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The Alperin weight conjecture and Uno's conjecture for the Monster $\mathbb{M}$, p odd

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Abstract

Suppose that p is 3, 5, 7, 11 or 13. We classify the radical p-chains of the Monster $\mathbb{M}$ and verify the Alperin weight conjecture and Uno's reductive conjecture for $\mathbb{M}$, the latter being a refinement of Dade's reductive conjecture and the Isaacs–Navarro conjecture.

1. *Introduction*

Recently, Isaacs and Navarro [**20**] proposed a new conjecture which is a refinement of the Alperin–McKay conjecture, and Uno [**23**] raised an alternating sum version of the conjecture which is a refinement of the Dade conjecture [**17**].

Dade's reductive conjecture [**17**] has been verified for all of the sporadic simple groups except $\mathbb{B}$ with $p=2$ and $\mathbb{M}$. The use of computer algebra systems, namely Magma [**12**] and GAP [**18**], to study permutation (or, in some cases, matrix) representations of the groups has been a central step in the program. Since the smallest faithful permutation representation of $\mathbb{M}$ has degree 97 239 461 142 009 186 000, it is difficult to verify the conjecture directly. However, from the classification in [**24**] of maximal p-local subgroups of $\mathbb{M}$, we know that when $p=3, 5, 7, 11$ or 13, the normalizer of each radical p-subgroup of $\mathbb{M}$ is a subgroup of one of precisely 23 maximal p-local subgroups. Thus we can classify radical chains in these maximal subgroups without performing any calculation in $\mathbb{M}$.

In this paper, we classify radical subgroups and radical chains of $\mathbb{M}$, and hence verify the Alperin weight conjecture and Uno's refinement of Dade's reductive conjecture for $\mathbb{M}$.

Note that the radical p-subgroups of the Monster $\mathbb{M}$ were given in [**26**], but one radical 3-subgroup of $\mathbb{M}$ is missing and the normalizers of six radical 3-subgroups are incorrect (see Remark 4.2).

The paper is organized as follows. In Section 2, we fix notation, state the conjectures in detail and state three lemmas. In Section 3, we recall the modified local strategy [**8, 9**]; we also explain how we applied it to determine the radical subgroups of each maximal subgroup, as well as how to determine the fusion of the radical subgroups in $\mathbb{M}$. Using the explicit representations of the maximal subgroups of $\mathbb{M}$ given by Bray and Wilson [**13**], in Section 4 we classify radical p-subgroups of $\mathbb{M}$ and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Uno's conjecture, and then determine radical chains (up to conjugacy) and their local structures. In the final section, we verify Uno's ordinary conjecture for $\mathbb{M}$. Details on the degrees of irreducible characters of the normalizers of radical chains are summarized in tabular form in the Appendix.

2. *Conjectures and lemmas*

Let p be a prime and let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R))=R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R)=N_G(R)$.

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Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks. Let $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-*weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|N(R)/R|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin [1] conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \ldots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \ldots < P_k$ and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \ldots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$; and

(b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. Let $B \in \mathrm{Blk}(G)$ and let $D(B)$ be a defect group of B. The p-local rank (see [6]) of B is the number

$$p\mathrm{lr}(B) = \max\{|C| : C \in \mathcal{R},\ C : P_0 < P_1 < \ldots < P_n \leqslant D(B)\}.$$

Let E be an extension of G, and let $F = E/G$. For $C \in \mathcal{R}(G)$ and $\psi \in \mathrm{Irr}(N_G(C))$, let $N_E(C, \psi)$ be the stabilizer of (C, ψ) in E. Then $N_F(C, \psi) = N_E(C, \psi)/N_G(C)$ is a subgroup of F. For a subgroup $U \leqslant F$, denote by $\mathrm{Irr}(N_G(C), B, d, U)$ the set of characters ψ in $\mathrm{Irr}(N_G(C))$ such that $\mathrm{d}(\psi) = d$, $B(\psi)^G = B$ and $N_F(C, \psi) = U$. Set $\mathrm{k}(N_G(C), B, d, U) = |\mathrm{Irr}(N_G(C), B, d, U)|$. In the notation above, the Dade invariant conjecture is stated as follows.

Dade's invariant conjecture [17]. If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then for any integer $d \geqslant 0$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, U) = 0$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

If $E = G$, then $F = U = 1$ and we set $\mathrm{k}(N_G(C), B, d) = \mathrm{k}(N_G(C), B, d, U)$. The invariant conjecture is then called the ordinary conjecture.

Dade's ordinary conjecture [16]. If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then for any integer $d \geqslant 0$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0.$$

Let H be a subgroup of a finite group G and let $\varphi \in \mathrm{Irr}(H)$. The p-remainder $r(\varphi) = r_p(\varphi)$ of φ is the integer $0 < r(\varphi) \leqslant p - 1$ such that the p'-part $(|H|/\varphi(1))_{p'}$ of $|H|/\varphi(1)$ satisfies

$$\left(\frac{|H|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) (\mathrm{mod}\ p).$$

Given an integer r with $1 \leqslant r < (p+1)/2$, let $\mathrm{Irr}(H, [r])$ be the subset of $\mathrm{Irr}(H)$ consisting of characters φ such that $r(\varphi) \equiv \pm r (\mathrm{mod}\ p)$, and let $\mathrm{Irr}(H, B, d, U, [r]) = \mathrm{Irr}(H, B, d, U) \cap \mathrm{Irr}(H, [r])$ and $\mathrm{k}(H, B, d, U, [r]) = |\mathrm{Irr}(H, B, d, U, [r])|$.

Let $B \in \mathrm{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \mathrm{Blk}(N_G(D))$. Then

$$\mathrm{k}(N_G(D), B, \mathrm{d}(B), [r]) = \sum_{U \leqslant F} \mathrm{k}(N_G(D), B, \mathrm{d}(B), U, [r])$$

is the number of characters $\varphi \in \operatorname{Irr}(b)$ such that φ has height 0 and $r(\varphi) \equiv \pm r(\operatorname{mod} p)$, where $\mathrm{d}(B)$ is the defect of B.

Isaacs–Navarro conjecture [20, Conjecture B]. In the notation above,

$$\mathrm{k}(G, B, \mathrm{d}(B), [r]) = \mathrm{k}(N_G(D), B, \mathrm{d}(B), [r]).$$

The following refinement of Dade's conjecture is due to Uno.

Uno's invariant conjecture [23, Conjecture 3.2]. If $O_p(G) = 1$ and $D(B) > 1$, then for any integers $d \geqslant 0$ and $1 \leqslant r < (p+1)/2$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_G(C), B, d, U, [r]) = 0. \tag{2.3}$$

If $E = G$, then $F = U = 1$ and we set $\mathrm{k}(N_G(C), B, d, [r]) = \mathrm{k}(N_G(C), B, d, U, [r])$. The invariant conjecture is then called the ordinary conjecture.

Note that if $p = 2$ or 3, then Uno's conjectures are equivalent to Dade's conjectures.

Let G be the Monster $\mathbb{M}$; then its Schur multiplier and outer automorphism group are trivial, so Dade's ordinary conjecture is equivalent to his reductive conjecture (and Uno's ordinary conjecture is also equivalent to his reductive conjecture). Thus it suffices to verify Uno's ordinary conjecture for $\mathbb{M}$.

The proofs of the following two lemmas are straightforward.

Lemma 2.1. *Let*

$$\sigma : O_p(G) < P_1 < \ldots < P_{m-1} < Q = P_m < P_{m+1} < \ldots < P_\ell$$

be a fixed radical p-chain of a finite group G, where $1 \leqslant m < \ell$. Suppose that

$$\sigma' : O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell-1)$th subchain $C_{\ell-1}$ is conjugate to σ' in G, and let $\mathcal{R}^0(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓth subchain C_ℓ is conjugate to σ in G. Then the map g sending any

$$O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell < \ldots$$

in $\mathcal{R}^-(\sigma, Q)$ to

$$O_p(G) < P_1 < \ldots < P_{m-1} < Q < P_{m+1} < \ldots < P_\ell < \ldots$$

induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$ and $N_G(C) = N_G(g(C))$.

Lemma 2.2. *Suppose that Q is a p-subgroup of G. Then Q is radical in G if and only if $N_G(Q) \leqslant M$ and Q is radical in M for some maximal p-local subgroup M of G. In particular, if $N_G(Q) \leqslant M$, then Q is radical in G if and only if Q is radical in M.*

The next lemma follows from [11, Lemma 7.1].

Lemma 2.3. *Let G be a finite group, and take $B \in \operatorname{Blk}(G)$ with $\operatorname{plr}(B) = 2$ and abelian defect group $D = D(B)$. Let $O_p(G) \neq R < D$ be radical, and let $b \in \operatorname{Blk}(N_G(R))$ with $b^G = B$. Then*

$$\mathrm{k}(N_G(R) \cap N_G(D), b, d, [r]) = \mathrm{k}(N_G(R), b, d, [r]).$$

LEMMA 2.4. *If Q is a p-subgroup of a finite group G, then there is a radical p-subgroup R such that*

$$Q \leqslant R \quad \text{and} \quad N_G(Q) \leqslant N_G(R).$$

Proof. This follows from [**6**, Lemma 2.1]. □

3. *A local subgroup strategy and fusions*

From [**24**], we know that each radical p-subgroup R of $\mathbb{M}$ is radical in one of the conjugates M of maximal p-local subgroups constructed in [**13**] and that, further, $N_{\mathbb{M}}(R) = N_M(R)$.

In [**8**] and [**9**], a (modified) local strategy was developed to classify the radical p-subgroups R. We review this method here. Suppose that M is a subgroup of a finite group G satisfying $N_G(R) = N_M(R)$.

Step 1. We first consider the case where M is p-local. Let $Q = O_p(M)$, so that $Q \leqslant R$. Choose a subgroup X of M. We explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X we have $K = Q$, and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W (or apply Step 2 below to W), and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step 2. Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a (maximal) subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step 1 to K. If K is not p-local, we can replace M by K and repeat Step 2.

Steps 1 and 2 constitute the modified local strategy. After applying the strategy, we list the radical subgroups of each M and do the fusions as follows.

Suppose that R is a radical p-subgroup of M. Using the local structure, we can determine whether or not $N_M(R)$ is a subgroup of another maximal subgroup Y. Suppose that $N_M(R)$ is a subgroup of Y. By Lemma 2.4, there is a radical subgroup P of Y such that $R \leqslant P$ and $N_M(R) \leqslant N_Y(P)$. Using local structure, we can determine whether or not R is radical in Y, and if so, we can identify R with a radical subgroup P of Y. Some details are given in the proof of Proposition 4.1.

The computations reported in this paper were carried out using MAGMA V2.11-1 on a Sun UltraSPARC Enterprise 4000 server.

4. *Radical subgroups and weights*

Let $\mathcal{R}_0(G,p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leqslant G$, we write $H \leqslant_G K$ if $x^{-1}Hx \leqslant K$ and $H \in_G \mathcal{R}_0(G,p)$ if $x^{-1}Hx \in \mathcal{R}_0(G,p)$ for some $x \in G$.

Let G be the Monster $\mathbb{M}$. Then

$$|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

and we may suppose $p \in \{2, 3, 5, 7, 11, 13\}$, since both conjectures hold for a block with a cyclic defect group by [**16**, Theorem 9.1] and [**7**, Theorem 5.2]. Suppose that p is odd, so that $p = 3, 5, 7, 11$ or 13.

Denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|_p)$. Given $R \in \mathcal{R}_0(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$.

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If $B_0 = B_0(G)$ is the principal p-block of G, then (cf. [8, (4.1)])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\mathcal{R}_0(G,p)$ such that $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$.

PROPOSITION 4.1. *The non-trivial radical p-subgroups R of $G = \mathbb{M}$ (up to conjugacy) and their local structures are given in Tables 1 and 2 according to whether $p \geqslant 5$ or $p = 3$, where $S \in \mathrm{Syl}_3(G)$ is a Sylow 3-subgroup of G. Here H^* denotes a subgroup of G such that $H^* \cong H$ but $H^* \neq_G H$.*

Proof. The maximal p-local subgroups of G were constructed by Bray and Wilson [13].

Case 1. Suppose $p = 11$ or 13. Then the radical p-subgroups of G are given in [24, Section 11].

Suppose $p = 7$. By [24, Theorem 7], G has the following five maximal 7-local subgroups:

$$\begin{aligned} M_1 &= N(7A) \cong (7{:}3 \times \mathrm{He}){:}2, \\ M_2 &= N(7A^2) \cong (7^2{:}(3 \times 2A_4) \times L_2(7)){:}2, \\ M_3 &= N(7B) \cong 7_+^{1+4}{:}(3 \times 2S_7), \\ M_4 &= N(7B^2) \cong 7^{2+1+2}{:}\,\mathrm{GL}_2(7) \end{aligned}$$

TABLE 1. *Non-trivial radical p-subgroups of $\mathbb{M}$ with $p \geqslant 5$.*

R	$C(R)$	N	$\|\mathrm{Irr}^0(N/C(R)R)\|$
13	$13 \times L_3(3)$	$(13{:}6 \times L_3(3)).2$	
13^2	13^2	$13^2{:}4.L_2(13).2$	4
13_+^{1+2}	13	$13_+^{1+2}{:}(3 \times 4.S_4)$	48
11	$11 \times M_{12}$	$(11{:}5 \times M_{12}){:}2$	
11^2	11^2	$11^2{:}(5 \times 2A_5)$	45
7	$7 \times \mathrm{He}$	$(7{:}3 \times \mathrm{He}){:}2$	
7^2	$7^2 \times L_2(7)$	$(7^2{:}(3 \times 2A_4) \times L_2(7)){:}2$	
$(7^2)^*$	7^2	$7^2{:}\mathrm{SL}_2(7)$	1
7_+^{1+4}	7	$7_+^{1+4}{:}(3 \times 2S_7)$	27
7^{2+1+2}	7^2	$7^{2+1+2}{:}\mathrm{GL}_2(7)$	6
$7_+^{1+4}.7$	7	$7_+^{1+4}.7.6^2$	36
5	$5 \times \mathrm{HN}$	$(D_{10} \times \mathrm{HN}).2$	
5^2	$5^2 \times U_3(5)$	$(5^2{:}4.2^2 \times U_3(5)){:}S_3$	
5^4	5^4	$5^4{:}(3 \times 2.L_2(25)){:}2$	3
5^{3+3}	5^3	$5^{3+3}.(2 \times L_3(5))$	2
5_+^{1+6}	5	$5_+^{1+6}.2J_2{:}4$	18
$5_+^{1+6}.5$	5	$5_+^{1+6}.5.(4 \times 2S_5)$	8
5^{2+2+4}	5^2	$5^{2+2+4}.(S_3 \times \mathrm{GL}_2(5))$	12
$5_+^{1+6}.5^2$	5	$5_+^{1+6}.5^2.(S_3 \times 4^2)$	48

and

$$M_5 = N(7B^2) = 7^2{:}\,\mathrm{SL}_2(7).$$

The radical 7-subgroups of He are given by [**2**, Proposition (2A)], so that

$$\mathcal{R}_0(M_1, 7) = \{7, 7^2, 7^3, 7 \times 7_+^{1+2}\}.$$

In addition, $C_G(7^2) \cong 7^2 \times L_3(2)$, $C_G(7^3) \cong 7^3$, $C_G(7 \times 7_+^{1+2}) = 7^2$ and

$$N_{M_1}(R) = \begin{cases} 7^2.(6 \times 3) \times L_3(2) & \text{if } R = 7^2, \\ 7^3{:}(3 \times 2.L_2(7).2) & \text{if } R = 7^3, \\ (7 \times 7_+^{1+2}).(6 \times 3 \times S_3) & \text{if } R = 7 \times 7_+^{1+2}. \end{cases} \tag{4.2}$$

By [**24**, p. 14], 7^2 is $7A$-pure, so that $N_{M_1}(7^2) \leqslant_G M_2$ and $N_{M_1}(7^2) \neq N_G(7^2)$. As shown in the proof of [**24**, Theorem 7, p. 14], 7^3 contains a $7B$-element, and $N_G(7^3) \leqslant_G M_3$. But then $O_7(M_3) \leqslant O_7(N_G(7^3))$, so 7^3 is non-radical in G. Since $(7 \times 7_+^{1+2}) \in \mathrm{Syl}_7(M_1)$ but $(7 \times 7_+^{1+2}) \notin \mathrm{Syl}_7(G)$, it follows that $N_G(7 \times 7_+^{1+2}) \nleqslant_G M_1$.

