Bull. Aust. Math. Soc. **96** (2017), 317–325
doi:10.1017/S0004972717000296

ON THE EXISTENCE OF f-MAXIMAL SPACELIKE HYPERSURFACES IN CERTAIN WEIGHTED MANIFOLDS

ARLANDSON M. S. OLIVEIRA and HENRIQUE F. DE LIMA✉

(Received 20 January 2017; accepted 23 February 2017; first published online 2 May 2017)

Abstract

We apply a mean-value inequality for positive subsolutions of the f-heat operator, obtained from a Sobolev embedding, to prove a nonexistence result concerning complete noncompact f-maximal spacelike hypersurfaces in a class of weighted Lorentzian manifolds. Furthermore, we establish a new Calabi–Bernstein result for complete noncompact maximal spacelike hypersurfaces in a Lorentzian product space.

2010 *Mathematics subject classification*: primary 53C42; secondary 53A07, 35P15.

Keywords and phrases: weighted Lorentzian product space, Bakry–Émery–Ricci tensor, f-Laplacian, f-maximal spacelike hypersurface, entire spacelike graph.

1. Introduction and statement of the main results

Let (Σ, g) be an n-dimensional complete Riemannian manifold. The Laplace operator $-\Delta$ on Σ can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \int_\Sigma |\nabla u|^2 \, dV, \quad u \in C_c^\infty(\Sigma) \subset L^2(dV),$$

where $|\cdot|$ is the norm induced by the Riemannian inner product $g = \langle \cdot , \cdot \rangle$ and dV is the volume element on Σ. Let $f \in C^\infty(\Sigma)$ be a weight function. If we replace the measure dV with the weighted measure $d\mu = e^{-f} dV$ in the definition of Q, we obtain a new quadratic form Q_f, and we denote by Δ_f the elliptic operator on $C_c^\infty(\Sigma) \subset L^2(d\mu)$ induced by Q_f. Thus, Δ_f is a natural generalisation of the Laplacian. It is symmetric and positive and extends to a positive operator on $L^2(d\mu)$. By Stokes' theorem,

$$\Delta_f u = \Delta u - \langle \nabla u, \nabla f \rangle.$$

Introducing a weight factor is the first step towards decoupling the leading term and the lower order terms of the operator, which in the case of the Laplace operator are completely determined by the metric of Σ.

The second author is partially supported by CNPq, Brazil, grant 303977/2015-9.

The triple $(\Sigma, g, d\mu)$ and the operator Δ_f defined above and acting over $C^\infty(\Sigma)$ will be called, respectively, the *weighted manifold*, Σ_f, associated with (Σ, g) and f, and the *f-Laplacian*. A notion of curvature for weighted manifolds goes back to Lichnerowicz [14] and it was later developed by Bakry and Émery in their seminal work [4], where they introduced the modified Ricci curvature

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f. \tag{1.1}$$

As is now common, we will refer to this tensor as the *Bakry–Émery–Ricci tensor* of Σ. The interplay between the geometry of Σ and the behaviour of the weight function f is mostly encoded in its Bakry–Émery–Ricci tensor Ric_f. This curvature is related to the f-Laplacian via the following Bochner formula:

$$\tfrac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + 2\langle \nabla u, \nabla\Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u) \quad \text{for all } u \in C^\infty(\Sigma). \tag{1.2}$$

The Bakry–Émery–Ricci curvature tensor arises in scalar–tensor gravitation theories in the conformal gauge known as the Jordan frame. In Lorentzian geometry, Case [6] has shown that a sign condition on timelike components of the Bakry–Émery–Ricci tensor, a so-called energy condition, will, in an analogous fashion to the Riemannian case, imply that singularity theorems and the timelike splitting theorem hold. Woolgar [20] used these theorems to obtain Jordan-frame singularity and timelike splitting theorems for the Brans–Dicke family of scalar–tensor theories and (1-loop) dilaton gravity, including totally skew torsion dilaton gravity. A connection between the theory of black holes and a Lorentzian Bakry–Émery formulation was established by Rupert and Woolgar [17]. Under an energy condition on the Bakry–Émery–Ricci curvature, Galloway and Woolgar [10] obtained singularity theorems of a cosmological type, for both zero and positive cosmological constant, that is, they found conditions under which every timelike geodesic is incomplete. The singularity theorems of general relativity (see, for instance, [12]) are some of the deepest statements in modern science, because they imply that the universe has a finite history, beginning in a so-called big bang singularity, provided that we can reliably extrapolate certain features of the known laws of physics back to early times and high energy scales.

