ON A THEOREM OF SOBCZYK

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In this paper the result of Sobczyk about complemented copies of $c_0$ is extended to a class of Banach spaces $X$ such that the unit ball of their dual endowed with the weak* topology has a certain topological property satisfied by every Corson-compact space. By means of a simple example it is shown that if Corson-compact is replaced by Rosenthal-compact, this extension does not hold. This example gives an easy proof of a result of Phillips and an easy solution to a question of Sobczyk about the existence of a Banach space $E$, $c_0 \subset E \subset \ell_\infty$, such that $E$ is not complemented in $\ell_\infty$ and $c_0$ is not complemented in $E$. Assuming the continuum hypothesis, it is proved that there exists a Rosenthal-compact space $K$ such that $C(K)$ has no projectional resolution of the identity.

There are two results that play an important role in deciding whether a copy of $c_0$ is complemented in a Banach space. The first one, due to Sobczyk, asserts that $c_0$ is complemented in every separable Banach space and the second one, due to Phillips, that $c_0$ is not complemented in $\ell_\infty$. In this paper the result of Sobczyk is extended to a class of Banach spaces $X$ such that the unit ball $U^*$ of their dual, endowed with the weak* topology $w^*$, has a certain topological property. In particular, every $X$ such that $(U^*, w^*)$ is a Corson-compact space, belongs to this class. By means of a simple example, it is shown that if Corson-compact is replaced by Rosenthal-compact, this extension does not hold. Moreover that example gives an easy proof of the result of Phillips. In fact a Banach space $E$ is obtained such that $c_0 \subset E \subset \ell_\infty$; $E$ is not complemented in $\ell_\infty$ and $c_0$ is not complemented in $E$. This gives an easy solution to a problem raised in [10] by Sobczyk. At the end of the paper, assuming the continuum hypothesis, it is proved that there exists a Rosenthal-compact space $K$ such that $C(K)$ has no projectional resolution of the identity.

If $X$ is a Banach space, we denote its dual by $X^*$, by $U^*$ the unit ball of $X^*$ and by $\sigma(X^*, X)$ the weak* topology. Given a subset $A$ of $X^*$, $(A, w^*)$ stands for the...
topological space obtained when $A$ is endowed with the topology induced by $\sigma(X^*, X)$. The cardinal of a set $J$ is denoted by $|J|$. A compact topological space $K$ is said to be Rosenthal-compact if there is a Polish space $X$ such that $K$ is homeomorphic to a subspace of $(B_1(X), J_p)$ where $B_1(X)$ is the set of functions of the first Baire class on $X$ and $J_p$ the topology of the pointwise convergence [5].

**Definition 1:** Let $X$ be a topological space and $\{F_n\}$ a sequence of disjoint subsets of $X$. A point $x$ in $X$ is said to be cofinitely near to $\{F_n\}$ if for every neighbourhood $V$ of $x$, the set

$$\{n \in \mathbb{N}: V \cap F_n = \emptyset\}$$

is finite.

**Definition 2:** A topological space is called cofinitely sequential if for every $x_0$ cofinitely near to a disjoint sequence of closed $G_\delta$ sets $\{F_n\}$ there exists a sequence $\{y_n\}$ such that $y_n \in F_n$ and $\{y_n\}$ converges to $x_0$.

It is easily seen that every metric space is cofinitely sequential. Let $X$ be a Fréchet space (that is, each point in the closure of a set $A$ in $X$ is the limit of a convergent sequence of points in $A$ [4]); given a point $x$ cofinitely near to a disjoint sequence of sets $\{F_n\}$, there exists a subsequence $\{F_{n_k}\}$ and points $x_k \in F_{n_k}$ such that $\lim x_k = x$. Nevertheless a Fréchet compact space will be constructed that is not cofinitely sequential.

**Theorem 3.** If $c_0$ is a closed linear subspace of a Banach space $X$ such that $(U^*, \omega^*)$ is cofinitely sequential, then there is a continuous projection of $X$ onto $c_0$ with norm not greater than two.