If $M = M_i$ with $i > 1$, then $|M/O_7(M)|_7 = 7$, so that a Sylow subgroup of M is its only radical 7-subgroup other than $O_7(M)$. Thus

$$\mathcal{R}_0(M_i, 7) = \begin{cases} \{7^2, 7^3\} & \text{if } i = 2, \\ \{7_+^{1+4}, 7_+^{1+4}.7\} & \text{if } i = 3, \\ \{7^{2+1+2}, 7_+^{1+4}.7\} & \text{if } i = 4, \\ \{7^2, 7_+^{1+2}\} & \text{if } i = 5. \end{cases} \tag{4.3}$$

In addition, $C(7_+^{1+2}) = C_{M_5}(7_+^{1+2}) = 7$ and

$$N_{M_i}(R) = \begin{cases} 7^3{:}(3^2 \times 2A_4){:}2 & \text{if } i = 2 \text{ and } R = 7^3, \\ 7_+^{1+4}.7.6^2 & \text{if } i = 3 \text{ or } 4 \text{ and } R = 7_+^{1+4}.7, \\ 7_+^{1+2}.6 & \text{if } i = 5 \text{ and } R = 7_+^{1+2}. \end{cases} \tag{4.4}$$

TABLE 2. *Non-trivial radical 3-subgroups of* $\mathbb{M}$.

R	$C(R)$	$N(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
3	$3\,\mathrm{Fi}_{24}'$	$3\,\mathrm{Fi}_{24}$	
3^*	$3 \times \mathrm{Th}$	$S_3 \times \mathrm{Th}$	
3^2	$3^2 \times O_8^+(3)$	$(3^2{:}2 \times O_8^+(3)).S_4$	
3_+^{1+2}	$3 \times G_2(3)$	$(3_+^{1+2}{:}2^2 \times G_2(3)){:}2$	
3^8	3^8	$3^8.O_8^-(3).2$	2
3_+^{1+12}	3	$3_+^{1+12}.2\,\mathrm{Suz}.2$	1
$3_+^{1+12}.3$	3	$3_+^{1+12}.3.2U_4(3).2^2$	4
$3_+^{1+12}.3^2$	3	$3_+^{1+12}.3^2.2.(A_6 \times 8).2$	7
3^{2+5+10}	3^2	$3^{2+5+10}.(M_{11} \times 2S_4)$	2
$3^{3+2+6+6}$	3^3	$3^{3+2+6+6}.(L_3(3) \times SD_{16})$	7
$3_+^{1+12}.3^5$	3	$3_+^{1+12}.3^5(2^2 \times M_{11})$	4
$3_+^{1+12}.3^{2+4}$	3	$3_+^{1+12}.3^{2+4}.(SD_{16} \times 2S_4)$	14
$3^{2+5+10}.3^2$	3^2	$3^{2+5+10}.3^2.(2S_4 \times SD_{16})$	14
S	3	$S.(SD_{16} \times 2^2)$	28

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Case 2. Suppose $p=5$. By [**24**, Theorem 5], $\mathbb{M}$ has six maximal 5-local subgroups as follows:

$$\begin{aligned} M_1 &= N(5A) \cong (D_{10} \times \mathrm{HN}).2, \\ M_2 &= N(5A^2) \cong (5^2{:}4.2^2 \times U_3(5)){:}S_3, \\ M_3 &= N(5B) \cong 5_+^{1+6}.2.J_2{:}4, \\ M_4 &= N(5B^2) = 5^2.5^2.5^4{:}(S_3 \times \mathrm{GL}_2(5)), \\ M_5 &= N(5B^3) = 5^{3+3}.(2 \times L_3(5)) \end{aligned}$$

and

$$M_6 = N(5A^4) = 5^4{:}(3 \times 2.L_2(25)){:}2.$$

Let $L = U_3(5)$, $\mathrm{GL}_2(5)$ or $L_2(25)$. Then Sylow 5-subgroups of L are the only radical subgroups, so

$$\mathcal{R}_0(M_i, 5) = \begin{cases} \{5^2, 5^2 \times 5_+^{1+2}\} & \text{if } i=2, \\ \{5^{2+2+4}, 5_+^{1+6}.5^2\} & \text{if } i=4, \\ \{5^4, 5^4.5^2\} & \text{if } i=6. \end{cases} \tag{4.5}$$

In addition, $C(5^2 \times 5_+^{1+2}) = 5^3$, $C(5^4.5^2) = 5^2$ and

$$N_{M_i}(R) = \begin{cases} (5^2 \times 5_+^{1+2}).(4.2^2 \times 8){:}S_3 & \text{if } i=2 \text{ and } R = 5^2 \times 5_+^{1+2}, \\ 5_+^{1+6}.5^2.(S_3 \times 4^2) & \text{if } i=4 \text{ and } R = 5^{1+6}.5^2, \\ 5^4.5^2.4.3^2.2^2 & \text{if } i=6 \text{ and } R = 5^4.5^2. \end{cases}$$

We may take

$$\mathcal{R}_0(M_3, 5) = \{5_+^{1+6}, 5_+^{1+6}.5, 5_+^{1+6}.5^2\}, \tag{4.6}$$

and so $C(R) = Z(R) = 5 = 5B$, $N_G(R) \leqslant M_3$ and $R \in_G \mathcal{R}_0(G,5)$ for each $R \in \mathcal{R}_0(M_3, 5)$.

The radical subgroups of HN are given by [**10**, Proposition 4.1], so that

$$\mathcal{R}_0(M_1, 5) = \{5, 5^2, 5 \times 5^2.5_+^{1+2}, 5 \times 5_+^{1+4}, 5 \times 5_+^{1+4}.5\}. \tag{4.7}$$

In addition,

$$C(5^2) = 5^2 \times U_3(5), \quad C(5 \times 5^2.5_+^{1+2}) = 5^3, \quad C(5 \times 5_+^{1+4}) = 5^2 = C(5 \times 5_+^{1+4}.5)$$

and

$$N_{M_1}(R) = \begin{cases} (D_{10} \times (D_{10} \times U_3(5).2).2 & \text{if } R = 5^2, \\ (D_{10} \times 5^2.5_+^{1+2}.4A_5).2 & \text{if } R = 5 \times 5^2.5_+^{1+2}, \\ (D_{10} \times 5_+^{1+4}.2_-^{1+4}.5.4).2 & \text{if } R = 5 \times 5_+^{1+4}, \\ (D_{10} \times 5_+^{1+4}.5.(2 \times 4)).2 & \text{if } R = 5 \times 5_+^{1+4}.5. \end{cases}$$

The fusions of elements of order 5 in HN are given in [**24**, p. 12]. Thus 5^2 is $5A$-pure, so that we may suppose $N_{M_1}(5^2) \leqslant M_2 = N(5A^2)$ and $N_{M_1}(5^2) \neq N(5^2)$.

If $R = 5 \times 5_+^{1+4}$ or $5 \times 5_+^{1+4}.5$, then the commutator subgroup $R' = [R,R] = 5$ is $5B$-pure, and so we may suppose $N_G(R) \leqslant M_3 = N(5B)$. By Lemma 2.2 and (4.6), $R \notin_G \mathcal{R}_0(G,5)$.

If $R = 5 \times 5^2.5_+^{1+2}$, then $[R, R'] = 5^2$ is $5B$-pure, so we may suppose $N_G(R) \leqslant M_4 = N(5B^2)$, and by Lemma 2.2 and (4.5) we have $R \notin_G \mathcal{R}_0(G,5)$. It follows that each $R \in \mathcal{R}_0(M_1,5) \backslash \{5, 5^2\}$ is non-radical in G and that $N_{M_1}(5^2) \neq N(5^2)$.

We may take

$$\mathcal{R}_0(M_5, 5) = \{5^{3+3}, 5^2.5^2.5^4, 5^{3+3}.5^2, 5^{3+3}.5_+^{1+2}\} \tag{4.8}$$

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and $C(5^2.5^2.5^4)=5^2$, $C(5^{3+3}.5^2)=C(5^{3+3}.5_+^{1+2})=5$ and

$$N_{M_5}(R)=\begin{cases}5^2.5^2.5^4.(2\times \mathrm{GL}_2(5)) & \text{if } R=5^2.5^2.5^4,\\ 5^{3+3}.5^2.(2\times \mathrm{GL}_2(5)) & \text{if } R=5^{3+3}.5^2,\\ 5^{3+3}.5_+^{1+2}.(2\times 4^2) & \text{if } R=5^{3+3}.5_+^{1+2}.\end{cases}$$

Since $C(R)$ is $5B$-pure, it follows from (4.5) and (4.6) that

$$N_{M_5}(R)\neq N_G(R)\quad \text{for } R\in \mathcal{R}_0(M_5,5)\backslash\{5^{3+3}\},$$

and this classifies the radical 5-subgroups of G.

Case 3. Suppose $p=3$. By [**24**, Theorem 3], $\mathbb{M}$ has seven maximal 3-local subgroups, namely

$$\begin{aligned}M_1&=N(3A)\cong 3.\mathrm{Fi}_{24},\\ M_2&=N(3A^2)\cong(3^2{:}2\times O_8^+(3)).S_4,\\ M_3&=N(3B)=3_+^{1+12}.2.\mathrm{Suz}{:}2,\\ M_4&=N(3B^2)=3^{2+5+10}{:}(M_{11}\times 2S_4),\\ M_5&=N(3B^3)=3^{3+2+6+6}.(SD_{16}\times L_3(2)),\\ M_6&=N(3^8)=3^8.O_8^-(3).2\end{aligned}$$

and

$$M_7=N(3C)=S_3\times \mathrm{Th},$$

where SD_{16} is a semidihedral group of order 16. We first classify radical subgroups of each M_i with $i\neq 7$ by applying modified local strategy, and then do the fusions in $\mathbb{M}$ using Lemmas 2.4 and 2.2.

Case 3.1. Let $M=M_3=3_+^{1+12}.2.\mathrm{Suz}.2$ and $R\in\mathcal{R}_0(M_3,3)$. Then $O_3(M)=3_+^{1+12}\leqslant R$ and $R/O_3(M)$ is a radical subgroup of 2.Suz. The group 2.Suz has a faithful permutation representation of degree 65 520. We may take

$$\mathcal{R}_0(2.\mathrm{Suz},3)=\{1,3,3^2,3^5,3^{2+4},3^{2+4}.3\},$$

and so

$$\mathcal{R}_0(M_3,3)=\{3^{1+12},3^{1+12}.3,3^{1+12}.3^2,3^{1+12}.3^5,3^{1+12}.3^{2+4},S\}. \tag{4.9}$$

Since $C_{M_3}(R)=3=Z(R)$ for each $R\in\mathcal{R}_0(M_3,3)$, it follows that $N_G(R)\leqslant_G M_3$, and so R is radical in G with $N_{M_3}(R)=N_G(R)$ for each $R\in\mathcal{R}_0(M_3,3)$. Thus we may suppose $\mathcal{R}_0(M_3,3)\subseteq\mathcal{R}_0(G,3)$.

Case 3.2. Applying local strategy, we get four classes of radical subgroups of M_4; one of them, R, has order 3^{18} and satisfies $C_{M_4}(R)=Z(R)=3$ and $N_{M_4}(R)=R.(2^2\times M_{11})$. Thus a generator of $Z(R)$ is a $3B$-element as $Z(O_3(M_4))$ is $3B$-pure, and we may suppose $N_G(R)\leqslant M_3$. By Lemma 2.4 and (4.9), R is radical in G and, by the local structures, $R=_G 3^{1+12}.3^5$.

Another radical subgroup Q of M_4 has order 3^{19} and satisfies $C_{M_4}(Q)=3^2$. So $N_G(Q)\leqslant_G M_4$ and Q is radical in G. We may take

$$\mathcal{R}_0(M_4,3)=\{3^{2+5+10},3^{1+12}.3^5,3^{2+5+10}.3^2,S\}, \tag{4.10}$$

and then $N_G(R)=N_{M_4}(R)$ for each $R\in\mathcal{R}_0(M_4,3)$, so we may suppose $\mathcal{R}_0(M_4,3)\subseteq\mathcal{R}_0(G,3)$.

Case 3.3. There are four classes of radical subgroups of M_5; one of them, R, has order 3^{19} with $C_{M_5}(R) = Z(R) = 3$ and $N_{M_5}(R) = R.(2S_4 \times SD_{16})$. Thus $Z(R)$ is $3B$-pure and we may suppose $N_{M_5}(R) \leqslant M_3$. By (4.9) and Lemma 2.2, R is radical in G such that $R =_G 3^{1+12}.3^{2+4}$.

Another radical subgroup Q of M_5 also has order 3^{19} and satisfies $C_{M_3}(Q) = Z(Q) = 3^2$, so that $Z(Q)$ is $3B$-pure. Since $|C_G(Z(Q))|_3 \geqslant 3^{19}$, it follows from [**24**, Proposition 5.1] that $N_G(Z(Q)) \leqslant_G M_4$. By Lemma 2.4 and (4.10), R is radical in G and $R =_G 3^{2+5+10}.3^2$.

We may take

$$\mathcal{R}_0(M_5, 3) = \{3^{3+2+6+6}, 3^{1+12}.3^{2+4}, 3^{2+5+10}.3^2, S\}, \tag{4.11}$$

and then $N_G(R) = N_{M_5}(R)$ for $R \in \mathcal{R}_0(M_5, 3)$. Thus we may suppose $\mathcal{R}_0(M_5, 3) \subseteq \mathcal{R}_0(G, 3)$.

Case 3.4. The radical subgroups of Fi_{24}' are given by [**5**, Proposition 4.1], so that the radical subgroups of M_1 and their local structures are as listed in Table 3.

The fusions in G of elements of order 3 of Fi_{24}' are given in [**24**, p. 3]. Thus 3^2 is $3A$-pure and $N_G(3^2) =_G M_2 = N(3A^2)$, so that $N_{M_1}(3^2) \neq N_G(3^2)$.

Since $Z(3_+^{1+2}) = 3$ is generated by a $3A$-element, we have $N_G(3_+^{1+2}) \leqslant M_1$ and so 3_+^{1+2} is radical in G with $N_G(3_+^{1+2}) = N_{M_1}(3_+^{1+2})$.

By [**24**, Proposition 2.2], 3^8 contains a $3B$-element, so we may suppose $3^8 \leqslant M_3$. As shown in the proof on [**24**, p. 6], 3^8 contains a subgroup of type $3A_2B_2$, so that $N_{M_1}(3^8) \leqslant M_6$ and $N_{M_1}(3^8) \neq N_G(3^8)$.

If $R = 3^{2+10}$, then $R' = 3$ is $3B$-pure, so that $N_G(R) \leqslant_G M_3$ and, by Lemma 2.2 and (4.9), R is non-radical in G.

Similarly, if $R \in \{3^{2+5+5}, 3 \times 3_+^{1+10}.3^4, 3 \times 3_+^{1+10}.(3 \times 3_+^{1+2}), 3.3^2.3^4.3^8.3^2\}$, then $Z(R) = Z(3^{2+10}) = 3^2$ and we may suppose $N_G(R) \leqslant N_G(3^{2+10}) \leqslant M_3$. By Lemma 2.2 and (4.9), R is non-radical in G.

If $R = 3^{4+4+3+3} = 3.3^3.[3^{10}]$, then $Q = [R, [R, R']] = 3^3$ is $3B$-pure and $Q \leqslant Z(R)$. Since $|C_G(3B_4(\mathrm{ii}))|_3 = |C_G(3B_4(\mathrm{iii}))|_3 = 3^{13}$, it follows from $|R| = 3^{14}$ that each subgroup of order 9 in $[R, [R, R']]$ is of type $3B_4(\mathrm{i})$, where the types $3B_4(\mathrm{i}), 3B_4(\mathrm{ii})$ and $3B_4(\mathrm{iii})$ of elementary subgroups of order 9 are as defined in [**24**, Proposition 5.1]. As shown in the proof of [**24**, Theorem 6.5], we may suppose $N_G(Q) = N_G(3B^3) = M_5$, so that $N_G(R) \leqslant M_5$ and, by (4.11), $N_G(R) \neq N_{M_1}(R)$.

TABLE 3. *Radical* 3*-subgroups of* $3.\mathrm{Fi}_{24}$.

R	$C(R)$	$N_{M_1}(R)$
3	$3.\mathrm{Fi}_{24}'$	$3.\mathrm{Fi}_{24}$
3^2	$3^2 \times O_8^+(3)$	$(3^2{:}2 \times O_8^+(3)){:}S_3$
3_+^{1+2}	$3 \times G_2(3)$	$(3_+^{1+2}{:}2^2 \times G_2(3)){:}2$
3^8	3^8	$3^8.O_7(3){:}2$
3^{2+10}	3^2	$3^{2+10}.(2 \times U_5(2){:}2)$
3^{2+5+5}	3^2	$3^{2+5+5}.(2 \times U_4(2)){:}2$
$3^{4+4+3+3}$	3^4	$3^{4+4+3+3}.(2^2 \times L_3(3))$
$3.3^2.2^4.3^8$	3^3	$3.3^2.2^4.3^8.(S_5 \times 2S_4)$
$3 \times 3_+^{1+10}.3^4$	3^2	$3 \times 3_+^{1+10}.3^4.(2^2 \times S_5)$
$3 \times 3_+^{1+10}.(3 \times 3_+^{1+2})$	3^2	$3 \times 3_+^{1+10}.(3 \times 3_+^{1+2}).(2^2 \times 2S_4)$
$3.3^2.3^4.3^8.3$	3^3	$3.3^2.3^4.3^8.3.(2^2 \times 2S_4)$
$3.3^2.3^4.3^8.3^2$	3^2	$3.3^2.3^4.3^8.3^2.2^4$

If $R = 3.3^2.3^4.3^8$, then $[R, R'] = 3^2$ is $3B$-pure. Since $|C_{M_1}([R, R'])|_3 = 3^{16}$, it follows that $[R, R']$ is of type $3B_4$(i) and so $N_G(R) \leqslant_G M_4$. By (4.10), R is non-radical in G.

