_0 is contained in $C(x)$ or disjoint from it. In other words, every element x of Q either lies in the centralizer of P_0 or commutes with no element of P_0 other than the identity.

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Suppose that q^{b-1} does not divide the order of $C(P_0)$, then Q contains a subgroup Q_0 of order q^2 which is disjoint from $C(P_0)$. As P_0 is normal in G, $G_0 = P_0Q_0$ is a group containing P_0 as a normal subgroup. Since Q_0 is a Sylow q-group of G_0, every element whose order divides q^2 is conjugate to an element of Q_0, and hence commutes with no element of order p. Therefore the order of every element divides either q^2 or p^{a_0}, where p^{a_0} is the order of P_0. Lemma 2.1 now implies that G_0 is a Frobenius group of type

$$(q^2, p^{a_0}),$$

consequently Q_0 is cyclic by Lemma 2.6. This is impossible since Q_0 contains no element of order q^2. Thus the assumption that q^{b-1} does not divide the order of $C(P_0)$ has led to a contradiction, which proves the result.

LEMMA 3.4. *Let G be a group which satisfies hypothesis* I. *Assume that there exists an automorphism σ of G of prime order r which sends some Sylow q group Q of G into itself. Furthermore suppose that $\sigma(s) \neq s$ where s is any proper subgroup of Q, or any proper subgroup of P which is normal in G, or any element of G other than the identity. Then either G is abelian*[5] *or* $b = 1$.

Proof. Suppose that G is not abelian and $b > 1$. Let P_1 be a minimal normal subgroup of G, if q^b divides the order of $C(P_1)$, then P_1 lies in the center of G, hence the p-Sylow group of the center of G must equal P since it is mapped into itself by σ, but then G is abelian contrary to assumption. Hence the Sylow q-group of $C(P_1)$ has order q^{b-1} by Lemma 3.3. By taking a group conjugate to P_1 if necessary, it may be assumed that there is a q-Sylow group Q_1 of $C(P_1)$ which is contained in Q.

Define $P_{i+1} = \sigma(P_i)$ for all i, let k be the smallest integer with the property that

$$\bar{P}_{k+1} \subset \bar{P}_1 + \ldots + \bar{P}_k,$$

where the bar denotes the mapping defined in Lemma 3.1. If y_1 is any element in P_1, and $y_{i+1} = \sigma(y_i)$, then σ maps the product $y_1 \ldots y_r$ into itself, hence by assumption this product is 1, and therefore y_r is contained in $P_1 \ldots P_{r-1}$. As y_1 was chosen arbitrarily in P_1, this states that $P_r \subset P_1 \ldots P_{r-1}$, and hence $\bar{P}_r \subset \bar{P}_1 + \ldots + \bar{P}_{r-1}$, consequently $k < r$. Since σ maps $P_1 \ldots P_k$ into itself and no proper subgroup of P which is normal in G has this property, $P = P_1 \ldots P_k$, and therefore $\bar{P} = \bar{P}_1 + \ldots + \bar{P}_k$. The representation is completely reducible, hence (**1**, Theorem 1.4C) $\bar{P} = \bar{P}_1 \oplus \ldots \oplus \bar{P}_k$.

Let $Q_i = C(P_i) \cap Q$, it is easily seen that $Q_{i+1} = \sigma^i(Q_1)$, hence by the choice of P_1, the order of Q_i is exactly q^{b-1}. Since $b > 1$, each group Q_i is a proper subgroup of Q, therefore $\sigma(Q_i) \neq Q_i$ for all i. As $k < r$, this implies that $Q_i \neq Q_j$ for $1 \leqslant i < j \leqslant k$, because otherwise $\sigma^{j-i}(Q_i) = Q_i$ which in turn implies $\sigma(Q_i) = Q_i$, as $0 < j - i < k < r$ and $j - i$ and r are relative prime.

[5]Actually G is abelian in all cases, this is a consequence of the Theorem below.