**Proof:** Let $\psi: c_0 \to X$ be the inclusion mapping, that is, $\psi(x) = x$, $\forall x \in c_0$. If $\phi^*: X^* \to c_0^*$ is the conjugate mapping, let us take

$$F_n = \psi^{-1}(\{e_n^*\}) \cap V^*,$$

where $e_n^*$ is the $n$th coordinate functional and

$$V^* = \{x^* \in X^*: \|x^*\| \leq 2\}.$$

According to the Hahn-Banach theorem, no $F_n \cap U^*$ is empty. Moreover each $F_n$ is a closed $G_\delta$ set in $(V^*, \omega^*)$ since each $e_n^*$ is a closed $G_\delta$ in $(c_0^*, \omega^*)$. We will show that $0$, the null functional, is cofinitely near to $\{F_n\}$. Indeed, otherwise there would exist a neighbourhood $W$ of $0$ and an infinite sequence of natural numbers $\{n_k\}$ such that

$$W \cap F_{n_k} = \emptyset, \quad \forall k \in \mathbb{N};$$

then
Let us take

\[ y_n^* \in [U^* \cap \psi^{-1}\{e_{n_k}\} - W] \]

Since \((U^*, w^*)\) is compact there must exist an accumulation point \(y_0^*\) of \(\{y_n^*\}\) and it is easily seen that \(y_0^*(x) = 0, \forall x \in c_0\). Then

\[ (y_n^*) - y_0^* \in V^* \cap \psi^{-1}\{e_{n_k}\} \]

and 0 is an accumulation point of this sequence, which contradicts (1).

It is obvious that \((V^*, w^*)\) is homeomorphic to \((U^*, w^*)\), so \((V^*, w^*)\) is cofinitely sequential. Therefore there exists a sequence \(x_n^* \in F_n\) such that \(\{x_n^*\}\) converges to 0. Then \(P: X \to c_0, P(x) = \{x_n^*(x)\}\) is a continuous projection and \(\|P\| \leq 2\).

Since every metric space is cofinitely sequential, every separable Banach space satisfies the hypothesis of Theorem 3, so we obtain the result of Sobczyk [10, 12]. (In fact the proof of Theorem 3 is an adaptation of the proof of Veech of that result [12].) There are cofinitely sequential spaces that are not metrisable. In fact, we will see that every Corson-compact space is cofinitely sequential. Let us recall some notation and results. For any set \(I\), we denote by \(\sum(I)\) the subset of \([0, 1]^I\) consisting of functions \(x(i)\) which are zero except on a countable subset of \(I\). A compact space is said to be a Corson-compact space if it is homeomorphic to a subset of \(\sum(I)\) for some set \(I\).

**Lemma 4.** [1]. Let \(K\) be a compact subset of \([0, 1]^I\) such that \(K \cap \sum(I)\) is dense in \(K\) and \(J_0 \subset I\). There exists a subset \(J_1\) of \(I\), containing \(J_0\), such that \(|J_1| = |J_0|\) and \(R_{J_1}(K) \subset K\), where, if \(J\) is a subset of \(I\), \(R_J: [0, 1]^I \to [0, 1]^I\) is defined by declaring \(R_J(x)(i)\) to be \(x(i)\) if \(i \in J\) and 0 otherwise.

**Proposition 5.** Every Corson-compact space is cofinitely sequential.

**Proof:** Let \(\{F_n\}\) be a disjoint sequence of closed sets in a Corson-compact space \(K\) and \(y_0\) a cofinitely near point to \(\{F_n\}\). We can assume that \(K\) is included in \(\sum(I)\) for some set \(I\). If

\[ J_0 = \{i \in I: y_0(i) \neq 0\}, \]

Lemma 4 enables us to construct inductively a sequence \(\{J_n\}\) of countable subsets of \(I\) such that

(i) \(J_0 \subset J_n \subset J_{n+1}, \forall n \in \mathbb{N}\).

(ii) \(R_{J_n}(F_n) \subset F_n, \forall n \in \mathbb{N}\).

Let us take \(J = U\{J_n: n \in \mathbb{N}\}\). Since \(J\) is countable, \([0, 1]^J\) is metrisable, so there exists \(x_n \in F_n\) such that \(\{x_n(i)\}\) converges to \(y_0(i)\) for every \(i \in J\). Now it is enough...
to show that \( \{R_{J_n}(x_n)\} \) converges to \( y_0 \) since from (ii), \( R_{J_n}(x_n) \in F_n \). Indeed, let \( I_0 \)
be a finite subset of \( I \). Then

\[
(1) \quad y_0(i) = R_{J_n}(x_n)(i) = 0, \quad \forall i \in I_0 - J.
\]