If $R = 3.3^2.3^4.3^8.3$, then $Z(R) =_G Z(3.3^2.3^4.3^8)$ and we may suppose $N_G(R) \leqslant M_4$. By (4.10), R is non-radical in G.

Case 3.5. Suppose R is a radical subgroup of $M_2 = (3^2{:}2 \times O_8^+(3)).S_4$ with $R \neq O_3(M_2)$. If $H = 3^2{:}2 \times O_8^+(3)$, then $Q = R \cap H$ is a radical subgroup of H, so that $Q = 3^2 \times Q_1$ for some radical subgroup Q_1 of $O_8^+(3)$. If $Q_1 \neq 1$, then $N_{O_8^+(3)}(Q_1)$ is a parabolic subgroup of $O_8^+(3)$. If $Q_1 = 1$, then $R/3^2 \cong 3$. Since $HR/H \cong R/Q \leqslant M_2/H$, it follows that $|R/Q| = 1$ or 3. So we can first classify radical subgroups of H and then, for each such subgroup Q, find $R \leqslant N_{M_2}(Q)$ such that $R \cap H = Q$ and $R = O_3(N_{M_2}(R))$.

Let $L_1 = 3^8.4.L_4(3).2^2$, $L_2 = (3^2{:}2 \times 3_+^{1+8}.2(A_4 \times A_4 \times A_4).2).S_4$ and $L_3 = 3_+^{1+2}.(2^2 \times G_2(3))$. By [**15**, p. 140], $N_H(Q) \leqslant L_i \cap H$ for some i.

We may take

$$\mathcal{R}_0(L_1, 3) = \{3^8, 3^5.3^6, 3^8.3^4, 3^8.3^3.3^2, 3^4.3^3.3^6, (3^2 \times 3_+^{1+8}).3^3\};$$

hence

$$C(3^8) = 3^8, \quad C(3^5.3^6) = 3^5,$$
$$C(3^8.3^4) = 3^3 = C(3^8.3^3.3^2) = C(3^8.3^3.3_+^{1+2}), \quad C(3^4.3^3.3^6) = 3^4$$

and

$$N_{L_1}(Q) = \begin{cases} 3^5.3^6.(Q_8 \times L_3(3)) & \text{if } Q = 3^5.3^6, \\ 3^8.3^4.(4 \times 2)2^3.3^2.D_8 & \text{if } Q = 3^8.3^4, \\ 3^8.3^3.3^2.(Q_8 \times 2S_4) & \text{if } Q = 3^8.3^3.3^2, \\ 3^4.3^3.3^6.(SD_{16} \times 2S_4) & \text{if } Q = 3^4.3^3.3^6, \\ (3^2 \times 3_+^{1+8}).3^3.(SD_{16} \times 2^2) & \text{if } Q = (3^2 \times 3_+^{1+8}).3^3. \end{cases}$$

Also, $N_{M_2}(Q) = N_{L_1}(Q)$ for $Q \in \{3^8, 3^8.3^4\}$, and if $Q \in \mathcal{R}_0(L_1, 3)\backslash\{3^8, 3^8.3^4\}$, then

$$N_{M_2}(R) = \begin{cases} 3^5.3^6.(SD_{16} \times L_3(3)) & \text{if } R = Q = 3^5.3^6, \\ 3^8.3^3.3^2.(SD_{16} \times 2S_4) & \text{if } R = Q = 3^8.3^3.3^2, \\ 3^4.3^3.3^6.2^2.2^4.3^2.2^2 & \text{if } R = Q = 3^4.3^3.3^6, \\ 3^4.3^3.3^6.3.2^3.S_4 & \text{if } Q = 3^4.3^3.3^6 \text{ and } R = 3^4.3^3.3^6.3, \\ (3^2 \times 3_+^{1+8}).3^3.2^3.S_4 & \text{if } R = Q = (3^2 \times 3_+^{1+8}).3^3. \end{cases}$$

We may take

$$\mathcal{R}_0(L_2, 3) = \{3^2 \times 3_+^{1+8}, 3^8.3^4, (3^2 \times 3_+^{1+8}).3, 3^8.3^3.3^2, (3^2 \times 3_+^{1+8}).3^3, (3^2 \times 3_+^{1+8}).3^3.3\};$$

then

$$C(3^2 \times 3_+^{1+8}) = 3^3 = C(3^8.3^3.3^2) = C((3^2 \times 3_+^{1+8}).3^3),$$
$$C((3^2 \times 3_+^{1+8}).3) = 3^2 = C((3^2 \times 3_+^{1+8}).3^3.3)$$

and

$$N_{L_2}(R) = \begin{cases} (3^2 \times 3_+^{1+8}).3.2^3.2^2.S_3 & \text{if } R = (3^2 \times 3_+^{1+8}).3, \\ 3^8.3^3.3^2.(SD_{16} \times 2S_4) & \text{if } R = 3^8.3^3.3^2, \\ (3^2 \times 3_+^{1+8}).3^3.2^3.S_4 & \text{if } R = (3^2 \times 3_+^{1+8}).3^3, \\ (3^2 \times 3_+^{1+8}).3^3.3.2^4 & \text{if } R = (3^2 \times 3_+^{1+8}).3^3.3. \end{cases}$$

In addition, $N_{M_2}(R) = N_{L_2}(R)$ for all $R \in \mathcal{R}_0(L_2, 3)$.

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We may take

$$\mathcal{R}_0(L_3,3)=\{3_+^{1+2},3^4.3^4,3_+^{1+2}.(3_+^{1+2}\times 3^2),3^3.3^4.3^2\};$$

hence

$$C(3_+^{1+2})=3\times G_2(3),\quad C(3^4.3^4)=3^4=C(3_+^{1+2}.(3_+^{1+2}\times 3^2)),\quad C(3^3.3^4.3^2)=3^3$$

and

$$N_{L_3}(R)=\begin{cases}3^4.3^4.(2^2\times 2S_4) & \text{if } R=3^4.3^4,\\ 3_+^{1+2}.(3_+^{1+2}\times 3^2)(2^2\times 2S_4) & \text{if } R=3_+^{1+2}.(3_+^{1+2}\times 3^2),\\ 3^3.3^4.3^2.2^4 & \text{if } R=3^3.3^4.3^2.\end{cases}$$

In addition, $N_{M_2}(R)\neq N_{L_3}(R)$ for all $R\in\mathcal{R}_0(L_3,3)\backslash\{3_+^{1+2}\}$.

It follows that

$$\mathcal{R}_0(M_2,3)=\{3^2,3_+^{1+2},3^8,3^5.3^6,3^2\times 3_+^{1+8},3^8.3^4,(3^2\times 3_+^{1+8}).3,3^4.3^3.3^6,\\ 3^8.3^3.3^2,3^4.3^3.3^6.3,(3^2\times 3_+^{1+8}).3^3,(3^2\times 3_+^{1+8}).3^3.3\}.$$

If $R=3^8$, then $L_1=N_{M_2}(R)<N_G(R)=_G M_6$. If $R=3_+^{1+2}$, then $Z(R)$ is $3A$-pure and

$$L_3=N_{M_2}(R)\leqslant N_G(R)\leqslant N_G(Z(R))=_G M_1$$

so that $L_3\neq N_G(R)$. If $R=3^2\times 3_+^{1+8}$, then $R'=3$ is generated by a $3A$-element of $O_8^+(3)$, which is a $3B$-element of Fi_{24}' and of G. Thus $L_2=N_{M_2}(R)\leqslant N_G(R)\leqslant_G M_3$ and, by Lemma 2.2 and (4.9), R is non-radical in G. For each $R\in\mathcal{R}_0(M_2,3)\backslash\{3^5.3^6,3^4.3^3.3^6,3^4.3^3.3^6.3\}$ with $R\neq O_3(L_i)$, we may suppose $R\in\mathcal{R}_0(L_2,3)$ and $N_{M_2}(R)=N_{L_2}(R)\leqslant L_2\leqslant_G M_3$. If R is radical in G with $N_G(R)\leqslant M_2$, then $N_{M_2}(R)=N_G(R)$ and $R\in_G\mathcal{R}_0(M_3,3)$, which is impossible.

If $R=3^5.3^6\in\mathcal{R}_0(M_2,3)$, then $R'=3B^3\leqslant O_8^+(3)\leqslant M_2$, $N_{M_2}(R)\leqslant_G N(3B^3)=M_5$ and, by Lemma 2.2 and (4.11), R is non-radical in G. If $R=3^4.3^3.3^6\in\mathcal{R}_0(M_2,3)$, then $Q=[R,R']=3B^2$, $N_{M_2}(R)\leqslant_G N(3B^2)=M_4$ and, by Lemma 2.2 and (4.10), R is non-radical in G. If $R=3^4.3^3.3^6.3$, then the last non-trivial term in its lower central series is conjugate in M_2 to Q, so $N_{M_2}(R)\leqslant_G N(3B^2)=M_4$ and, by Lemma 2.2 and (4.10), R is non-radical in G. It follows that for each $R\in\mathcal{R}_0(M_2,3)\backslash\{3^2\}$, $N_G(R)\neq N_{M_2}(R)$.

Case 3.6. There are eight classes of radical subgroups of M_6; one of them, R, has order 3^{14} with $C_{M_6}(R)=Z(R)=3$ and $N_{M_6}(R)=R.2.U_4(3).2^2$. Since $|C(3A)|_3=3^{17}$ and $|C(3C)|_3=3^{11}$, it follows that $Z(R)$ is $3B$-pure and $N_{M_6}(R)\leqslant N_G(R)\leqslant N_G(Z(R))=_G M_3$. By Lemma 2.2 and (4.9), $R=_G 3_+^{1+12}.3$.

A radical subgroup Q of M_6 has order 3^{17} and is such that $C_{M_6}(Q)=Z(Q)=3^2$ and $N_{M_6}(Q)=Q.(2S_4\times A_6).2$. As shown above, $Z(Q)$ is $3B$-pure. Since $|C(3B_4(\mathrm{ii}))|_3=|C(3B_4(\mathrm{iii}))|_3=3^{13}$, it follows that $Z(Q)=3B_4(\mathrm{i})$ and $N_G(Q)\leqslant N_G(Z(Q))=_G M_4$. In particular, $Q=_G 3^{2+5+10}$ and $N_G(Q)\neq N_{M_6}(Q)$.

Another radical subgroup W of M_6 has order 3^{17} and is such that $C_{M_6}(W)=Z(W)=3^3$ and $N_{M_6}(W)=W.(L_3(3)\times Q_8)$. A similar proof to that above shows that $Z(W)=3B^3$, $N_G(W)\leqslant N_G(Z(W))=_G M_5$ and $W=_G 3^{3+2+6+6}$, so $[N_G(W){:}N_{M_6}(W)]=2$.

For other radical subgroups U, we have $Z(U)=Z(R)$ or $Z(Q)$, and so $N_{M_6}(U)\leqslant N_G(Z(U))\leqslant_G M_4$ or M_5. By Lemma 2.4, (4.10) and (4.11), U is radical in G but $N_G(U)\neq N_{M_6}(U)$.

The radical subgroups of M_6 and their local structures are given in Table 4.

Case 3.7. Suppose that $R\in\mathcal{R}_0(G,3)$ with $N_G(R)\leqslant M_7=S_3\times\mathrm{Th}$. If $R\neq 3^*=O_3(M_7)$, then by [**24**, Proposition 2.1] we may suppose $N_G(R)\leqslant M_i$, and so $R\in\mathcal{R}_0(M_i,3)$ for $1\leqslant i\leqslant 6$.

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Thus the radical 3-subgroups of G are as listed in Table 2, and the centralizers and normalizers are given by MAGMA. □

REMARK 4.2. The radical 3-subgroups of $G=\mathbb{M}$ were given in [**26**, Theorem 4]. However, the radical 3-subgroup 3_+^{1+2} was missing and the normalizers of the radical subgroups $3_+^{1+12}.3$, $3_+^{1+12}.3^2$, $3_+^{1+12}.3^5$, $3_+^{1+12}.3^{2+4}$, $3^{2+5+10}.3^2$ and S, which are denoted in [**26**, Theorem 4] by $V_3^{(1)}, V_3^{(4)}, V_3^{(3)}, V_3^{(2)}, V_4^{(1,0)}$ and $V_3^{(5)}$, respectively, are incorrect. Some of the structures of radical 3-subgroups of the Baby Monster given in [**26**, Theorem 2] are also not correct. See [**11**, Proposition 5.1] for the classifications of the radical 3-subgroups of the Baby Monster.

LEMMA 4.3. *Let $G=\mathbb{M}$ and $B_0=B_0(G)$, and let $\mathrm{Blk}^+(G,p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ the characters of $\mathrm{Irr}(G)$ with positive p-defect. If a defect group $D(B)$ of B is cyclic, then $\mathrm{Irr}(B)$ is given by* [**19**, p. 451].

(a) *If $p=13$, then $\mathrm{Blk}^+(G,p)=\{B_i \mid 0\leqslant i\leqslant 4\}$ such that $D(B_i)\cong 13$ when $1\leqslant i\leqslant 4$. In the notation of* [**15**, p. 220],

$$\mathrm{Irr}(B_0)=\mathrm{Irr}^+(G)\Big\backslash\left(\bigcup_{i=1}^{4}\mathrm{Irr}(B_i)\right).$$

Moreover, $\ell(B_0)=52$, $\ell(B_i)=12$ for $1\leqslant i\leqslant 3$ and $\ell(B_4)=6$.

(b) *If $p=11$, then $\mathrm{Blk}^+(G,p)=\{B_i \mid 0\leqslant i\leqslant 6\}$ such that $D(B_i)\cong 11$ when $1\leqslant i\leqslant 6$. In the notation of* [**15**, p. 220],

$$\mathrm{Irr}(B_0)=\mathrm{Irr}^+(G)\Big\backslash\left(\bigcup_{i=1}^{6}\mathrm{Irr}(B_i)\right).$$

Moreover, $\ell(B_0)=45$, $\ell(B_i)=10$ for $1\leqslant i\leqslant 4$ and $\ell(B_j)=5$ for $j=5,6$.

(c) *If $p=7$, then $\mathrm{Blk}^+(G,p)=\{B_i \mid 0\leqslant i\leqslant 5\}$ such that $D(B_1)\cong 7^2$ and $D(B_j)\cong 7$ for $2\leqslant i\leqslant 5$. In the notation of* [**15**, p. 220],

$$\begin{aligned}\mathrm{Irr}(B_1) = \{&\chi_{10},\chi_{13},\chi_{15},\chi_{24},\chi_{37},\chi_{38},\chi_{49},\chi_{67},\chi_{78},\chi_{91},\\ &\chi_{93},\chi_{105},\chi_{106},\chi_{111},\chi_{115},\chi_{133},\chi_{139},\chi_{142},\chi_{144},\\ &\chi_{156},\chi_{161},\chi_{163},\chi_{165},\chi_{170},\chi_{175},\chi_{187},\chi_{188}\}\end{aligned}$$

TABLE 4. *Radical 3-subgroups of $3^8.O_8^-(3).2$.*

R	$C(R)$	$N_{M_6}(R)$
3^8	3^8	$3^8.O_8^-(3).2$
$3_+^{1+12}.3$	3	$3_+^{1+12}.3.2U_4(3).2^2$
$3^{3+2+6+6}$	3^3	$3^{3+2+6+6}.(L_3(3)\times Q_8)$
3^{2+5+10}	3^2	$3^{2+5+10}.(2S_4\times A_6).2$
$3_+^{1+12}.3^5$	3	$3_+^{1+12}.3^5.(2^2\times M_{10})$
$3_+^{1+12}.3^{2+4}$	3	$3_+^{1+12}.3^{2+4}.(2S_4\times Q_8)$
$3^{2+5+10}.3^2$	3^2	$3^{2+5+10}.3^2.(2S_4\times Q_8)$
S	3	$S.(Q_8\times 2^2)$

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and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\Big\backslash\Big(\bigcup_{i=1}^{5}\mathrm{Irr}(B_i)\Big).$$

Moreover, $\ell(B_0) = 70$, $\ell(B_1) = 24$, $\ell(B_i) = 6$ *for* $2 \leqslant i \leqslant 4$ *and* $\ell(B_5) = 3$.