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Suppose that there exists an operator isomorphism ρ from the Γ module $\bar{P}_i$ onto the Γ module $\bar{P}_j$ with $1 \leqslant i < j \leqslant k$. As Q_i and Q_j are distinct subgroups of Q of order q^{b-1} it is possible to find an element x in Q_j which is not contained in Q_i. Then

$$\rho\,\{A(x)\bar{y}\} = A(x)\,\{\rho(\bar{y})\}$$

for y in $\bar{P}_i$. Since x is in Q_j, $\rho(y)$ is in $\bar{P}_j$, $A(x)\{\rho(\bar{y})\} = \rho(\bar{y})$, therefore $\rho\{A(x)\bar{y}\} = \rho(\bar{y})$ which implies that $A(x)\bar{y} = \bar{y}$ which finally yields that x is in Q_i contrary to the choice of x. Consequently the Γ module $\bar{P}_i$ is not operator isomorphic to the Γ module $\bar{P}_j$ for $1 \leqslant i < j \leqslant k$. This implies that the irreducible subspaces of $\bar{P}$ are unique (**1**, Theorem 1.6C), in other words the only irreducible subspaces of $\bar{P}$ are $\bar{P}_1, \ldots, \bar{P}_k$, hence the only minimal normal subgroups of G which are contained in P are $P_1, \ldots, P_k$. As σ is an isomorphism of G mapping P into itself, $\sigma(P_k)$ is a minimal normal subgroup of G contained in P, therefore $\sigma(P_k) = P_i$ for some i between 1 and k, hence $\sigma^{k+1-i}(P_i) = P_i$. As $0 < k + 1 - i \leqslant k < r$, $k + 1 - i$ and r must be relatively prime and $\sigma(P_i) = P_i$, therefore $\sigma(Q_i) = Q_i$, which was shown to be impossible. Hence the assumption that G is non-abelian and $b > 1$ has led to a contradiction which proves the Lemma.

4. The regular subgroup of a Frobenius group. Before proceeding to investigate the structure of the regular subgroup of a Frobenius group it is necessary to make the following definition.

Definition. A group G is said to be *exceptional* if G is a non-cyclic simple group in which the normalizer of every characteristic subgroup $\neq \{1\}$ of a Sylow p-group P of G is P, for all primes p dividing the order of G.

No known groups are exceptional in the sense defined above. A special case of a conjecture of Zassenhaus (**7**, footnote on p. 6) would be sufficient to prove that exceptional groups do not exist. The case treated in the theorem below is concerned with groups in which no subgroup has a composition factor which is an exceptional group. This is a large class of groups as is shown by the following Lemma.

LEMMA 4.1. *If G is a solvable group, or if every Sylow group of G is abelian, then no subgroup of G has a composition factor which is an exceptional group.*

Proof. If G is solvable, the result is immediate as no simple group can occur as a composition factor of a subgroup. If every Sylow group of G is abelian then this is also the case for every composition factor of a subgroup, hence it suffices to show that a group H in which every Sylow group is abelian cannot be exceptional. Let P be a Sylow p-group of H for some prime p dividing the order of H. Suppose that P is its own normalizer, then by a theorem of Burnside (**5**, p. 139), H cannot be simple and consequently not exceptional.

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LEMMA 4.2. *Let M be the regular subgroup of a Frobenius group G of type (h,m). Suppose that M is not a direct product of exceptional groups, and that the regular subgroup of every proper subgroup of G which is a Frobenius group of type (h,m_1) is nilpotent, then M contains a normal subgroup of prime power order.*

Proof. Let H be a subgroup of G of order h. If M contains a proper characteristic subgroup M_1 of order m_1, then M_1 is normal in G, hence by Lemma 2.2, M_1H is a Frobenius group of type (h,m_1), consequently M_1 is nilpotent and any Sylow subgroup of M_1 is normal in M.

Suppose now that M is characteristically simple and contains no normal subgroup of prime power order. Then M is the direct product of isomorphic non-cyclic simple groups $M_1, \ldots, M_s$. By assumption these are not exceptional therefore there is a prime p such that the Sylow p-group P_1 of M_1 contains a characteristic subgroup T_1 such that $N(T_1) \cap M_1 \neq P_1$, where $N(T_1)$ denotes the normalizer of T_1 in G. Let P_i and T_i denote the images in M_i of P_1 and T_1 respectively, under an isomorphism mapping M_1 onto M_i. Let

$$P = P_1 \otimes \ldots \otimes P_s, \qquad T = T_1 \otimes \ldots \otimes T_s;$$