Moreover, if \( i \in I_0 \cap J \) there exists an \( n_0 \) such that \( i \in J_n, n \geq n_0, \) so \( x_n(i) = R_{J_n}(x_n)(i) \) and

\[
(2) \quad y_0(i) = \lim x_n(i) = \lim R_{J_n}(x_n)(i).
\]

From (1) and (2), \( \lim R_{J_n}(x_n)(i) = y_0(i), \quad \forall i \in I_0. \)

**COROLLARY 6.** Let \( X \) be a Banach space that contains \( c_0 \). If \( (U^*, w^*) \) is a Corson-compact space then there is a continuous projection \( P \) of \( X \) onto \( c_0 \) with \( \|P\| \leq 2 \).

**REMARK 7.** In [11] a result is proved that improves Corollary 6 since its applications are not restricted to copies of \( c_0 \). A compact space \( K \) is said to be a Valdivia-compact space if there exists a set \( I \) such that \( K \) is homeomorphic to a closed subset \( F \) of \( [0, 1]^I \) such that \( F \cap \bigcup (I) \) is dense in \( F \) [3]. Then in [11] it is shown that if \( K \) is a Valdivia-compact space then every separable subspace of \( C(K) \) is contained in a complemented separable subspace \( S \) of \( C(K) \). In fact, there is a projection \( P \) from \( C(K) \) onto \( S \) with \( \|P\| \leq 1 \).

Therefore if \( X \) is a Banach space containing \( c_0 \), such that \( (U^*, w^*) \) is a Valdivia-compact space, there is a projection \( P \) of \( X \) onto \( c_0 \) with \( \|P\| \leq 2 \). Indeed it is enough to consider \( c_0 \subset X \subset C(U^*) \) and apply the previous observation and the result of Sobczyk.

If Corson-compact is changed for Rosenthal-compact in Corollary 6, the assertion becomes false; this and other facts will be derived from the following example.

**EXAMPLE 8.** Let us take \( L = \bigcup \{\{0, 1\}^\alpha : 0 \leq \alpha \leq \omega \} \). An element of \( L \) is a function whose domain is \( \alpha \) with \( 0 \leq \alpha \leq \omega \), where \( \omega \) is the first infinite ordinal. When \( \alpha = 0 \) there is exactly one element of \( \{0, 1\}^\alpha \), namely the empty mapping from \( 0 \) to \( \{0, 1\} \); we shall write \( 0 \) for this trivial object. We will define an order \( \leq \) on \( L \) (the usual order in the real numbers is denoted by \( \leq \)).

\[
[s \leq t] \leftrightarrow [\text{dom } s \leq \text{dom } t \text{ and } t|_{\text{dom } s} = s].
\]

We equip \( L \) with a topology (the order-topology) by declaring the element \( 0 \) to be an isolated point while taking basic neighbourhoods of points \( t \neq 0 \) to be intervals \( (s, t] \) with \( s < t \). Thus \( L \) is scattered and locally compact. Let \( K \) be the Alexandroff compactification of \( L \); \( K = L \cup \{\infty\} \).

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We will show that $K$ is a Rosenthal-compact space such that there exists a non-complemented copy of $c_0$ in $C(K)$. Let $E$ be the closed linear subspace of $C(K)$ spanned by $\{1_{\{t\}}: t \in L_0\}$, where $1_A$ stands for the indicator function of the set $A$ and $L_0 = L - \{0, 1\}^\omega$. Then if we write $L_0 = \{x_n: n \in \mathbb{N}\}$ it is easy to check that $\psi: c_0 \to E$, $\psi(\{t_n\}) = \sum t_n 1_{\{x_n\}}$ is a linear isometry. Moreover $E$ is not complemented in $C(K)$. Indeed, otherwise there would exist a projection $R: C(K) \to E$; then if $P = \psi^{-1} \circ R$, by considering $P^*(e_i^*)$ we would obtain measures $\{\mu_t: t \in L_0\}$ such that

1. $\langle 1_{\{t\}}, \mu_t \rangle = 1, \forall t \in L_0$.
2. $\langle 1_{\{t\}}, \mu_s \rangle = 0, \forall s, t \in L_0, s \neq t$.
3. $\{\mu_t: t \in L_0\}$ weak*-converges to zero.