(d) *If* $p = 5$, *then* $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 4\}$ *such that* $D(B_1) \cong 5^2$ *and* $D(B_j) \cong 5$ *for* $2 \leqslant i \leqslant 4$. *In the notation of* [**15**, p. 220],

$$\begin{aligned}\mathrm{Irr}(B_1) = \{&\chi_{21}, \chi_{28}, \chi_{30}, \chi_{31}, \chi_{58}, \chi_{63}, \chi_{67}, \chi_{79}, \chi_{92}, \chi_{104},\\ &\chi_{140}, \chi_{144}, \chi_{151}, \chi_{155}, \chi_{161}, \chi_{167}, \chi_{178}, \chi_{184}, \chi_{189}, \chi_{194}\}\end{aligned}$$

and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\Big\backslash\Big(\bigcup_{i=1}^{4}\mathrm{Irr}(B_i)\Big).$$

Moreover, $\ell(B_0) = 91$, $\ell(B_1) = 16$, $\ell(B_2) = 2$ *and* $\ell(B_i) = 4$ *for* $i = 3, 4$.

(e) *If* $p = 3$, *then* $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 6\}$ *such that* $D(B_1) \cong 3_+^{1+2}$ *and* $D(B_j) \cong 3$ *for* $2 \leqslant j \leqslant 6$. *In the notation of* [**15**, p. 220],

$$\mathrm{Irr}(B_1) = \{\chi_k \mid k \in \{66, 71, 72, 77, 78, 131, 141, 143, 161, 168, 178, 186, 189\}\}$$

and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\Big\backslash\Big(\bigcup_{i=1}^{6}\mathrm{Irr}(B_i)\Big).$$

Moreover, $\ell(B_0) = 83$, $\ell(B_1) = 7$ *and* $\ell(B_j) = 2$ *for* $2 \leqslant j \leqslant 6$.

Proof. If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [**19**, p. 451], we may suppose that D is non-cyclic, so that by Proposition 4.1 we know $p \neq 13$ or 11, and $D = p^2$ or 3_+^{1+2}.

If $D = p^2$ and $L = L_2(7)$, $U_3(5)$ or $O_8^+(3)$ depending on whether $p = 7, 5$ or 3, then $C(D) = p^2 \times L$ and $\theta = 1 \times \mathrm{St}$, where $1 \in \mathrm{Irr}(p^2)$ is the trivial character and $\mathrm{St} \in \mathrm{Irr}(L)$ is the Steinberg character. If $p \neq 3$, then G has a unique block B_1 with $D = p^2$, as $N(D)/C(D)$ is a p'-group. If $p = 3$, then $D = 3^2$ and $N(D)/C(D) = 2.S_4$. By the uniqueness, an element of order 3 in $N(D)/C(D)$ stabilizes θ, so that G has no block B with a defect group 3^2.

If $D = 3_+^{1+2}$, then $C(D) = 3 \times G_2(3)$ and so $\theta = 1 \times \mathrm{St}$. Since $N(D)/C(D)D = D_8$ is a $3'$-group, it follows that G has a unique block B with $D = 3_+^{1+2}$.

Using the method of central characters, $\mathrm{Irr}(B)$ is given as above. If $D(B)$ is cyclic, then $\ell(B)$ is as given in [**19**, p. 451].

If $p = 7$ or 5 and $B = B_1$, then $D(B) = p^2$, the non-trivial elements of $D(B)$ are conjugate in G and $C(x) = p \times H$ for any $1 \neq x \in D(B)$, where $H = \mathrm{He}$ or HN according to whether $p = 7$ or 5. It follows that

$$\mathrm{k}(B) = \ell(B) + \sum_{b \in \mathrm{Blk}(C(x), B)} \ell(b), \tag{4.12}$$

where $\mathrm{Blk}(C(x), B) = \{b \in \mathrm{Blk}(C(x)) : b^G = B\}$. In particular, for $b \in \mathrm{Blk}(C(x), B)$, we have $b = B_0(p) \times b'$ for some block $b' \in \mathrm{Blk}(H)$ with cyclic defect group p. By [**19**, p. 139 or p. 248], H has a unique such block b' with $\ell(b') = 3$ or 4 depending on whether $p = 7$ or 5; so $\ell(B) = \mathrm{k}(B) - \ell(b') = 24$ or 16.

If $p = 3$ and $B = B_1$, then $D(B) = 3_+^{1+2}$, $C_G(D(B)) = 3 \times G_2(3)$ and $Z(D(B)) = 3A$. Since $D(B)$ contains elements of type $3E$ of Fi_{24}', it follows that $D(B)$ has a $3C$-element of G. If $D(B)$ contains a $3B$-element of G, then $G_2(3)$ is conjugate to a subgroup

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of $M_3 = 3_+^{1+12}.2\,\mathrm{Suz}.2$. Since $G_2(3)$ is not a subgroup of Suz, it follows that $D(B)$ contains no element of $3B$. Thus

$$\mathrm{k}(B) = \ell(B) + \sum_{b_1 \in \mathrm{Blk}(C(3A),B)} \ell(b_1) + \sum_{b_2 \in \mathrm{Blk}(C(3C),B)} \ell(b_2).$$

Now, $C(3C) = 3 \times \mathrm{Th}$ and Th has a unique block b_2' with $D(b_2') = 3$ and $\ell(b_2') = 2$ (see [**19**, p. 273]); so $\mathrm{Blk}(C(3C), B) = \{b_2\}$ with $\ell(b_2) = \ell(b_2') = 2$. Similarly, $C(3A) = 3.\mathrm{Fi}_{24}$ and Fi_{24} has a unique block b_1' with $D(b_1') = 3^2$. Since $N_{\mathrm{Fi}_{24}'}(D(b_1')) = (3^2{:}2 \times G_2(3)){:}2$, it follows that $\ell(b_1') = 4$ and so $\mathrm{Blk}(C(3A), B) = \{b_1\}$ with $\ell(b_1) = 4$. Thus $\ell(B_1) = 13 - 4 - 2 = 7$.

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell_{13}(G) = 179$, $\ell_{11}(G) = 181$, $\ell_7(G) = 164$, $\ell_5(G) = 148$ and $\ell_3(G) = 101$. Thus $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \sum_{B \in \mathrm{Blk}^+(G,p)} \ell(B) + |\mathrm{Irr}^0(G)|$$

where $|\mathrm{Irr}^0(G)| = 85, 86, 49, 31$ or 1 when $p = 13, 11, 7, 5$ or 3, respectively. □

THEOREM 4.4. *Let $G = \mathbb{M}$ and let B be a p-block of G with a non-cyclic defect group. If $p \geqslant 3$, then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof. We may suppose $p = 3, 5, 7, 11$ or 13. If $B = B_0$, then Theorem 4.4 follows from Lemma 4.3, equation (4.1) and Tables 1 and 2.

Suppose $B \neq B_0$, so that $p = 7, 5$ or 3. Now suppose $p = 7$ or 5; then $B = B_1$ and $D(B) = p^2$ is abelian. Thus each B-weight has the form (p^2, φ) for some character $\varphi \in \mathrm{Irr}^0(N(p^2)/p^2)$. So φ covers a character $\theta \in \mathrm{Irr}^0(C(p^2)/p^2)$. Since $C(p^2)/p^2 = L_2(7)$ or $U_3(5)$ according to whether $p = 7$ or 5, θ is the Steinberg character St of $C(p^2)/p^2$ and θ has an extension to $N(p^2)/p^2$, so that the number of B-weights equals $|\mathrm{Irr}(N(p^2)/C(p^2))|$. Now $N(7^2)/C(7^2) = (3 \times 2A_4){:}2$ and $N(5^2)/C(5^2) = 4.2^2{:}S_3$ have 24 and 16 irreducible characters, respectively, so that by Lemma 4.3, $\mathcal{W}(B) = \ell(B)$.

Suppose $p = 3$, so that $B = B_1$ with $D = D(B) = 3_+^{1+2}$. If (R, φ) is a B-weight, then we may suppose $R \leqslant D$, so that $R = 3, 3^*, 3^2$ or D.

Now $C(D(B)) = 3 \times G_2(3)$ and $N(D)/C(D)D = D_8$ has five irreducible characters, so B has five weights of the form (D, φ).

If $R = 3$ or 3^*, then R is a proper subgroup of $D \cap C(R)$ and so B has no B-weight of the form (R, φ). If $R = 3^2$, then $C(R) = R \times O_8^+(3)$, $R = D \cap C(R)$ and $\mathrm{Irr}^0(C(R)/R) = \{\mathrm{St}\}$. Since $N(R)/C(R) = 2.S_4$ has exactly two irreducible characters of degree 3, it follows that B has two B-weights of the form (R, φ). □

5. *Radical chains*

Let $G = \mathbb{M}$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. We will do some cancellations in the alternating sum of Uno's conjecture. We first list some radical p-chains $C(i)$ and their normalizers for certain integers i, and then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0 = \mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union of G-orbits of all the $C(i)$. The subgroups of the p-chains in Tables 5 and 6 are given either by Tables 1 and 2 or in the proofs of Proposition 4.1 and Lemma 5.1. The radical 13-chains are also given in Table 5.

LEMMA 5.1. *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leqslant i \leqslant 8\} & \text{with } C(i) \text{ as given in Table 5 if } p = 7, \\ \{C(i) : 1 \leqslant i \leqslant 12\} & \text{with } C(i) \text{ as given in Table 5 if } p = 5, \\ \{C(i) : 1 \leqslant i \leqslant 32\} & \text{with } C(i) \text{ as given in Table 6 if } p = 3. \end{cases}$$

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Then

$$\sum_{C\in\mathcal{R}(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B,d,[r])=\sum_{C\in\mathcal{R}^0(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B,d,[r]),$$

where $B=B_0$ *when* $p=3$. *If* $p=3$ *and* $B=B_1$, *then Dade's ordinary conjecture for* B *is equivalent to*

$$\mathrm{k}(G,B_1,d)+\mathrm{k}(N(C(33)),B_1,d)=\mathrm{k}(N(C(2)),B_1,d)+\mathrm{k}(N(C(34)),B_1,d),$$

where the $C(i)$ *are as given in Table 6.*

Proof. Let $C\in\mathcal{R}(G)$ be given by (2.1), so that we may suppose $P_1\in\mathcal{R}_0(G,p)$.

Case 1. Suppose $p=7$ and let $V=O_7(M_1)$, so that $N(V)=M_1=(7{:}3\times \mathrm{He}){:}2$. Let $\mathcal{R}(G,V)$ be the subfamily of $\mathcal{R}(G)$ consisting of radical chains whose first non-trivial subgroup is V. If $C\in\mathcal{R}(G,V)$ is given by (2.1), then $P_1=V$ and $P_i=V\times Q_i$ for some $Q_i\leqslant \mathrm{He}$ when $i\geqslant 2$. In particular, $C_{\mathrm{He}}:1<Q_2<\ldots<Q_n$ is a radical chain of He.

TABLE 5. *Some radical p-chains of* $\mathbb{M}$ *with* $p=13,11,7$ *or* 5.

C		$N(C)$
$C(1)$	1	$\mathbb{M}$
$C(2)$	$1<13^2$	$13^2{:}4L_2(3).2$
$C(3)$	$1<13^2<13^{1+2}$	$13^{1+2}.4^2.3$
$C(4)$	$1<13$	$(13{:}6\times L_3(3)).2$
$C(5)$	$1<13<13^2$	$13^2.(12\times 3)$
$C(6)$	$1<13^{1+2}$	$13^{1+2}.(3\times 4S_4)$
$C(1)$	1	$\mathbb{M}$
$C(2)$	$1<7^2$	$7^2{:}((3\times 2A_4)\times L_2(7)){:}2$
$C(3)$	$1<7^2<7^3$	$7^3.(3^2\times 2A_4){:}2$
$C(4)$	$1<7_+^{1+4}$	$7_+^{1+4}.(3\times 2S_7)$
$C(5)$	$1<7^{2+1+2}<7_+^{1+4}.7$	$7_+^{1+4}.7.6^2$
$C(6)$	$1<7^{2+1+2}$	$7^{2+1+2}{:}\mathrm{GL}_2(7)$
$C(7)$	$1<(7^2)^*<7_+^{1+2}$	$7_+^{1+2}.6$
$C(8)$	$1<(7^2)^*$	$7^2{:}\mathrm{SL}_2(7)$
$C(1)$	1	$\mathbb{M}$
$C(2)$	$1<5^2$	$(5^2{:}4.2^2\times U_3(5)){:}S_3$
$C(3)$	$1<5^2<5^2\times 5_+^{1+2}$	$(5^2\times 5_+^{1+2}).(4.2^2\times 8){:}S_3$
$C(4)$	$1<5_+^{1+6}$	$5_+^{1+6}{:}2.J_2.4$
$C(5)$	$1<5^{2+2+4}<5_+^{1+6}.5^2$	$5_+^{1+6}.5^2.(S_3\times 4^2)$
$C(6)$	$1<5^{2+2+4}$	$5^{2+2+4}.(S_3\times \mathrm{GL}_2(5))$
$C(7)$	$1<5^{3+3}<5^2.5^2.5^4$	$5^2.5^2.5^4.(2\times \mathrm{GL}_2(5))$
$C(8)$	$1<5^{3+3}$	$5^{3+3}.(2\times L_3(5))$
$C(9)$	$1<5^{3+3}<5^{3+3}.5^2$	$5^{3+3}.5^2.(2\times \mathrm{GL}_2(5))$
$C(10)$	$1<5^{3+3}<5^{3+3}.5^2<5^{3+3}.5_+^{1+2}$	$5^{3+3}.5_+^{1+2}.(2\times 4^2)$
$C(11)$	$1<5^4<5^4.5^2$	$5^4.5^2.(6\times 3).(4\times 2)$
$C(12)$	$1<5^4$	$(5^4.(3\times 2L_2(25))){:}2$

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Conversely, if $C_{\mathrm{He}}: 1 < Q_2 < \ldots < Q_n$ is a chain of $\mathcal{R}(\mathrm{He})$, then $C: 1 < V < V \times Q_2 < \ldots < V \times Q_n$ is a chain of $\mathcal{R}(G, V)$. The map $\varphi: \mathcal{R}(G, V) \to \mathcal{R}(\mathrm{He})$ given by $\varphi(C) = C_{\mathrm{He}}$ is a bijection.

Let $H = 7{:}3 \times \mathrm{He}$ and let $\tau \in N(V) \backslash H$ be an involution. Then $\mathrm{He} = [H, H']$ and $7{:}3$ is the largest normal solvable subgroup of $N(V)$, so that τ stabilizes He and $7{:}3$, respectively. In addition, for $C \in \mathcal{R}(G, V)$,

$$N_H(C) = 7{:}3 \times N_{\mathrm{He}}(C_{\mathrm{He}}).$$

TABLE 6. *Some radical* 3-*chains of* $\mathbb{M}$.