it is clear then that P is a Sylow p-group of M, T is a normal subgroup of $N(P)$ and $P \neq N(T) \cap M$. By assumption T is not normal in M, hence $N(T) \neq G$, then by Lemma 2.5 and the assumption of this Lemma $N(T) \cap M$ is nilpotent, hence P is a normal subgroup of $N(T) \cap M$ consequently $P \neq N(P) \cap M$. Let C be the center of P, C is a normal subgroup of $N(P)$, therefore $P \neq N(C) \cap M$. As $N(C) \neq G$ Lemma 2.5 once again yields that $N(C) \cap M$ is nilpotent, hence the p-commutator subgroup of $N(C) \cap M$ is a proper subgroup of $N(C) \cap M$. If it can be established that $N(C) \cap M$ is p-normal, a result of Grün (**5**, Theorem 6, p. 141) will imply that $M \neq M'$, where M' is the commutator subgroup of M. As M is characteristically simple this yields that $M' = \{1\}$, hence M is abelian in contradiction with the assumption that M contains no normal subgroup of prime power order, and the Lemma is proved. We now proceed to show that M is p-normal.

Suppose $C \subset xPx^{-1}$ for some x, then $x^{-1}Cx \subset P$. As $N(C) \cap M \neq P$, there is a prime q different from p which divides the order of $N(C) \cap M$, let Q be a Sylow q-group of $N(C) \cap M$. As $N(C) \cap M$ is nilpotent and contains both P and Q they commute elementwise. Since $x^{-1}Cx \subset P$, Q commutes elementwise with $x^{-1}Cx$ and therefore is contained in $N(x^{-1}Cx)$. Since $x^{-1}Px$ and Q are contained in the nilpotent group $N(x^{-1}Cx) \cap M$ they also commute elementwise, hence P and $x^{-1}Px$ are both contained in $N(Q)$. As Q is characteristic in $N(C) \cap M$ it is normal in $N(C)$, hence h divides the order of $N(Q)$, as $N(Q) \neq G$, the assumptions of the Lemma yield that $N(Q) \cap M$ is nilpotent, consequently $P = x^{-1}Px$ as both P and $x^{-1}Px$ are Sylow p-groups of a nilpotent group. Therefore the center C of P is contained in no other Sylow p-groups of M, hence M is p-normal, which suffices to prove the Lemma.

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THEOREM. *Let M be the regular subgroup of a Frobenius group G. Suppose that no subgroup of M has an exceptional group as a composition factor, then M is nilpotent.*

Proof. Suppose that the theorem is false. Let M of order m be a non-nilpotent group of minimum order in which no subgroup has an exceptional group as a composition factor, and which can be represented as the regular subgroup of some Frobenius group G. Pick a prime r which divides the order of G but does not divide m and let R be a subgroup of G of order r, then $G_0 = RM$ is a Frobenius group of type (r,m) by Lemma 2.2. Suppose M has a non-trivial center C, then C is characteristic in M and therefore normal in G_0. It is clear that M/C satisfies the assumption of the theorem and has order less than m, hence M/C is nilpotent. It follows directly from the definition of a nilpotent group that this implies that M is nilpotent contradicting the choice of M. Thus the center of M is $\{1\}$.

As a first step in the proof it will be shown that $m = p^a q^b$, where p and q are primes and where the Sylow p-group of M is normal in M. The group M satisfies the assumption of Lemma 4.2, hence for some prime p dividing m, M contains a normal subgroup whose order is a power of p. Let P be a maximal normal subgroup of M whose order is a power of p, then P is characteristic in M and hence normal in G_0. By Lemma 2.3, G/P is a Frobenius group whose regular subgroup is M/P. It is clear that M/P satisfies the assumptions of the theorem and has order less than m, hence M/P is nilpotent. The Sylow p-group of M/P is a normal subgroup of M/P, hence its inverse image in M is normal in M and contains P, it follows from the way P was chosen that P is the Sylow p-group of M. Let $q_1, \ldots, q_s$ be the distinct primes, other than p, which divide m, let Q_i be a Sylow q_i-group of M. Since M/P is nilpotent, Q_iP/P is normal in M/P, hence Q_iP is normal in M, therefore Q_iP is characteristic in M and hence normal in G_0. Consequently, Lemma 2.2. implies that Q_iPR is a Frobenius group whose regular subgroup is Q_iP. If $s > 1$, $Q_iP \neq M$, hence Q_iP is nilpotent, therefore every element of P commutes with every element of Q_i. Since this is the case for all i, the center of P lies in the center of M, which leads to a contradiction since M has no non-trivial center. Therefore[6] $s = 1$ and $m = p^a q^b$.

