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In [Kis08], the author investigated a maximal center Z appearing in the first elementary link of a factorization of Φ_θ, which is the Cremona transformation on $\mathbb{P}^3$ obtained from an automorphism θ on $\mathbb{C}^3$, by an application of the Sarkisov Program (see [Cor95, Cor00]). As a consequence, he asserted that Z had to be either a point or a line lying on the hyperplane at infinity, provided that θ is tame [Kis08, Theorem 1.1, p. 964]. Then, he made use of this theorem to obtain an alternative proof of the famous result due to Shestakov and Umirbaev [SU04a, SU04b] concerning non-tameness of the Nagata automorphism σ (see [Kis05, Kis08] for the explicit equation of σ), in combination with a birationally geometric argument to confirm that Z for Φ_σ is a smooth conic (see [Kis08, § 4]). However, the argument in [Kis08, Steps 1 and 2, p. 972] used to obtain [Kis08, Theorem 1.1] contains a crucial gap, which results from the usage of an ambiguous notion, namely that of *being isomorphic along a valuation*. As a result, [Kis08, Theorem 1.1] does not follow at this stage. In this erratum, we shall first correct the statement of [Kis08, Proposition 3.1, p. 968] and point out an inaccuracy in [Kis08, Lemma 3.2, p. 971]; then we discuss the most crucial gap in [Kis08, pp. 972–973] arising from the aforementioned ambiguous expression. The author is grateful to the editor for giving him the opportunity to write this erratum; he also thanks A. Dubouloz and S. Lamy for helpful suggestions.

In what follows, we use the same notation and conventions as in [Kis08]. We begin with [Kis08, Proposition 3.1, p. 968], from which one case is missing. The author stated there that *if* $\deg C_0 \geqslant 3$, *then* C_0 *has a singular point, so that the resulting* 3*-fold* Z *has a singular locus of dimension one*; but this is, in fact, false. For instance, in the case where C_0 is a cuspidal cubic (respectively, a cubic with an ordinary double point), the 3-fold $\mathrm{Bl}_{C_0}(\mathbb{P}^3)$ has an *isolated* Gorenstein terminal singularity whose analytic type is $o \in (xy - z^2 - t^3 = 0)$ (respectively, $o \in (xy - zt = 0)$). Indeed, [Kis08, Proposition 3.1] must be replaced by the following.

PROPOSITION 0.1. *Let $\theta \in G_3$ be an automorphism on the affine 3-space $\mathbb{C}^3$ (which is not necessarily tame), and let $\Phi_\theta : \mathbb{P}^3 \cdots \to \mathbb{P}^3$ be the Cremona transformation induced by θ in a natural way. Then the maximal center of the first elementary link of the Sarkisov factorization of Φ_θ is either a point, a line, a smooth conic or a singular cubic on the hyperplane at infinity.*

Proof. Unless the maximal center, say C_0, is a point, it must be an irreducible rational curve contained in the hyperplane H_∞ at infinity. In the case of $\deg C_0 \geqslant 4$, the first elementary link

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is obtained by the blow-up along C_0 followed by the contraction of the proper transform of H_∞ to a point which is no longer terminal. This is in contradiction to the mechanism of the Sarkisov Program (see [Cor95, Cor00]); hence we have $\deg C_0 \leqslant 3$ as desired. □

Next, we discuss the incorrect statement in [Kis08, Lemma 3.2, p. 971], where the author asserted that for any discrete valuation ν whose center $\mathrm{Center}_X(\nu)$ on X is either a point or a line on H_∞ or H_∞ itself, its center $\mathrm{Center}_{X'}(\nu)$ on X' must also be either a point or a line on H'_∞ or H'_∞ itself. However, there exists a gap in the proof of this assertion. More precisely, the eighth line of the proof contains the statement that *φ_α is extended to an automorphism on U*, but this is impossible unless the centers of $h : U \to V$ (using the notation in [Kis08, Lemma 3.2]) are invariant under application of φ_α. For instance, let us consider the following very simple example.

Example 0.1. Let $\alpha \in J_3$ be defined by $\alpha(x) = x + yz(y+z)$, $\alpha(y) = y$ and $\alpha(z) = z$. Let C be an irreducible curve of degree greater than or equal to two that is contained in H'_∞ on the target projective 3-space $X' = \mathbb{P}^3$, and let ν_C be the valuation corresponding to the exceptional divisor of the blow-up along C. Then it is easy to see that $\mathrm{Center}_X(\nu_C)$ is a point on H_∞ whereas $\mathrm{Center}_{X'}(\nu_C)$ equals C.

Meanwhile, the most crucial gap in the proof of [Kis08, Theorem 1.1] is found in [Kis08, Claims 1 and 2 in Steps 1 and 2, p. 972]; more precisely, it lies in the sentence *the (strong) maximal singularity of χ'_1, say ν, is extracted in a suitable procedure $\chi_k^{(j)}$ in $(***)$, and ν is also extracted in a suitable elementary transformation*. This gap results from the abuse of the ambiguous notion of *being isomorphic along ν*.

Example 0.2. The Cremona transformation Φ_α induced by $\alpha \in J_3$ in Example 0.1 has four maximal centers, namely the three lines $L_1 := (y = w = 0)$, $L_2 := (z = w = 0)$ and $L_3 := (y + z = w = 0)$ in H_∞ and the point $P := [1 : 0 : 0 : 0]$. However, we can construct a Sarkisov factorization of Φ_α, which starts with the blow-up at P and is simultaneously a factorization into elementary transformations (cf. [Fre95] and [Kis08, Remark 3.1]), where none of the strong maximal singularities corresponding to the lines L_i are extracted.

As this example indicates, when there are several possibilities for the maximal singularities of the first elementary link, there may exist a factorization into elementary transformations in which some of singularities are not extracted. Nonetheless, it seems reasonable to expect the existence of a suitable factorization of Φ_θ by elementary transformations where at least one of the maximal singularities of the first elementary link in the Sarkisov Program is extracted, provided that θ is tame. Once this conjecture is verified, we could obtain [Kis08, Theorem 1.1] with the condition 'for any Sarkisov factorization of Φ_θ' replaced by 'for a suitable Sarkisov factorization of Φ_θ', after preparing a supplementary lemma about $\mathbb{P}^2$-bundles that takes into consideration Proposition 0.1 in this erratum instead of the wrong statement [Kis08, Lemma 3.2]. At least, for σ (the Nagata automorphism), since Φ_σ has a unique maximal center which is a smooth conic on H_∞ (see [Kis08, Claim on p. 974]), the aforementioned attempt is enough to ascertain that σ is not tame. But, in any case, the argument in Steps 1 and 2 of [Kis08, p. 972] is incorrect, and hence the problem of finding a geometric alternative proof of Shestakov and Umirbaev's result remains open.

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