C		$N(C)$
$C(1)$	1	$\mathbb{M}$
$C(2)$	$1 < 3^2$	$(3^2{:}2 \times O_8^+(3)).S_4$
$C(3)$	$1 < 3^2 < 3^2 \times 3_+^{1+8}$	$(3^2 \times 3_+^{1+8}).2^2.2^6.3^3.2^3.S_3$
$C(4)$	$1 < 3^2 < 3^8 < 3^5.3^6$	$3^5.3^6.(Q_8 \times L_3(3))$
$C(5)$	$1 < 3^2 < 3^8$	$3^8 4L_4(3).2^2$
$C(6)$	$1 < 3^2 < 3^8 < 3^8.3^4$	$3^8.3^4(4 \times 2).2^3.3^2.D_8$
$C(7)$	$1 < 3^2 < 3^8 < 3^5.3^6 < 3^4.3^3.3^6$	$3^4.3^3.3^6.(Q_8 \times 2S_4)$
$C(8)$	$1 < 3^2 < 3^8 < 3^4.3^3.3^6$	$3^4.3^3.3^6.(SD_{16} \times 2S_4)$
$C(9)$	$1 < 3^2 < 3^8 < 3^8.3^4 < 3^8.3^3.3^2$	$3^8.3^3.3^2.(Q_8 \times 2S_4)$
$C(10)$	$1 < 3^2 < 3^8 < 3^8.3^4 < 3^8.3^3.3^2 < (3^2 \times 3_+^{1+8}).3^3$	$(3^2 \times 3_+^{1+8}).3^3.(Q_8 \times 2^2)$
$C(11)$	$1 < 3^2 < 3^8 < 3^4.3^3.3^6 < (3^2 \times 3_+^{1+8}).3^3$	$(3^2 \times 3^{1+8}).3^3.(SD_{16} \times 2^2)$
$C(12)$	$1 < 3^2 < 3^5.3^6 < 3^4.3^3.3^6$	$3^4.3^3.3^6.(SD_{16} \times 2S_4)$
$C(13)$	$1 < 3^2 < 3^5.3^6 < 3^4.3^3.3^6 < (3^2 \times 3_+^{1+8})3^3$	$(3^2 \times 3_+^{1+8})3^3.(SD_{16} \times 2^2)$
$C(14)$	$1 < 3^2 < 3^5.3^6 < 3^8.3^3.3^2$	$3^8.3^3.3^2.(SD_{16} \times 2S_4)$
$C(15)$	$1 < 3^2 < 3^5.3^6$	$3^5.3^6.(SD_{16} \times L_3(3))$
$C(16)$	$1 < 3^2 < 3^4.3^3.3^6 < (3^2 \times 3_+^{1+8}).3^3$	$(3^2 \times 3_+^{1+8}).3^3.2^3.S_4$
$C(17)$	$1 < 3^2 < 3^4.3^3.3^6$	$3^4.3^3.3^6.2^2.2^4.3^2.2^2$
$C(18)$	$1 < 3_+^{1+12}$	$3_+^{1+12}.2.\mathrm{Suz}.2$
$C(19)$	$1 < 3^{2+5+10} < 3_+^{1+12}.3^5$	$3_+^{1+12}.3^5.(2^2 \times M_{11})$
$C(20)$	$1 < 3^{2+5+10}$	$3^{2+5+10}.(2S_4 \times M_{11})$
$C(21)$	$1 < 3^{3+2+6+6} < 3_+^{1+12}.3^{2+4}$	$3_+^{1+12}.3^{2+4}.(2S_4 \times SD_{16})$
$C(22)$	$1 < 3^{3+2+6+6} < 3_+^{1+12}.3^{2+4} < S$	$S.(SD_{16} \times 2^2)$
$C(23)$	$1 < 3^{3+2+6+6} < 3^{2+5+10}.3^2$	$3^{2+5+10}.3^2.(SD_{16} \times 2S_4)$
$C(24)$	$1 < 3^{3+2+6+6}$	$3^{3+2+6+6}.(L_3(3) \times SD_{16})$
$C(25)$	$1 < 3^8 < 3_+^{1+12}.3$	$3_+^{1+12}.3.2U_4(3).2^2$
$C(26)$	$1 < 3^8$	$3^8.O_8^-(3).2$
$C(27)$	$1 < 3^8 < 3^{3+2+6+6}$	$3^{3+2+6+6}.(Q_8 \times L_3(3))$
$C(28)$	$1 < 3^8 < 3^{3+2+6+6} < 3^{1+12}.3^{2+4}$	$3^{1+12}.3^{2+4}.(2S_4 \times Q_8)$
$C(29)$	$1 < 3^8 < 3^{2+5+10}$	$3^{2+5+10}.(2S_4 \times M_{10})$
$C(30)$	$1 < 3^8 < 3^{2+5+10} < 3^{1+12}.3^5$	$3^{1+12}.3^5.(2^2 \times M_{10})$
$C(31)$	$1 < 3^8 < 3^{2+5+10} < 3^{1+12}.3^5 < 3^8.3_+^{1+8}.3^2.3$	$3^8.3_+^{1+8}.3^2.3.(Q_8 \times 2^2)$
$C(32)$	$1 < 3^8 < 3^{2+5+10} < 3^{2+5+10}.3^2$	$3^{2+5+10}.3^2.(2S_4 \times Q_8)$
$C(33)$	$1 < 3^2 < 3_+^{1+2}$	$3_+^{1+2}{:}2^2 \times G_2(3)$
$C(34)$	$1 < 3_+^{1+2}$	$(3_+^{1+2}{:}2^2 \times G_2(3)){:}2$

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By [**2**, (5A)], Dade's invariant conjecture holds for He. Using [**2**, Tables I–III], it is easy to check that Uno's invariant conjecture also holds for He when $p=7$.

Let $\mathcal{R}(G,V)_{\rm e}$ and $\mathcal{R}(G,V)_{\rm o}$ be the subfamilies of $\mathcal{R}(G,V)$ consisting of chains C such that $|C|$ is even or odd, respectively. Let

$$\mathcal{X}_K^+ = \bigcup_{C\in\mathcal{R}(G,V)_{\rm e}/K} \mathrm{Irr}(B_0(N_K(C))) \quad \text{and} \quad \mathcal{X}_K^- = \bigcup_{C\in\mathcal{R}(G,V)_{\rm o}/K} \mathrm{Irr}(B_0(N_K(C))),$$

where $K=H$ or $N(V)$.

By [**4**, Lemma (3B)(b)] and the truth of Uno's invariant conjecture for He, there is a defect-preserving bijection ϕ from $\mathcal{X}_H^+$ to $\mathcal{X}_H^-$ such that for each $\chi\in\mathcal{X}_H^+$,

$$r(\phi(\chi)) \equiv \pm r(\phi(\chi)) \pmod p \quad \text{and} \quad \phi(\chi^\tau)=\phi(\chi)^\tau.$$

It follows that ϕ can be extended as a defect- and r-preserving bijection from $\mathcal{X}_{N(V)}^+$ to $\mathcal{X}_{N(V)}^-$. Thus

$$\sum_{C\in\mathcal{R}(G,V)/N(V)} (-1)^{|C|}\mathrm{k}(N_{N(V)}(C),B_0,d,[r])=0. \tag{5.1}$$

Therefore we may suppose $P_1\neq_G V$. If $P_1=7^{1+4}=O_7(M_3)$, let $C':1<7^{1+4}<7^{1+4}.7$ and $g(C'):1<7^{1+4}.7$. Then $N(C')=N(g(C'))=7^{1+4}.7.6^2$ and

$$\mathrm{k}(N(C'),B,d,[r])=\mathrm{k}(N(g(C')),B,d,[r]), \tag{5.2}$$

so we may suppose $P_1\neq 7^{1+4}.7$ and that if $P_1=7^{1+4}$, then $C=_G C(4)$.

If $P_1=7^2=O_7(M_2)$, then $C\in_G\{C(2),C(3)\}$; if $P_1=7^{2+1+2}=O_7(M_4)$, then $C\in\{C(5),C(6)\}$; and if $P_1=(7^2)^*=O_7(M_5)$, then $C\in\{C(7),C(8)\}$.

Case 2. Suppose $p=5$, and let $V=O_5(M_1)$ so that $N(V)=(D_{10}\times \mathrm{HN}).2$. Uno's invariant conjecture for HN is verified by [**10**, Theorem 6.1]. A similar proof to that in Case 1 shows that we may suppose (5.1) holds.

Let $R\in\mathcal{R}_0(M_3,5)\backslash\{5^{1+6}\}$ and $\sigma(R):1<Q=5^{1+6}<R$, so that $\sigma(R)':1<R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1, so there is a bijection g from $\mathcal{R}^-(\sigma(R),5^{1+6})$ onto $\mathcal{R}^0(\sigma(R),5^{1+6})$ such that $N(C')=N(g(C'))$ and $|C'|=|g(C')|-1$ for each $C'\in\mathcal{R}^-(\sigma(R),5^{1+6})$. So we may suppose

$$C\notin \bigcup_{R\in\mathcal{R}_0(M_3,5)\backslash\{5^{1+6}\}} (\mathcal{R}^-(\sigma(R),5^{1+6})\cup\mathcal{R}^0(\sigma(R),5^{1+6})). \tag{5.3}$$

In particular, we may suppose $P_1\notin\mathcal{R}_0(M_3,5)\backslash\{5^{1+6}\}$ and that if $P_1=5^{1+6}$, then $C=C(4)$.

If $P_1=5^2=O_5(M_2)$, then $C\in\{C(2),C(3)\}$; if $P_1=5^{2+2+4}=O_5(M_4)$, then $C\in\{C(5),C(6)\}$; if $P_1=5^4=O_5(M_6)$, then $C\in\{C(11),C(12)\}$.

Suppose $P_1=5^{3+3}=O_5(M_5)$. Let

$$C':1<5^{3+3}<5^2.5^2.5^4<5^{3+3}.5^{1+2} \quad\text{and}\quad g(C'):1<5^{3+3}<5^{3+3}.5^{1+2},$$

so that $N(C')=N(g(C'))$ and (5.2) holds. Thus $C\in\{C(7),C(8),C(9),C(10)\}$.

Case 3. Suppose $p=3$ and $B=B_0$. Let $V\in\{3,3^*,3_+^{1+2}\}$ so that $N(V)=3.\mathrm{Fi}_{24}$, $S_3\times\mathrm{Th}$ or $(3_+^{1+2}{:}2\times G_2(3)){:}2$. Uno's projective invariant conjecture was verified for Th, Fi_{24}' and $G_2(3)$ by [**5, 23**] and [**3**], respectively. A proof similar to that in Case 1 shows that (5.1) holds, so we may suppose $P_1\neq_G 3$, 3^* or 3_+^{1+2}. In the following, the groups L_1, L_2 and L_3 are the same as those appearing in the proof of Case 3.5 in Proposition 4.1.

Case 3.1. Let $R\in\mathcal{R}_0(M_3,3)\backslash\{3^{1+12}\}$ and $\sigma(R):1<Q=3^{1+12}<R$, so that $\sigma(R)':1<R$, where $\mathcal{R}_0(M_3,3)$ is given by (4.9). A similar proof to that in Case 1 shows that we may

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suppose (5.3) holds with $\mathcal{R}_0(M_3, 5)$ replaced by $\mathcal{R}_0(M_3, 3)$ and 5^{1+6} replaced by 3^{1+12}. In particular, $P_1 \notin \mathcal{R}_0(M_3, 3)\backslash\{3^{1+12}\}$, and if $P_1 = 3^{1+12}$, then $C = C(18)$.

We may suppose

$$P_1 \in_G \{3^2, 3^8, 3^{2+5+10}, 3^{3+2+6+6}, 3^{2+5+10}.3^2\}.$$

Case 3.2. Let $\sigma : 1 < Q = 3^{2+5+10} < 3^{2+5+10}.3^2$ so that $\sigma' : 1 < 3^{2+5+10}.3^2$, where $3^{2+5+10}, 3^{2+5+10}.3^2 \in \mathcal{R}_0(M_4, 3)$, which is given by (4.10). Then σ and σ' satisfy the conditions of Lemma 2.1. A similar proof to that in Case 1 shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 3^{2+5+10}) \cup \mathcal{R}^0(\sigma, 3^{2+5+10})). \tag{5.4}$$

In particular, $P_1 \neq_G 3^{2+5+10}.3^2$, and if $P_1 = 3^{2+5+10}$, then $P_2 \neq_G 3^{2+5+10}.3^2$.

Let $C' : 1 < 3^{2+5+10} < S$ and $g(C') : 1 < 3^{2+5+10} < 3^{1+12}.3^5 < S$. Then $N(C') = N(g(C')) = S.(SD_{16} \times 2^2)$ and so (5.2) holds. Thus, if $P_1 = 3^{2+5+10} = O_3(M_4)$, we may suppose $C \in_G \{C(19), C(20)\}$.

Case 3.3. Let $C' : 1 < 3^{3+2+6+6} < S$ and $g(C') : 1 < 3^{3+2+6+6} < 3^{2+5+10}.3^2 < S$, where $3^{3+2+6+6}, 3^{2+5+10}.3^2 \in \mathcal{R}_0(M_5, 3)$. Then $N(C') = N(g(C'))$, and (5.2) holds. We may suppose $C \neq_G C'$ or $g(C')$, so that if $P_1 = 3^{3+2+6+6}$, we may suppose

$$C \in_G \{C(21), C(22), C(23), C(24)\}.$$

Case 3.4. If $P_1 = 3^8$, then $N(P_1) = M_6 = 3^8.O_8^-(3).2$. Applying the Borel–Tits theorem [**14**] to $O_8^-(3)$, it follows that $C \in_G \{C(j) : 25 \leqslant j \leqslant 32\}$.

Case 3.5. Finally, suppose $P_1 = 3^2 = O_3(M_2)$. Let δ be the radical 3-chain $1 < 3^2 < 3_+^{1+2}$, and let $\mathcal{R}(G, \delta)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C such that $C_1 =_G \delta$. Then $N(\delta) = L_3 = 3_+^{1+2}{:}2^2 \times G_2(3)$. The ordinary conjecture for $G_2(3)$ was verified in [**3**]. A similar proof to that in Case 1 shows that we may suppose (5.1) holds with $\mathcal{R}(G, V)$ replaced by $\mathcal{R}(G, \delta)$ and $N(V)$ replaced by $N(\delta)$.

Let $R \in \mathcal{R}_0(L_2, 3)\backslash\{3^2 \times 3^{1+8}\}$ and $\sigma(R) : 1 < 3^2 < Q = 3^2 \times 3^{1+8} < R$, so that $\sigma(R)' : 1 < 3^2 < R$. A proof similar to that in Case 2 shows that we may suppose (5.3) holds with $\mathcal{R}_0(M_3, 5)$ replaced by $\mathcal{R}_0(L_2, 3)$ and 5^{1+6} replaced by $3^2 \times 3^{1+8}$. Thus we may suppose $P_2 \notin_G \mathcal{R}_0(L_2, 5)\backslash\{3^2 \times 3^{1+8}\}$ and that if $P_2 = 3^2 \times 3^{1+8}$, then $C = C(3)$.

Let $L_4 := 3^5.3^6.(Q_8 \times L_3(3)) \leqslant L_1 \leqslant M_2$. We may take

$$\mathcal{R}_0(L_4, 3) = \{3^5.3^6, 3^4.3^3.3^6, 3^8.3^3.3^2, (3^2 \times 3^{1+8}).3^3\} \subseteq \mathcal{R}_0(L_1, 3).$$

Moreover, $N_{L_1}(R) = N_{L_4}(R)$ for $R \neq 3^4.3^3.3^6$ or $(3^2 \times 3^{1+8}).3^3$, and

$$N_{L_4}(R) = \begin{cases} 3^4.3^3.3^6.(Q_8 \times 2S_4) & \text{if } R = 3^4.3^3.3^6, \\ (3^2 \times 3^{1+8}).3^3.(Q_8 \times 2^2) & \text{if } R = (3^2 \times 3^{1+8}).3^3. \end{cases}$$

Let

$$\sigma : 1 < 3^2 < 3^8 < Q = 3^5.3^6 < 3^8.3^3.3^2$$

so that

$$\sigma' : 1 < 3^2 < 3^8 < 3^8.3^3.3^2.$$

A similar proof to that in Case 1 shows that we may suppose (5.4) holds with 3^{2+5+10} replaced by $3^5.3^6$.

Let

$$C' : 1 < 3^2 < 3^8 < 3^5.3^6 < 3^4.3^3.3^6 < (3^2 \times 3^{1+8}).3^3$$

and

$$g(C'): 1 < 3^2 < 3^8 < 3^5.3^6 < (3^2 \times 3^{1+8}).3^3.$$

Then $N(C') = N(g(C'))$ and (5.2) holds. In particular, if $P_1 = 3^2$, $P_2 = 3^8$ and $P_3 = 3^5.3^6$, then $C \in_G \{C(4), C(7)\}$.

Let $L_5 := 3^8.3^4.(4 \times 2).2^3.3^2.D_8 \leqslant L_1 \leqslant M_2$. We may take

$$\mathcal{R}_0(L_5, 3) = \{3^8.3^4, 3^8.3^3.3^2, (3^2 \times 3^{1+8}).3^3\} \subseteq \mathcal{R}_0(L_1, 3)$$

and, moreover, $M_{L_1}(R) = N_{L_5}(R)$ for all $R \in \mathcal{R}_0(L_5, 3)$. Let

$$C' : 1 < 3^2 < 3^8 < 3^8.3^4 < (3^2 \times 3^{1+8}).3^3 \quad \text{and} \quad g(C'): 1 < 3^2 < 3^8 < (3^2 \times 3^{1+8}).3^3.$$

Then $N(C') = N(g(C'))$ and (5.2) holds. Thus, if $P_1 = 3^2$ and $P_2 = 3^8$, then $C \in_G \{C(i) : 4 \leqslant i \leqslant 11\}$.

Let $L_6 := 3^5.3^6.(SD_{16} \times L_3(3)) \leqslant M_2$. We may take

$$\mathcal{R}_0(L_6, 3) = \{3^5.3^6, 3^4.3^3.3^6, 3^8.3^3.3^2, (3^2 \times 3^{1+8}).3^3\} \subseteq \mathcal{R}_0(L_1, 3)$$

and, moreover, $M_{L_1}(R) = N_{L_6}(R)$ for $R \neq 3^8.3^3.3^2$ or $3^5.3^6$ and

$$N_{L_2}(3^8.3^3.3^2) = N_{L_6}(3^8.3^3.3^2) = 3^8.3^3.3^2.(SD_{16} \times 2S_4).$$

Let

$$C' : 1 < 3^2 < 3^5.3^6 < 3^8.3^3.3^2 < (3^2 \times 3^{1+8}).3^3$$

and

$$g(C'): 1 < 3^2 < 3^5.3^6 < (3^2 \times 3^{1+8}).3^3.$$

Then $N(C') = N(g(C'))$ and (5.2) holds.