As all the Sylow q-groups of G lie in M and are conjugate in M, the index of the normalizer $N(Q)$ of a Sylow q-group Q of G divides m, therefore r divides the order of $N(Q)$. Let R_0 be a subgroup of $N(Q)$ of order r, the group Q_0 consisting of all elements in the center of Q whose order divides q is a characteristic subgroup of Q, hence $R_0 \subset N(Q_0)$, therefore R_0Q_0 is a group and also R_0Q_0P is a group. By Lemma 2.2 this is a Frobenius group whose regular subgroup is Q_0P. If $Q_0 \neq Q$, then $Q_0P \neq M$, hence Q_0P is nilpotent, therefore both P and Q lie in $C(Q_0)$, thus Q_0 is in the center of M which is impossible,

[6] $s \leqslant 1$, hence either $s = 1$ or $s = 0$, in the latter case M is a p-group, which is impossible, hence only the case $s = 1$ needs to be considered.

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hence $Q = Q_0$. Let P_0 be the group consisting of all the elements in the center of P whose order divides p, then as before R_0QP_0 is a Frobenius group whose regular subgroup is QP_0, if $P_0 \neq P$, then $QP_0 \neq M$ and QP_0 is nilpotent, hence P_0 lies in the center of M which is impossible. Therefore $P = P_0$. Consequently the group G satisfies hypothesis I of section 3.

Pick an element x in R, then the mapping $\sigma(y) = xyx^{-1}$ defines an automorphism σ of M of prime order r with the property that $\sigma(y) \neq y$ for all $y \neq 1$ in M. We wish to show that M satisfies the hypothesis of Lemma 3.4. Suppose $Q_0 \neq \{1\}$ is a subgroup of Q such that $\sigma(Q_0) = Q_0$, then $R \subset N(Q_0)$, therefore RQ_0 is a group and by Lemma 2.2 RQ_0P is a Frobenius group whose regular subgroup is Q_0P. If $Q \neq Q_0$, then $Q_0P \neq M$, hence Q_0P is nilpotent, therefore Q_0 lies in the center of M (as Q is abelian) which is impossible, therefore $Q = Q_0$. Suppose $P_0 \neq \{1\}$ is subgroup of P which is normal in M such that $\sigma(P_0) = P_0$, then $R \subset N(P_0)$, therefore P_0 is normal in G, hence RQP_0 is a group and Lemma 2.2 can once again be applied to show that QP_0 is nilpotent if $P_0 \neq P$, this leads to the fact P_0 is contained in the center of M which cannot be the case and P_0 must equal P. Therefore M satisfies the assumption of Lemma 3.4, hence, that Lemma implies that the order of Q is q, since M was assumed to be non-abelian.

If any element x in P commutes with any element of order q, then $C(x)$ contains P and is divisible by q, therefore x lies in the center of M, hence $x = 1$. In other words, the order of every element of M divides either p^a or q, hence the order of every element in G divides either p^a or q or r, since no element of order r commutes with any element whose order is not r. Therefore every element of G has an order dividing p^a or qr and by Lemma 2.1, G is a Frobenius group of type (qr,p^a), consequently Lemma 2.6 implies that QR is a cyclic group. This is impossible since G contains no elements of order qr. The assumption that the Theorem is false has led to a contradiction and the proof is complete.

COROLLARY. *Let M be a group which admits a group of automorphisms A in which no automorphism other than the identity leaves any element of M other than the identity invariant. Furthermore assume that no subgroup of M has an exceptional group as a composition factor, then M is nilpotent.*

Proof. Let G be the group defined by extending M by A (**5**, pp. 94-98), then it is easily seen that G is a Frobenius group whose regular subgroup is M, hence Theorem 1 yields the desired result.

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Cornell University

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