Let us take

$$B_t = \{p \in K: \mu_t(\{p\}) \neq 0\}.$$

According to (i) and (ii) we have $B_t \cap L_0 = \{t\}, \forall t \in L_0$. Moreover each $B_t$ must be countable so $\bigcup \{B_t: t \in L_0\}$ is countable; then

4. $H = \{0, 1\}^\omega - (\bigcup \{B_t: t \in L_0\}) \neq \emptyset$.

Let $s_0$ be an element of $H$ and let us consider the clopen set

$$C = \{s \in L: s \leq s_0\}.$$

Since $s_0 \in H$, according to (iv) we have

$$\langle 1_C, \mu_t \rangle = 1, \forall t \in C \cap L_0,$$

which contradicts (iii).

In order to see that $K$ is a Rosenthal-compact space we will define a function $\varphi: K \to C(\Delta)$, where $\Delta$ stands for $\{0, 1\}^\omega$ endowed with the pointwise topology. If $\alpha \in L$ we take

$$\varphi(\alpha) = 1_{U(\alpha)}$$

where $U(\alpha) = \{\beta \in \{0, 1\}^\omega: \alpha \leq \beta\}$, and

$$\varphi(\infty) = 0,$$

the null function.

If $\alpha \in L_0$, $\varphi(\alpha)$ is the characteristic function of a clopen set so $\varphi(\alpha)$ is continuous in $\{0, 1\}^\omega$ and $\varphi(\alpha) = \lim \varphi(\alpha | n)$, for $\alpha \in \{0, 1\}^\omega$. Therefore every element of $\varphi(K)$ is a function of the first Baire class on $\Delta$. Moreover, if $\varphi(K)$ is endowed with the topology of the pointwise convergence it is easy to check that $\varphi$ is continuous. Since $\varphi$ is injective and $K$ is compact we have that $K$ is homeomorphic to $\varphi(K)$ which shows that $K$ is a Rosenthal-compact space.

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REMARK 9. This example gives a simple proof of the fact that $c_0$ is not complemented in $\ell_\infty$. Indeed, let us write $L_0 = \{x_n : n \in \mathbb{N}\}$. Since $L_0$ is dense in $K$ the mapping $\varphi : C(K) \to \ell_\infty$ defined by $\varphi(f) = \{f(x_n)\}$ is an isometric embedding. Moreover we have that $\varphi(E) = c_0$. Then $c_0$ is not complemented in $\ell_\infty$ since $E$ is not complemented in $C(K)$. (For other simple proofs see [8] and [13].) Since every $\ell_\infty$-valued continuous linear mapping defined in a subspace of a Banach space can be extended to a linear continuous mapping in the whole space, it is easy to deduce that there is no complemented copy of $c_0$ in $\ell_\infty$.

REMARK 10. In [10] p.945 it is asked if there exists a closed linear subspace $S$ of $\ell_\infty$, $c_0 \subset S$, such that there is no projection of $\ell_\infty$ onto $S$, and no projection of $S$ onto $c_0$. By means of the previous example it is easy to construct a subspace with these properties. Indeed, let $\varphi$ be the mapping defined in Remark 9 and $\varphi(C(K)) = S$; it has been shown that $c_0 \subset S$ and $c_0$ is not complemented in $S$, so we have only to show that $S$ is not complemented in $\ell_\infty$. By construction there are infinite convergent sequences in $K$ so $C(K)$ is not a Grothendieck space [6]. Since $\varphi$ is a linear isometry, $S$ is not a Grothendieck space, therefore $S$ is not complemented in $\ell_\infty$.

REMARK 11. If $K$ is the Rosenthal-compact space constructed in Example 8, we have that $(U^*, w^*)$, the unit ball of the dual of $C(K)$, is a Rosenthal-compact space [5], so it is a Fréchet topological space [2]. Therefore, according to Theorem 3, we have an example of a Fréchet compact space that is not cofinitely sequential. Moreover $C(K)$ is a Banach space with a non-complemented copy of $c_0$, such that the unit ball of its dual $(U^*, w^*)$ is a Rosenthal-compact space; this fact shows that if Corson-compact is changed for Rosenthal-compact in Corollary 6, the assertion becomes false.