Let $L_7 := 3^4.3^3.3^6.2^2.2^4.3^2.2^2 \leqslant M_2$. We may take

$$\mathcal{R}_0(L_7, 3) = \{3^4.3^3.3^6, 3^4.3^3.3^6.3, (3^2 \times 3^{1+8}).3^3, (3^2 \times 3^{1+8}).3^3.3\}$$

and, moreover, $M_{M_2}(R) = N_{L_7}(R)$ for $R \in \mathcal{R}_0(L_7, 3) \backslash \{3^4.3^3.3^6.3\}$ and

$$N_{L_7}(3^4.3^3.3^6.3) = N_{M_2}(3^4.3^3.3^6.3) = 3^4.3^3.3^6.3.2^3.S_4.$$

Let $\sigma : 1 < 3^2 < Q = 3^4.3^3.3^6 < 3^4.3^3.3^6.3$ so that $\sigma' : 1 < 3^2 < 3^4.3^3.3^6.3$. A proof similar to that in Case 2 shows that we may suppose (5.4) holds with 3^{2+5+10} replaced by $3^4.3^3.3^6$. Thus we may suppose $P_2 \neq_G 3^4.3^3.3^6.3$ and that if $P_2 = 3^4.3^3.3^6$, then $P_3 \neq_G 3^4.3^3.3^6.3$.

Let

$$C' : 1 < 3^2 < 3^4.3^3.3^6 < (3^2 \times 3^{1+8}).3^3 < (3^2 \times 3^{1+8}).3^3.3$$

and

$$g(C'): 1 < 3^2 < 3^4.3^3.3^6 < (3^2 \times 3^{1+8}).3^3.3.$$

Then $N(C') = N(g(C'))$ and (5.2) holds.

It follows that if $P_1 = 3^2$, then we may suppose $C \in_G \{C(i) : 2 \leqslant i \leqslant 17\}$.

Now suppose $p = 3$ and $B = B_1$, so that $D(B) = 3_+^{1+2}$. Let C be a radical chain such that there exists a block $b \in \mathrm{Blk}(N(C))$ with $b^G = B$. Then we may suppose that the last subgroup of C is a subgroup of $D(B)$. If P_1 is the first non-trivial subgroup of C, then

$$P_1 \in \{3, 3^2, 3_+^{1+2}\}.$$

If $P_1 = 3 = V$, then $N(P_1) = 3.\mathrm{Fi}_{24}$ and the same proof as above shows that (5.1) holds with B_0 replaced by B, so that we may suppose $P_1 \neq_G 3$. If $P_1 = 3^2$, then $C =_G C(2)$ or $C(33)$. If $P_1 = 3_+^{1+2}$, then $C = C(34)$. □

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6. *The proof of Uno's ordinary conjecture*

Suppose $B \in \mathrm{Blk}(\mathbb{M})$ with $D(B) \cong p^2$, so that $p\mathrm{lr}(B) = 2$. By Lemma 2.3, Uno's ordinary conjecture for B is equivalent to the equation

$$\mathrm{k}(\mathbb{M}, B, d, [r]) = \mathrm{k}(N_{\mathbb{M}}(D(B)), B, d, [r]). \tag{6.1}$$

Tables listing the degrees of irreducible characters referenced in the proof of Theorem 6.1 are given in the Appendix.

THEOREM 6.1. *Let B be a p-block of the Monster $G = \mathbb{M}$ with a positive defect. If p is odd, then B satisfies Uno's ordinary conjecture.*

Proof. We may suppose that $D(B)$ is non-cyclic; then, by Lemma 4.3, $B = B_0$ when $p = 13$ or 11 and $B \in \{B_0, B_1\}$ when $p = 7, 5$ or 3.

Case 1. If $p = 13$, then $B = B_0$ and $D(B) \cong 13_+^{1+2}$. The representatives of radical 13-chains are given in Table 5. We set $\mathrm{k}(i, d, r) = \mathrm{k}(N(C(i)), B, d, [r])$ for integers i, d and r. The values of $\mathrm{k}(i, d, r)$ are given in Table 7.

It follows that

$$\sum_{i=1}^{6} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, [r]) = 0.$$

Case 2. If $p = 11$, then $B = B_0$, $D(B) \cong 11^2$ and $N_{\mathbb{M}}(D(B)) \cong (11^2{:}(5 \times 2A_5))$. Thus

$$\mathrm{k}(\mathbb{M}, B, d, [r]) = \mathrm{k}(N_{\mathbb{M}}(D(B)), B, d, [r]) = \begin{cases} 10 & \text{if } d = 2 \text{ and } r = 1, \\ 10 & \text{if } d = 2 \text{ and } r = 2, \\ 10 & \text{if } d = 2 \text{ and } r = 3, \\ 10 & \text{if } d = 2 \text{ and } r = 4, \\ 10 & \text{if } d = 2 \text{ and } r = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Thus (6.1) holds.

Case 3. If $p = 7$ and $B = B_1$, then $D(B) \cong 7^2$,

$$N_{\mathbb{M}}(D(B)) \cong (7^2{:}(3 \times 2A_4) \times L_2(7)){:}2 \leqslant (7{:}3 \times \mathrm{He}){:}2$$

and

$$\mathrm{k}(\mathbb{M}, B, d, [r]) = \mathrm{k}(N(D(B)), B, d, [r]) = \begin{cases} 9 & \text{if } d = 2 \text{ and } r = 1, \\ 9 & \text{if } d = 2 \text{ and } r = 2, \\ 9 & \text{if } d = 2 \text{ and } r = 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Thus (6.1) holds.

TABLE 7. *Values of* $\mathrm{k}(i, d, r)$ *when* $p = 13$ *and* $B = B_0$.

Defect d / Value r	3 / 1	3 / 2	3 / 3	3 / 4	3 / 5	3 / 6	2 / 1	2 / 2	2 / 3	2 / 4	2 / 5	2 / 6	Otherwise / Otherwise
$\mathrm{k}(1, d, r) = \mathrm{k}(6, d, r)$	18	12	3	4	12	6	3	3	0	0	1	0	0
$\mathrm{k}(2, d, r) = \mathrm{k}(3, d, r)$	3	0	0	56	0	0	0	0	0	4	0	0	0
$\mathrm{k}(4, d, r) = \mathrm{k}(5, d, r)$	0	0	0	0	0	0	4	0	39	0	0	12	0

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Suppose $B = B_0$, The values of $\mathrm{k}(i, d, r)$ are given in Table 8. It follows that

$$\sum_{i=1}^{8} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, [r]) = 0.$$

Case 4. Suppose $p = 5$ and $B = B_1$. Then

$$D(B) \cong 5^2, \quad N(D(B)) = N(C(2)) \cong (5^2{:}4.2^2 \times U_3(5)){:}S_3$$

and Theorem 6.1 follows from

$$\mathrm{k}(\mathbb{M}, B, d, [r]) = \mathrm{k}(N(C(2)), B, d, [r]) = \begin{cases} 10 & \text{if } d = 2 \text{ and } r = 1, \\ 10 & \text{if } d = 2 \text{ and } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $B = B_0$, The non-zero $\mathrm{k}(i, d, r)$ values are given in Table 9. It follows that

$$\sum_{i=1}^{12} (-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, [r]) = 0.$$

Case 5. Suppose $p = 3$, so that Uno's ordinary conjecture is equivalent to Dade's ordinary conjecture.

TABLE 8. *Values of* $\mathrm{k}(i, d, r)$ *when* $p = 7$ *and* $B = B_0$.

Defect d	6	5	5	5	4	4	3	3	3	2	Otherwise
Value r	1	1	2	3	1	3	1	2	3	1	Otherwise
$\mathrm{k}(1, d, r)$	49	0	24	12	1	0	0	5	1	0	0
$\mathrm{k}(2, d, r) = \mathrm{k}(3, d, r)$	0	0	0	0	0	0	36	36	36	0	0
$\mathrm{k}(4, d, r)$	49	0	24	12	6	4	0	5	1	0	0
$\mathrm{k}(5, d, r)$	49	24	9	8	6	4	0	0	0	0	0
$\mathrm{k}(6, d, r)$	49	24	9	8	1	0	0	0	0	0	0
$\mathrm{k}(7, d, r) = \mathrm{k}(8, d, r)$	0	0	0	0	0	0	13	0	4	1	0

TABLE 9. *Values of* $\mathrm{k}(i, d, r)$ *when* $p = 5$ *and* $B = B_0$.

Defect d	9	9	8	8	7	7	6	6	5	5	4	4
Value r	1	2	1	2	1	2	1	2	1	2	1	2
$\mathrm{k}(1, d, r)$	40	40	10	5	12	12	0	4	0	0	2	4
$\mathrm{k}(2, d, r) = \mathrm{k}(3, d, r)$	0	0	0	0	0	0	0	0	66	66	20	20
$\mathrm{k}(4, d, r)$	40	40	0	20	12	12	24	4	2	2	2	4
$\mathrm{k}(5, d, r)$	40	40	0	20	44	29	24	4	0	0	0	0
$\mathrm{k}(6, d, r)$	40	40	10	5	44	29	0	4	0	0	0	0
$\mathrm{k}(7, d, r)$	10	60	0	10	50	50	2	4	0	0	0	0
$\mathrm{k}(8, d, r)$	10	60	0	10	16	16	2	4	0	0	0	0
$\mathrm{k}(9, d, r)$	10	60	10	20	16	16	16	16	2	2	0	0
$\mathrm{k}(10, d, r)$	10	60	10	20	50	50	16	16	0	0	0	0
$\mathrm{k}(11, d, r) = \mathrm{k}(12, d, r)$	0	0	0	0	0	0	16	51	0	0	2	1

If $B = B_1$, then $D(B) = 3_+^{1+2}$, $N(D(B)) = N(C(34)) = (3_+^{1+2}{:}2^2 \times G_2(3)){:}2$ and

$$\mathrm{k}(\mathbb{M}, B, d) = \mathrm{k}(N(D(B)), B, d) = \begin{cases} 9 & \text{if } d = 3, \\ 4 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

TABLE 10. *Values of* $\mathrm{k}(i, d)$ *when* $p = 3$ *and* $\mathrm{d}(N(C(i))) = 14$ *or* 15.

Defect d	15	14	13	12	11	10	9	8	7	Otherwise
$\mathrm{k}(2, d)$	81	99	24	108	30	2	0	27	2	0
$\mathrm{k}(3, d)$	81	99	93	108	75	48	24	27	2	0
$\mathrm{k}(4, d)$	0	162	54	108	60	6	0	0	0	0
$\mathrm{k}(5, d)$	0	171	57	108	72	9	0	18	0	0
$\mathrm{k}(6, d)$	0	171	180	108	72	114	24	18	0	0
$\mathrm{k}(7, d)$	0	162	54	216	234	6	0	0	0	0
$\mathrm{k}(8, d)$	0	171	57	180	255	9	0	0	0	0
$\mathrm{k}(9, d)$	0	162	216	108	60	108	24	0	0	0
$\mathrm{k}(10, d)$	0	162	216	216	234	108	0	0	0	0
$\mathrm{k}(11, d)$	0	171	180	180	255	114	0	0	0	0
$\mathrm{k}(12, d)$	0	171	57	180	255	9	0	0	0	0
$\mathrm{k}(13, d)$	0	171	180	180	255	114	0	0	0	0
$\mathrm{k}(14, d)$	0	171	180	90	78	114	24	0	0	0
$\mathrm{k}(15, d)$	0	171	57	90	78	9	0	0	0	0
$\mathrm{k}(16, d)$	81	99	93	243	156	48	0	0	0	0
$\mathrm{k}(17, d)$	81	99	24	243	111	2	0	0	0	0

TABLE 11. *Values of* $\mathrm{k}(i, d)$ *when* $p = 3$ *and* $\mathrm{d}(N(C(i))) = 20$.

Defect d	20	19	18	17	16	15	14	13	12	11	8	7	Otherwise
$\mathrm{k}(1, d)$	81	27	18	13	3	0	9	2	0	2	8	2	0
$\mathrm{k}(18, d)$	81	27	36	19	3	9	9	38	10	11	8	2	0
$\mathrm{k}(19, d)$	81	45	36	19	90	15	63	53	10	0	0	0	0
$\mathrm{k}(20, d)$	81	45	18	13	90	15	63	17	0	0	0	0	0
$\mathrm{k}(21, d)$	81	27	81	181	3	9	108	55	40	11	0	0	0
$\mathrm{k}(22, d)$	81	45	81	181	90	15	162	115	40	0	0	0	0
$\mathrm{k}(23, d)$	81	45	63	61	90	15	162	79	0	0	0	0	0
$\mathrm{k}(24, d)$	81	27	63	61	3	0	108	19	0	2	0	0	0
$\mathrm{k}(25, d)$	54	18	72	92	3	6	36	25	32	13	3	0	0
$\mathrm{k}(26, d)$	54	18	63	35	3	0	36	7	0	1	3	0	0
$\mathrm{k}(27, d)$	54	18	81	62	3	0	90	17	0	1	0	0	0
$\mathrm{k}(28, d)$	54	18	90	221	3	6	90	35	44	13	0	0	0
$\mathrm{k}(29, d)$	54	27	63	35	108	18	63	55	0	0	0	0	0
$\mathrm{k}(30, d)$	54	27	72	92	108	18	63	73	32	0	0	0	0
$\mathrm{k}(31, d)$	54	27	90	221	108	18	117	101	44	0	0	0	0
$\mathrm{k}(32, d)$	54	27	81	62	108	18	117	83	0	0	0	0	0

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Thus Theorem 6.1 follows from Lemma 5.1 and

$$\mathrm{k}(N(C(2), B, d) = \mathrm{k}(N(C(33)), B, d) = \begin{cases} 9 & \text{if } d = 3, \\ 2 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $B = B_0$ and suppose $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 14$ or 15, so that $C =_G C(i)$ for $2 \leqslant i \leqslant 17$. Set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B, d)$. The values of $\mathrm{k}(i, d)$ are given in Table 10.
It follows that

$$\sum_{i=2}^{17} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0.$$

Suppose $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 20$, so that $C =_G C(i)$ for $i = 1$ or $18 \leqslant i \leqslant 32$. The values of $\mathrm{k}(i, d)$ are given in Table 11. It follows that

$$\sum_{\mathrm{d}(N(C(i)))=20} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) = 0.$$

Theorem 6.1 follows for $\mathbb{M}$. □

Appendix. *Degrees of character tables for chain normalizers of* $\mathbb{M}$

TABLE A.1. *The degrees of characters in* $\mathrm{Irr}((3^2{:}2 \times O_8^+(3)).S_4)$.

Degree	1	2	3	4	8	16	300	600
Number	2	3	2	1	2	1	2	3
Degree	900	1200	1560	2400	2457	2808	3120	4800
Number	2	1	4	2	4	2	1	1
Degree	4914	5616	6240	8424	9100	9450	11232	17550
Number	3	3	4	2	4	2	1	2
Degree	18200	18900	19656	22464	24192	27300	28350	32760
Number	6	3	2	2	2	2	2	4
Degree	35100	36400	37800	44928	48384	52650	54600	65520
Number	3	4	1	1	3	2	7	1
Degree	70200	72576	72800	75600	81900	96768	109200	131040
Number	1	2	5	2	2	1	1	4
Degree	139776	140400	145600	151200	163800	174720	193536	199017
Number	4	6	6	1	4	4	2	4
Degree	218400	218700	245700	279552	280800	291200	294840	327600
Number	6	2	6	3	2	4	4	3
Degree	332800	349440	387072	398034	436800	437400	491400	531441
Number	2	1	1	3	7	3	10	2
Degree	561600	582400	589680	656100	665600	698880	716800	737100
Number	4	3	1	2	3	4	4	6
Degree	873600	874800	982800	998400	1062882	1118208	1137240	1164800
Number	3	1	11	6	3	2	4	5
Degree	1179360	1257984	1310400	1331200	1397760	1433600	1474200	1572480
Number	4	4	4	1	8	6	6	4
Degree	1592136	1594323	1749600	1965600	1996800	2125764	2150400	2274480
Number	2	2	2	3	3	1	4	1
Degree	2329600	2515968	2662400	2795520	2867200	2948400	3144960	3499200
Number	6	3	2	4	2	1	1	1
Degree	3931200	4251528	4422600	4548960	4659200	5324800	5591040	5734400
Number	5	2	2	4	2	1	4	4
Degree	6289920	6988800	7862400	7987200	8503056	10063872	11182080	11468800
Number	4	2	1	2	1	2	2	2

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TABLE A.2. *The degrees of characters in* $\mathrm{Irr}((3^2 \times 3^{1+8}).2^2.2^6.3^3.2^3.S_3)$.

Degree	1	2	3	4	6	8	9	12	16
Number	4	6	4	2	4	12	8	7	12
Degree	18	24	27	32	36	48	54	64	72
Number	6	5	12	3	8	12	16	8	6
Degree	81	96	108	128	144	162	192	216	256
Number	4	7	5	10	4	2	1	8	7
Degree	288	324	384	432	486	512	576	648	768
Number	1	3	12	4	2	2	8	9	4
Degree	864	972	1024	1152	1296	1458	1536	1728	1944
Number	4	4	8	10	12	4	15	14	5
Degree	2048	2304	2592	2916	3072	3456	3888	4096	4374
Number	6	14	10	3	13	12	14	1	6
Degree	4608	5184	5832	6144	6912	7776	8748	9216	10368
Number	12	13	8	14	19	5	10	9	15
Degree	11664	12288	13122	13824	15552	17496	18432	20736	23328
Number	6	1	2	8	5	2	6	7	3
Degree	27648	31104	34992	36864	55296	62208	69984		
Number	4	12	8	1	2	1	1		

TABLE A.3. *The degrees of characters in* $\mathrm{Irr}(3^5.3^6.(Q_8 \times L_3(3)))$.

Degree	1	2	8	12	13	16	24	26	27	32	39
Number	4	1	1	4	4	16	1	21	4	4	4
Degree	52	54	78	96	104	128	156	208	216	234	312
Number	25	1	9	1	17	4	6	25	1	4	1
Degree	416	468	624	702	832	936	1248	1404	1664	1872	2496
Number	26	17	8	4	2	17	9	9	12	23	2
Degree	2808	3744	4992	5616	7488	8424	11232	14976	16848		
Number	6	28	9	23	4	2	12	15	4		

TABLE A.4. *The degrees of characters in* $\mathrm{Irr}(3^8.4.L_4(3).2^2)$.

Degree	1	2	8	39	52	78	90	104	130
Number	4	3	2	4	8	3	4	4	4
Degree	180	208	260	312	351	390	416	468	520
Number	3	4	9	2	4	4	10	12	14
Degree	702	720	729	780	832	936	1040	1170	1280
Number	3	2	4	7	10	5	17	4	8
Degree	1458	1560	1664	1872	2080	2340	2560	2808	3120
Number	3	7	1	8	18	5	2	6	7
Degree	3328	3744	4160	4680	5616	5832	6240	6656	7020
Number	4	2	11	11	1	2	2	1	8
Degree	8320	9360	10240	11232	12480	14040	16640	18720	22464
Number	12	12	2	8	7	6	6	8	1
Degree	24960	28080	29952	33280	37440	37908	42120	44928	49920
Number	5	17	4	8	13	4	4	4	8
Degree	56160	59904	66560	74880	75816	84240	99840	112320	119808
Number	9	1	6	5	1	4	1	5	4
Degree	133120	149760	151632	168480					
Number	3	5	4	1					

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Table A.5. *The degrees of characters in* $\mathrm{Irr}(3^8.3^4.(4\times 2).2^3.3^2.D_8)$.

Degree	1	2	4	6	8	9	12	16	18	24	32
Number	8	10	19	8	24	8	10	21	10	6	26
Degree	36	48	64	72	96	128	144	162	192	256	288
Number	3	10	20	4	22	15	18	4	20	14	12
Degree	324	384	432	512	576	648	768	864	972	1024	1152
Number	7	36	8	10	18	21	31	10	4	4	16
Degree	1296	1458	1536	1728	1944	2304	2592	2916	3072	3456	3888
Number	27	4	22	26	9	12	26	7	15	17	2
Degree	4608	5184	5832	6912	7776	9216	10368	11664	13824		
Number	6	20	1	9	9	1	9	6	2		

Table A.6. *The degrees of characters in* $\mathrm{Irr}(3^4.3^3.3^6.(Q_8\times 2S_4))$.

Degree	1	2	3	4	6	8	16	18	24	32	36
Number	8	14	8	7	2	27	45	12	2	20	39
Degree	48	54	64	72	96	108	128	144	192	216	288
Number	12	12	28	33	15	15	13	45	2	14	54
Degree	384	432	576	648	864	1152	1296	1728	3456		
Number	13	57	6	2	45	27	4	64	27		

Table A.7. *The degrees of characters in* $\mathrm{Irr}(3^4.3^3.3^6.(SD_{16}\times 2S_4))$.

Degree	1	2	3	4	6	8	16	18	24	32	36	48
Number	8	18	8	13	6	15	32	4	4	33	19	4
Degree	54	64	72	96	108	128	144	192	216	256	288	384
Number	4	30	30	11	11	18	35	10	26	4	41	10
Degree	432	576	648	768	864	1152	1296	1728	2304	2592	3456	6912
Number	48	25	4	4	59	16	4	60	10	1	38	9

Table A.8. *The degrees of characters in* $\mathrm{Irr}(3^8.3^3.3^2.(Q_8\times 2S_4))$.

Degree	1	2	3	4	6	8	12	16	24	32	48	64
Number	8	22	8	29	10	27	6	31	18	28	48	4
Degree	72	96	128	144	162	192	288	324	384	432	486	576
Number	8	40	13	30	4	38	21	17	48	8	4	34
Degree	648	864	972	1152	1296	1728	1944	2592	3456	3888		
Number	23	22	9	15	37	24	2	27	6	9		

Table A.9. *The degrees of characters in* $\mathrm{Irr}((3^2\times 3^{1+8}).3^3.(Q_8\times 2^2))$.

Degree	1	2	4	6	8	12	16	18	24	32	36	48	54
Number	16	36	44	24	23	54	30	24	46	13	66	40	8
Degree	72	96	108	144	162	216	288	324	432	648	864	1296	
Number	39	52	42	60	8	60	27	26	88	20	36	54	

Table A.10. *The degrees of characters in* $\mathrm{Irr}((3^2\times 3^{1+8}).3^3.(SD_{16}\times 2^2))$.

Degree	1	2	4	6	8	12	16	18	24	32
Number	16	28	32	8	39	22	34	8	44	18
Degree	36	48	54	64	72	96	108	144	162	192
Number	34	50	8	4	45	36	22	47	8	20
Degree	216	288	324	432	576	648	864	1296	1728	2592
Number	76	36	14	87	10	34	50	40	12	18

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TABLE A.11. *The degrees of characters in* $\mathrm{Irr}(3^4.3^3.3^6.(SD_{16} \times 2S_4))$.

Degree	1	2	3	4	6	8	16	18	24	32	36	48
Number	8	18	8	13	6	15	32	4	4	33	19	4
Degree	54	64	72	96	108	128	144	192	216	256	288	384
Number	4	30	30	11	11	18	35	10	26	4	41	10
Degree	432	576	648	768	864	1152	1296	1728	2304	2592	3456	6912
Number	48	25	4	4	59	16	4	60	10	1	38	9

TABLE A.12. *The degrees of characters in* $\mathrm{Irr}((3^2 \times 3^{1+8}).3^3.(SD_{16} \times 2^2))$.

Degree	1	2	4	6	8	12	16	18	24	32
Number	16	28	32	8	38	22	34	8	44	18
Degree	36	48	54	64	72	96	108	144	162	192
Number	34	50	8	4	45	36	22	47	8	20
Degree	216	288	324	432	576	648	864	1296	1728	2592
Number	76	36	14	87	10	34	50	40	12	18

TABLE A.13. *The degrees of characters in* $\mathrm{Irr}(3^8.3^3.3^2.(SD_{16} \times 2S_4))$.

Degree	1	2	3	4	6	8	12	16	24	32
Number	8	18	8	21	6	33	8	34	6	25
Degree	48	64	72	96	128	144	162	192	256	288
Number	20	18	8	37	10	18	4	44	4	17
Degree	324	384	432	486	576	648	768	864	972	1152
Number	11	32	8	4	21	24	19	30	3	20
Degree	1296	1728	1944	2304	2592	3456	3888	5184	6912	7776
Number	37	28	8	6	29	10	6	9	2	3

TABLE A.14. *The degrees of characters in* $\mathrm{Irr}(3^5.3^6.(SD_{16} \times L_3(3)))$.

Degree	1	2	8	12	13	16	24	26	27	32	39
Number	4	3	2	4	4	16	3	15	4	12	4
Degree	52	54	78	96	104	128	156	208	216	234	312
Number	17	3	3	2	20	8	8	24	2	4	4
Degree	416	468	624	702	832	936	1248	1404	1664	1872	2496
Number	20	11	8	4	14	12	4	3	8	17	8
Degree	2808	3328	3744	4992	5616	7488	8424	9984	11232	14976	16848
Number	16	4	21	6	26	11	4	3	17	8	4
Degree	22464	29952	33696								
Number	3	6	1								

TABLE A.15. *The degrees of characters in* $\mathrm{Irr}((3^2 \times 3^{1+8}).3^3.2^3.S_4)$.

Degree	1	2	3	4	6	8	12	16	18	24	32	36	48
Number	8	16	8	10	12	18	18	20	8	17	8	14	29
Degree	54	64	72	96	108	144	162	192	216	288	324	432	486
Number	12	1	18	14	34	28	8	1	68	22	20	87	4
Degree	576	648	864	972	1296	1728	1944	2592	3888	5184	7776		
Number	3	39	38	8	62	4	10	26	24	1	2		

Table A.16. *The degrees of characters in* $\mathrm{Irr}(3^4.3^3.3^6.2^2.2^4.3^2.2^2)$.

Degree	1	2	3	4	6	8	9	12	16	24
Number	4	12	8	13	12	10	4	4	9	12
Degree	32	48	54	64	96	108	128	144	162	192
Number	13	20	4	13	9	15	6	4	4	21
Degree	216	256	288	324	384	432	576	648	768	864
Number	33	1	7	7	12	56	1	9	1	62
Degree	1152	1296	1728	2592	3456	3888	5184	6912	10368	20736
Number	8	26	39	21	30	2	29	4	14	1

Table A.17. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.2.\mathrm{Suz}.2)$.

Degree	1	143	220	364	728	780	1001
Number	2	2	2	2	1	2	2
Degree	1144	3432	4928	5940	10010	10725	12012
Number	1	2	2	2	1	2	2
Degree	14300	15795	17496	18954	20020	25025	30030
Number	2	2	1	2	4	2	1
Degree	32032	40040	50050	54054	64064	65520	66560
Number	1	2	1	2	2	2	2
Degree	70200	75075	79872	80080	88452	96228	100100
Number	2	2	2	4	2	1	5
Degree	102400	113724	120120	122472	128128	128700	133056
Number	2	1	1	1	2	1	2
Degree	137280	146432	159744	163800	168960	187110	189540
Number	2	2	1	2	2	1	2
Degree	192192	193050	197120	208494	228800	243243	248832
Number	2	2	4	2	2	2	2
Degree	277200	288288	315392	465920	625482	655200	1137240
Number	2	1	2	2	1	2	1
Degree	1347192	1441440	1990170	2501928	2882880	3127410	3603600
Number	1	1	1	1	2	1	2
Degree	4264650	4378374	4659200	5125120	6254820	6368544	7207200
Number	2	1	6	2	1	2	3
Degree	7440174	7454720	7862400	8648640	9797760	10810800	11531520
Number	2	4	2	1	1	2	2
Degree	12509640	14414400	16166304	17513496	20500480	20966400	25625600
Number	1	2	2	3	2	4	2
Degree	27634932	28828800	31274100	33679800	34594560	35026992	39405366
Number	1	4	1	2	1	1	1
Degree	40030848	43243200	45034704	51175800	51251200	57657600	58378320
Number	2	3	1	2	4	1	2
Degree	61501440	62548200	64864800	67092480	75779550	87567480	89282088
Number	2	2	2	2	1	1	2
Degree	92252160	93822300	102502400	109459350	112586760	115315200	116756640
Number	1	1	3	2	1	1	1
Degree	125096400	129729600	151992126	153964800	157621464	159213600	163762560
Number	1	1	1	1	1	1	1
Degree	193995648	197026830	203793408	210161952	233834040	246005760	250192800
Number	1	1	2	2	1	2	1
Degree	262702440	276349320	281466900	303984252	307507200	328378050	369008640
Number	2	1	1	1	2	1	1
Degree	410009600	437837400	483611310	636854400			
Number	2	1	1	1			

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TABLE A.18. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.3^5.(2^2 \times M_{11}))$.

Degree	1	2	10	11	16	20	22	32
Number	4	2	12	4	8	6	2	4
Degree	44	45	55	88	90	110	132	220
Number	4	4	4	2	2	10	4	10
Degree	264	330	396	440	528	660	792	880
Number	2	8	4	3	4	10	2	4
Degree	1056	1320	1584	1760	1782	1980	2376	2640
Number	2	2	5	2	4	4	2	8
Degree	3564	3960	5280	5832	7128	7920	10560	10692
Number	4	2	4	2	1	9	1	2
Degree	11664	11880	15840	16038	17496	17820	23760	28512
Number	1	14	4	4	2	18	3	6
Degree	29160	32076	35640	57024	58320	64152	69984	71280
Number	4	8	21	3	6	5	4	22
Degree	80190	87480	96228	106920	116640	128304	139968	142560
Number	4	4	6	7	2	5	2	10
Degree	160380	171072	174960	192456	209952	213840	240570	256608
Number	12	1	4	4	4	1	8	2
Degree	285120	320760	384912	427680	481140	577368	641520	721710
Number	1	7	5	4	12	2	1	4
Degree	962280							
Number	2							

TABLE A.19. *The degrees of characters in* $\mathrm{Irr}(3^{2+5+10}.3^5.(2S_4 \times M_{11}))$.

Degree	1	2	3	4	10	11	16	20
Number	2	3	2	1	6	2	4	9
Degree	22	30	32	33	40	44	45	48
Number	3	6	6	2	3	3	2	4
Degree	55	64	88	90	110	132	135	165
Number	2	2	3	3	3	2	2	2
Degree	176	180	220	440	528	880	1056	1760
Number	1	1	1	8	4	10	2	3
Degree	1782	2112	2640	3168	3520	3564	4224	5280
Number	2	4	4	2	4	3	2	2
Degree	5346	6336	7040	7128	7920	10560	14256	15840
Number	2	1	2	1	2	4	2	5
Degree	16038	17820	21120	23328	23760	28512	31680	32076
Number	2	9	4	2	6	4	2	3
Degree	35640	42240	46656	47520	48114	53460	57024	64152
Number	15	1	1	5	2	7	3	1
Degree	71280	85536	106920	116640	128304	142560	213840	228096
Number	14	1	1	4	6	18	4	3
Degree	233280	256608	279936	285120	320760	466560	513216	559872
Number	6	5	4	7	4	2	5	2
Degree	570240	641520	769824	962280	1026432	1140480	1283040	1539648
Number	8	12	2	4	2	1	7	1
Degree	1924560	2566080						
Number	2	1						

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TABLE A.20. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.3^{2+4}.(SD_{16} \times 2S_4))$.

Degree	1	2	3	4	6	8	16	32	36	48
Number	8	18	8	13	6	11	10	11	8	4
Degree	64	72	96	108	128	144	192	216	288	384
Number	8	22	3	8	2	18	4	6	13	2
Degree	432	576	864	1152	1296	1728	2304	2592	3456	3888
Number	24	15	45	4	2	47	1	1	32	4
Degree	5832	6912	7776	11664	13122	17496	23328	26244	31104	34992
Number	12	19	3	24	4	16	43	11	2	16
Degree	39366	46656	52488	69984	78732	93312	104976	139968	157464	209952
Number	4	25	16	3	3	4	7	20	4	2

TABLE A.21. *The degrees of characters in* $\mathrm{Irr}(S.(SD_{16} \times 2^2))$.

Degree	1	2	4	6	8	12	16	24	32
Number	16	28	16	8	11	6	8	8	2
Degree	36	48	72	96	108	144	192	216	288
Number	16	16	36	6	16	22	1	44	6
Degree	324	432	576	648	864	1296	1458	1728	1944
Number	16	62	1	28	40	26	8	19	8
Degree	2592	2916	3888	4374	5184	5832	7776	8748	11664
Number	16	18	2	8	4	73	4	22	53
Degree	13122	15552	17496	23328	26244	34992	52488	69984	104976
Number	8	1	44	10	14	37	10	4	8

TABLE A.22. *The degrees of characters in* $\mathrm{Irr}(3^{2+5+10}.3^2.(SD_{16} \times 2S_4))$.

Degree	1	2	3	4	6	8	16	24	32	64
Number	8	18	8	13	6	15	16	4	5	4
Degree	96	128	144	192	288	324	384	432	576	648
Number	8	2	16	12	28	8	6	8	14	18
Degree	768	864	972	1152	1296	1458	1728	1944	2304	2592
Number	1	14	8	4	18	4	22	2	1	20
Degree	2916	3456	3888	4374	5184	5832	6912	7776	8748	10368
Number	11	14	4	4	14	28	3	1	3	8
Degree	11664	17496	20736	23328	34992	46656	69984	93312	139968	279936
Number	29	20	4	35	18	45	17	10	13	4

TABLE A.23. *The degrees of characters in* $\mathrm{Irr}(3^{3+2+6+6}.(SD_{16} \times L_3(3)))$.

Degree	1	2	12	13	16	24	26	27	32
Number	4	3	4	4	16	3	15	4	12
Degree	39	52	54	78	104	208	312	416	468
Number	4	9	3	3	4	6	4	2	8
Degree	624	832	936	1248	1404	1664	1872	2808	3744
Number	4	4	18	3	8	2	12	2	11
Degree	4992	5616	5832	7488	11232	14976	16848	18954	22464
Number	2	12	2	11	12	2	2	4	11
Degree	29952	33696	37908	44928	56862	69984	75816	89856	93312
Number	1	1	11	6	4	2	26	3	8
Degree	113724	151632	157464	227448	303264	606528	1213056		
Number	3	29	2	10	9	17	2		

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TABLE A.24. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.3.2U_4(3).2^2)$.