Let us recall that a projectional resolution of identity on a Banach space $X$ is a set of projections $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ where $\mu$ is the first ordinal whose cardinality equals the density character $\text{d}ens(X)$ of $X$, which satisfies:

1. $\|P_\alpha\| = 1$, $\forall \alpha$.
2. $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\omega \leq \alpha \leq \beta \leq \mu$.
3. $\text{d}ens(P_\alpha(X)) \leq |\alpha|$, $\forall \alpha$.
4. $\bigcup\{P_{\beta+1}(X) : \beta < \alpha\}$ is dense in $P_\alpha(X)$.
5. $P_\mu = \text{Id}_X$.

THEOREM 12. Assuming the Continuum Hypothesis, there exists a Rosenthal-compact space such that there is no projectional resolution of the identity in $C(K)$.

PROOF: Let $K$ be the Rosenthal-compact space constructed in Example 8. We will suppose there is a projectional resolution of the identity on $C(K)$ and obtain a contradiction. The density character in $C(K)$ is $\omega$, the continuum. Therefore, assuming the continuum hypothesis, there exists a set of projections $\{P_\alpha : \omega \leq \alpha \leq \omega_1\}$ which
satisfies the above conditions. In Example 8 a non-complemented copy \( E \) of \( c_0 \) was obtained. Let \( \{e_n\} \) be a basis of this copy and \( \{f_n\} \) linear functionals on \( E \) satisfying

\[
x = \sum f_n(x_n)e_n, \forall x \in E.
\]

According to (iv) and (v) there exist \( \alpha_n, \omega \leq \alpha_n \leq \omega_1 \) and \( a_n \in P_{\alpha_n}(C(K)) \) such that

\[
\|e_n - a_n\| < 2^{-(n+1)} \|f_n\|^{-1}.
\]

So

\[
\sum \|f_n\| \|e_n - a_n\| < 1.
\]

Then \( \{a_n\} \) is a basis equivalent to \( \{e_n\} \) and its closed linear span \( [a_n] \) is not complemented in \( C(K) \) [7].

On the other hand, the supremum \( \alpha \) of the sequence \( \alpha_n \) must satisfy \( \omega \leq \alpha < \omega_1 \), so \( a_n \in P_{\alpha_n}(C(K)) \subseteq P_{\alpha}(C(K)) \) and \( [a_n] \subseteq P_{\alpha}(C(K)) \). Then \( [a_n] \) is a copy of \( c_0 \) in the separable space \( P_{\alpha}(C(K)) \) so \( [a_n] \) is complemented in \( P_{\alpha}(C(K)) \) [10]. Since \( P_{\alpha}(C(K)) \) is complemented in \( C(K) \), \( [a_n] \) must be complemented in \( C(K) \), a contradiction.

According to the result of Sobczyk, whenever \( c_0 \) is a closed linear subspace of a Banach space \( X \) with countable density character (that is, separable), \( c_0 \) is complemented in \( X \) [10]. On the other hand, \( c_0 \) is not complemented in \( \ell_\infty \) [9] and \( \text{dens}(\ell_\infty) = c \), the continuum. Then the following question arises: Which are the cardinal numbers \( \alpha \) such that \( c_0 \) is complemented in every Banach space \( X \), containing \( c_0 \) as a closed subspace and \( \text{dens}(X) = \alpha \)? According to the previous remarks such a cardinal \( \alpha \) satisfies \( \omega \leq \alpha < c \), but this does not give a complete answer unless we assume the continuum hypothesis. In the following example we will show that there exists no cardinal \( \alpha \) with this property such that \( \omega < \alpha < c \).

**Example 13.** Let \( \rho \) be a cardinal number such that \( \omega \leq \rho \leq c \). Let \( A \) be a subset of \( \{0, 1\}^\omega \), \( |A| = \rho \), and let us consider

\[
M = \bigcup \{D(\alpha) : 0 \leq \alpha < \omega\} \cup A, \text{ where } D(\alpha) = \{0, 1\}^\alpha.
\]

Thus \( M \) is included in \( L \) of Example 8, and we will consider \( M \) endowed with the subspace topology, for which \( M \) is scattered and locally compact. Let \( K_0 \) be the Alexandroff compactification of \( M \). Then \( \text{dens}(C(K_0)) = \rho \) and we can construct a non-complemented copy of \( c_0 \) in \( C(K_0) \). Indeed, a copy of \( c_0 \) is spanned by \( \{\delta_t : t \in L_0\} \) where \( L_0 \) was defined in Example 8. Reasoning as in Example 8, we obtain that this copy is not complemented.
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