Degree	1	21	40	70	72	90	112	140
Number	4	4	1	6	2	4	2	4
Degree	180	189	210	224	240	252	280	378
Number	3	4	4	4	1	2	1	2
Degree	420	504	560	630	729	896	1008	1080
Number	6	4	4	8	4	8	5	4
Degree	1120	1260	1280	1440	1458	1512	1890	2016
Number	2	5	4	4	2	2	2	4
Degree	2240	2520	3024	3584	4480	5040	6048	7560
Number	6	2	2	2	4	3	1	4
Degree	8960	10080	12096	13440	15120	16128	17496	20160
Number	1	5	2	2	4	4	1	5
Degree	24192	26880	30240	32256	35840	40320	40824	43740
Number	9	1	14	2	1	8	4	1
Degree	45360	52488	60480	61236	64512	81648	96768	108864
Number	3	2	7	1	2	1	2	2
Degree	120960	131220	163296	183708	204120	241920	244944	275562
Number	8	3	2	2	4	14	2	2
Degree	306180	326592	349920	367416	408240	459270	483840	612360
Number	3	4	1	3	5	2	10	4
Degree	734832	787320	816480	918540	967680	979776	1049760	1062882
Number	2	5	2	6	1	4	6	2
Degree	1088640	1102248	1119744	1224720	1306368	1377810	1632960	1741824
Number	3	2	2	2	4	2	4	1
Degree	1837080	1959552	2125764	2204496	2449440	2755620		
Number	6	2	1	1	2	1		

TABLE A.25. *The degrees of characters in* $\mathrm{Irr}(3^8.O_8^-(3).2)$.

Degree	1	246	574	819	1066	7462	7749
Number	2	2	3	2	1	1	4
Degree	14391	14924	21320	22386	29848	44772	51660
Number	4	2	2	6	1	2	3
Degree	59696	66339	67158	74620	76752	95940	119556
Number	6	2	2	1	2	1	1
Degree	127920	134316	149240	179334	191880	201474	223860
Number	2	2	1	2	3	6	2
Degree	238784	268632	358176	367360	402948	418446	447720
Number	6	3	2	8	4	3	1
Degree	531441	537264	575640	596960	604422	671580	682240
Number	2	8	2	3	1	3	8
Degree	716352	734720	777114	931840	955136	1074528	1151280
Number	1	2	1	5	2	3	3
Degree	1343160	1535040	1554228	1611792	1673784	2014740	2686320
Number	5	4	4	2	1	2	2
Degree	5372640	6447168	7651584	10745280	10879596	12433824	16117920
Number	2	1	2	2	3	2	2
Degree	17192448	21490560	21759192	23313420	25788672	31084560	32235840
Number	4	5	2	1	1	1	3
Degree	32638788	33475680	41964156	42981120	43518384	48353760	54397980
Number	2	2	1	5	1	2	5
Degree	64471680	65277576	68769792	74602944	87036768	87156324	99470592
Number	1	1	2	1	3	1	2
Degree	107122176						
Number	1						

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TABLE A.26. *The degrees of characters in* $\mathrm{Irr}(3^{3+2+6+6}.(Q_8 \times L_3(3)))$.

Degree	1	2	12	13	16	24	26	27	32
Number	4	1	4	4	16	1	13	4	4
Degree	39	52	54	78	104	208	234	312	416
Number	4	3	1	1	2	3	8	2	1
Degree	468	624	702	832	936	1248	1404	1664	1872
Number	22	4	8	2	16	1	6	1	13
Degree	3744	4992	5616	5832	7488	11232	14976	16848	18954
Number	17	1	14	1	2	18	3	3	4
Degree	22464	37908	44928	56862	69984	75816	93312	113724	151632
Number	2	17	9	4	1	24	4	9	17
Degree	157464	227448	303264	606528					
Number	1	3	13	10					

TABLE A.27. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.3^{2+4}.(Q_8 \times 2S_4))$.

Degree	1	2	3	4	6	8	16	18
Number	8	14	8	7	2	9	6	8
Degree	32	36	48	54	64	72	96	108
Number	5	22	4	8	4	18	1	6
Degree	128	144	192	216	288	384	432	576
Number	1	16	2	18	18	1	54	4
Degree	864	1152	1296	1728	3456	3888	5832	7776
Number	49	4	3	38	48	4	10	1
Degree	11664	13122	17496	23328	26244	31104	34992	39366
Number	33	4	8	34	17	1	10	4
Degree	46656	52488	69984	78732	93312	104976	139968	
Number	12	19	9	9	1	4	8	

TABLE A.28. *The degrees of characters in* $\mathrm{Irr}(3^{2+5+10}.(2S_4 \times M_{10}))$.

Degree	1	2	3	4	9	10	16	18	20
Number	4	6	4	2	4	6	2	6	9
Degree	27	30	32	36	40	48	64	80	160
Number	4	6	3	2	3	2	1	8	6
Degree	162	240	288	320	324	360	480	486	576
Number	2	4	8	1	3	8	4	2	4
Degree	640	648	720	960	1152	1280	1296	1440	1458
Number	2	1	14	1	2	1	2	7	2
Degree	1620	1920	2160	2304	2592	2880	2916	3240	3840
Number	9	2	8	1	4	1	3	19	4
Degree	4320	4374	4860	5184	5760	5832	6480	7776	8640
Number	12	2	7	3	4	1	12	1	9
Degree	9720	11520	11664	12960	14580	17280	20736	23328	25920
Number	5	2	2	20	4	2	3	3	19
Degree	29160	38880	43740	46656	51840	58320	69984	87480	93312
Number	9	3	4	1	2	9	12	9	2
Degree	103680	116640	139968	174960	186624	233280	279936	349920	466560
Number	9	10	6	14	1	11	2	5	4
Degree	559872	933120							
Number	1	1							

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TABLE A.29. *The degrees of characters in* $\mathrm{Irr}(3^{1+12}.3^5.(2^2 \times M_{10}))$.

Degree	1	2	9	10	16	18	20	32	40
Number	8	4	8	12	4	4	14	2	6
Degree	60	72	80	90	120	144	160	162	180
Number	12	8	1	8	6	4	2	4	18
Degree	216	240	270	288	320	324	360	480	540
Number	8	1	8	2	1	4	9	4	10
Degree	576	648	720	864	960	972	1080	1440	1458
Number	1	1	1	2	4	2	22	6	4
Degree	1620	2160	2592	2880	2916	3240	4320	5184	5832
Number	18	29	6	3	4	29	12	3	3
Degree	6480	8640	8748	9720	11664	12960	14580	15552	17496
Number	14	1	2	7	1	20	12	1	14
Degree	19440	21870	23328	25920	29160	34992	43740	46656	52488
Number	5	8	2	9	20	6	18	1	12
Degree	58320	65610	69984	77760	87480	116640	131220	139968	174960
Number	11	8	4	3	13	4	10	1	5
Degree	209952	233280	349920						
Number	2	1	2						

TABLE A.30. *The degrees of characters in* $\mathrm{Irr}(3^8.3^{1+8}.3^2.3.(Q_8 \times 2^2))$.

Degree	1	2	4	6	8	12	16	18
Number	16	20	8	8	5	2	4	16
Degree	24	32	36	48	54	72	96	108
Number	4	1	36	8	16	22	5	52
Degree	144	162	216	288	324	432	648	864
Number	12	16	46	4	36	58	18	49
Degree	972	1296	1458	1944	2592	2916	4374	5832
Number	8	26	8	6	12	30	8	49
Degree	7776	8748	11664	13122	17496	23328	26244	34992
Number	4	34	26	8	42	4	26	16
Degree	52488	69984	104976					
Number	8	1	2					

TABLE A.31. *The degrees of characters in* $\mathrm{Irr}(3^{2+5+10}.3^2.(Q_8 \times 2S_4))$.

Degree	1	2	3	4	6	8	16	24	32	64
Number	8	14	8	7	2	11	9	2	2	2
Degree	72	96	128	144	162	192	288	324	384	432
Number	16	4	1	36	8	6	18	22	5	16
Degree	486	576	648	864	972	1152	1296	1458	1728	2592
Number	8	8	16	16	6	3	22	4	20	26
Degree	2916	3456	3888	4374	5184	5832	8748	10368	11664	17496
Number	17	10	4	4	2	25	9	12	17	12
Degree	23328	34992	46656	69984	93312	139968	279936			
Number	26	30	24	21	4	6	1			

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TABLE A.32. *The degrees of characters in* $\mathrm{Irr}(3_+^{1+2}{:}2^2 \times G_2(3))$.

Degree	1	2	4	6	14	28	56	64	78	84	91
Number	4	4	1	2	4	4	1	8	4	2	12
Degree	104	128	156	168	182	208	256	273	312	336	364
Number	4	8	4	4	20	4	2	8	1	4	11
Degree	384	416	448	546	624	672	728	729	819	832	896
Number	4	1	8	22	2	1	10	4	4	4	8
Degree	1008	1092	1456	1458	1638	1664	1792	2184	2688	2912	3276
Number	2	14	6	4	8	4	2	2	4	2	5
Degree	3328	4368	4374	4914	4992						
Number	1	4	2	2	2						

TABLE A.33. *The degrees of characters in* $\mathrm{Irr}((5^2{:}4.2^2 \times U_3(5)){:}S_3)$.

Degree	1	2	3	4	20	21	24	40	42	60
Number	4	6	4	2	4	4	4	6	6	4
Degree	63	80	84	105	125	126	168	210	250	252
Number	4	2	14	4	4	4	12	6	6	16
Degree	288	315	336	375	378	420	480	500	504	576
Number	6	4	2	4	4	2	4	2	12	6
Degree	672	756	864	1344	2016	2520	3000	3024	6048	6912
Number	4	2	2	2	4	4	4	4	2	2

TABLE A.34. *The degrees of characters in* $\mathrm{Irr}((5^2 \times 5^{1+2})(4.2^2 \times 8){:}S_3)$.

Degree	1	2	3	4	6	20	24	40	48	60	80	192	384	480
Number	16	36	16	20	4	8	24	12	10	8	4	4	2	8

TABLE A.35. *The degrees of characters in* $\mathrm{Irr}((5_+^{1+6}{:}2.J_2{:}4)$.

Degree	1	12	14	28	36	42	63	84	90
Number	4	2	4	2	4	2	4	4	4
Degree	100	112	126	128	140	160	175	216	225
Number	2	2	4	2	2	4	4	4	4
Degree	252	288	300	336	350	378	448	500	3000
Number	6	4	4	8	4	2	6	1	2
Degree	7000	7560	8064	10500	16128	18000	25000	28000	30240
Number	3	4	8	2	4	1	2	2	5
Degree	31500	32000	35000	37800	42000	45000	63000	75600	80000
Number	1	2	2	4	1	1	3	2	1
Degree	87500	94500	96768	108000	112000	112500	120960	126000	144000
Number	1	2	4	1	2	1	1	1	1
Degree	150000	168000	175000	224000					
Number	1	2	1	1					

TABLE A.36. *The degrees of characters in* $\mathrm{Irr}(5^{1+6}.5^2.(S_3 \times 4^2))$.

Degree	1	2	4	8	12	48	60	100	200	240
Number	32	16	8	4	16	4	16	8	4	4
Degree	300	400	500	600	800	1000	1200	2400	3000	
Number	16	2	8	32	1	4	8	2	16	

TABLE A.37. *The degrees of characters in* $\mathrm{Irr}(5^{2+2+4}.(S_3 \times \mathrm{GL}_2(5)))$.

Degree	1	2	4	5	6	8	10	12	72	288	300	480
Number	8	4	20	8	12	10	4	6	16	4	4	2
Degree	600	900	960	1200	1500	1800	2400	3600	4800	7200	14400	
Number	16	8	1	12	4	4	2	16	1	8	2	

TABLE A.38. *The degrees of characters in* $\mathrm{Irr}(5^{2+2+4}.(2 \times \mathrm{GL}_2(5)))$.

Degree	1	4	5	6	24	48	96	100	192	200
Number	8	20	8	12	16	8	4	4	2	10
Degree	300	400	480	500	600	800	1000	1200	2400	4800
Number	8	12	2	4	16	4	2	18	18	10

TABLE A.39. *The degrees of characters in* $\mathrm{Irr}(5^{3+3}.(2 \times L_3(5)))$.

Degree	1	30	31	96	124	125	155	186
Number	2	2	6	20	22	2	6	2
Degree	248	372	496	620	744	2976	3100	5952
Number	4	4	4	2	2	2	2	2
Degree	6200	9300	12400	15500	18600	24800	31000	37200
Number	6	4	8	2	6	4	2	2

TABLE A.40. *The degrees of characters in* $\mathrm{Irr}(5^{3+3}.5^2(2 \times \mathrm{GL}_2(5)))$.

Degree	1	4	5	6	8	12	16	20	24
Number	8	22	8	12	4	4	4	2	10
Degree	96	120	192	240	480	500	600	960	1000
Number	4	8	2	8	2	2	8	2	6
Degree	1200	1500	2000	2400	2500	3000	4000	5000	6000
Number	16	4	8	8	2	6	4	2	2

TABLE A.41. *The degrees of characters in* $\mathrm{Irr}(5^{3+3}.5^{1+2}.(2 \times 4^2))$.

Degree	1	4	8	16	20	32	40	80	100	160	200	400	500	800	1000	2000
Number	32	24	8	4	16	2	8	4	24	2	40	26	8	10	16	8

TABLE A.42. *The degrees of characters in* $\mathrm{Irr}(5^4.5^2.(6 \times 3).(4 \times 2))$.

Degree	1	2	12	24	48	72	144	600	1200
Number	8	34	8	6	1	8	2	2	1

TABLE A.43. *The degrees of characters in* $\mathrm{Irr}(5^4{:}(3 \times 2.L_2(25)){:}2)$.

Degree	1	2	12	13	24	25	26	48	50	52	624	1248	1872	3744
Number	2	1	4	4	2	2	8	18	1	15	2	1	8	2

TABLE A.44. *The degrees of characters in* $\mathrm{Irr}(7^2{:}(3 \times 2A_4) \times L_2(7)){:}2)$.

Degree	1	2	3	4	6	7	8	12	14	16
Number	6	9	6	3	15	6	6	18	9	9
Degree	18	21	24	28	32	48	144	288	336	384
Number	9	6	9	3	3	3	6	3	3	3

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TABLE A.45. *The degrees of characters in* $\mathrm{Irr}(7^3.(3^2 \times 2A_4){:}2)$.

Degree	1	2	3	4	6	12	18	48	144
Number	18	27	18	9	9	9	3	9	6

TABLE A.46. *The degrees of characters in* $\mathrm{Irr}(7^{1+4}.(3 \times 2S_7))$.

Degree	1	6	8	14	15	20	21	28
Number	6	6	3	12	6	15	6	3
Degree	35	36	294	720	1176	1680	1764	2940
Number	6	6	1	6	2	6	1	2
Degree	3360	4116	4320	4410	5880	6174	10290	10584
Number	3	4	1	1	2	1	1	1

TABLE A.47. *The degrees of characters in* $\mathrm{Irr}(7^{1+4}.7.6^2)$.

Degree	1	6	36	42	84	126	252	294	882
Number	36	12	1	18	9	8	6	6	4

TABLE A.48. *The degrees of characters in* $\mathrm{Irr}(7^{2+1+2}.\mathrm{GL}_2(7))$.

Degree	1	6	7	8	42	48	126	168	252	288	294	336	672	1008	2016
Number	6	21	6	15	1	6	2	2	3	1	1	8	9	4	6

TABLE A.49. *The degrees of characters in* $\mathrm{Irr}(7^{1+2}.6)$.

Degree	1	3	6	42
Number	6	4	7	1

TABLE A.50. *The degrees of characters in* $\mathrm{Irr}(7^2.\mathrm{SL}_2(7))$.

Degree	1	3	4	6	7	8	48
Number	1	2	2	3	1	2	7

TABLE A.51. *The degrees of characters in* $\mathrm{Irr}(11^2{:}(5 \times 2A_5))$.

Degree	1	2	3	4	5	6	120
Number	5	10	10	10	5	5	5

TABLE A.52. *The degrees of characters in* $\mathrm{Irr}(13^2{:}4L_2(13).2)$.

Degree	1	12	13	14	168	672
Number	4	26	4	22	4	3

TABLE A.53. *The degrees of characters in* $\mathrm{Irr}(13^{1+2}{:}4^2.3)$.

Degree	1	12	48	156
Number	48	8	3	4

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TABLE A.54. *The degrees of characters in* $\mathrm{Irr}((13{:}6 \times L_3(3)).2)$.

Degree	1	12	13	26	27	32	39	52	144	156	192	312	324	468
Number	12	13	12	12	12	12	12	6	1	1	4	3	1	1

TABLE A.55. *The degrees of characters in* $\mathrm{Irr}((13^2.(12 \times 3))$.

Degree	1	6	12	36
Number	36	12	3	4

TABLE A.56. *The degrees of characters in* $\mathrm{Irr}((13^{1+2}.(3 \times 4S_4))$.

Degree	1	2	3	4	72	96	156	312	468
Number	12	18	12	6	4	3	3	3	1

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