In 1970, Calabi [5] stated the well-known Calabi–Bernstein theorem: *The only complete maximal surfaces in the three-dimensional Lorentz–Minkowski spacetime, that is, spacelike surfaces with zero mean curvature, are the spacelike planes.*

The nonparametric version of this theorem asserts that the only entire maximal graphs in $\mathbb{L}^3$ are the affine functions. Cheng and Yau [8] extended this result to complete maximal hypersurfaces in $\mathbb{L}^{n+1}$. A natural generalisation of the Lorentz–Minkowski spacetime is the class of Lorentzian product manifolds of the form $-I \times M$, where M is an n-dimensional Riemannian manifold and $I \subseteq \mathbb{R}$ is an open interval. Several authors have obtained Calabi–Bernstein-type results for maximal spacelike hypersurfaces in such Lorentzian manifolds (see, for instance, [2]).

Let (M, g_M) be an n-dimensional connected Riemannian manifold and $I \subseteq \mathbb{R}$ an open interval. Consider the product manifold $\overline{M} = I \times M$, and denote by π_I and π_M the projections onto the factors I and M, respectively. The class of Lorentzian manifolds which will concern us is the one obtained by furnishing $\overline{M}$ with the Lorentzian metric

$$\overline{g}_p(v,w) = -(\pi_I)_*(v)(\pi_I)_*(w) + g_M((\pi_M)_*(v),(\pi_M)_*(w))|_{\pi_M(p)}$$

for all $p \in \overline{M}$ and $v, w \in T_p\overline{M}$. A standard computation shows that ∂_t is a globally defined closed Killing vector field on $\overline{M}$. The family of spacelike hypersurfaces $S_t = \{t\} \times M, t \in I$ constitutes a foliation of $\overline{M}$ by totally geodesic leaves that we call slices.

Let S be an n-dimensional connected manifold. A smooth immersion $\iota : S \to \overline{M}$ is said to be a spacelike hypersurface if S, furnished with the metric induced from $\overline{g}$ via ι, is a Riemannian manifold. If this is so, we shall always assume that the metric on S is the induced one. In this setting, it follows from the connectedness of S that we can uniquely choose a globally defined timelike unit normal vector field $N \in \mathfrak{X}(S)^\perp$, having the same time orientation as ∂_t, that is such that $\overline{g}(N, \partial_t) \le -1$. We say that N is the future-pointing Gauss map of S. A function naturally attached to a spacelike hypersurface S immersed into a Lorentzian product $\overline{M}$ is the height function defined by $h = \pi_I|_S$. Inspired by Gromov [11], we define the f-mean curvature H_f of S by

$$nH_f = nH - \overline{g}(\nabla f, N), \tag{1.3}$$

where H stands for the mean curvature of S with respect to its Gauss map N. We will say that S is f-maximal if $H_f \equiv 0$.

From a splitting theorem due to Case (see [6, Theorem 1.2]), it follows that if $\overline{M}$ is a weighted Lorentzian product space endowed with a weight function f bounded from above such that $\overline{\mathrm{Ric}}_f(V, V) \ge 0$ for all timelike vector fields V, then f does not depend on t. Motivated by this result, we will consider weighted Lorentzian products $\overline{M}_f = -I \times M_f$, where the weight function f does not depend on the parameter $t \in I$ or, in other words, $\overline{g}(\overline{\nabla} f, \partial_t) = 0$. In the trivial case, f = constant, the assumption on the Ricci tensor in [6] reduces to the so-called *timelike convergence condition* (which means that the Ricci curvature is nonnegative on timelike directions) and the splitting theorem generalises to a well-known one (see [9]).

In this setting, we prove the following nonexistence result.

Theorem 1.1. *Let $\overline{M}_f = -I \times M_f$ be a weighted Lorentzian product space whose fibre M is noncomplete with nonnegative sectional curvature and such that the weight function f is bounded and convex. Then there are no complete noncompact f-maximal spacelike hypersurfaces immersed in $\overline{M}_f$.*

Recall that the only functions convex and bounded from above on a complete Riemannian manifold are the constant ones (see, for instance, [19, Corollary 6.2.5]). On the other hand, taking into account examples given in [19, pages 84–86], we see that there exist nonconstant convex and bounded functions on noncomplete Riemannian manifolds.

The proof of Theorem 1.1 is presented in Section 3. As a consequence, we get the following Calabi–Bernstein result.

https://doi.org/10.1017/S0004972717000296 Published online by Cambridge University Press

Theorem 1.2. *Let $\overline{M} = -I \times M$ be a Lorentzian product space whose fibre M is complete with nonnegative sectional curvature. The only maximal noncompact spacelike hypersurfaces of $\overline{M}$ are the slices.*

Let $\Omega \subseteq M$ be a connected domain and let $u \in C^\infty(\Omega)$ be a smooth function such that $u(\Omega) \subseteq I$. We denote by $S(u)$ the graph over Ω determined by u, that is,

$$S(u) = \{(u(x), x) : x \in \Omega\} \subset -I \times M.$$

The graph is said to be entire if $\Omega = M$. The metric induced on Ω from the Lorentzian metric of the ambient space via $S(u)$ is

$$g_{S(u)} = -du^2 + g_M.$$

It can be easily seen that a graph $S(u)$ is a spacelike hypersurface if and only if $|Du|_M < 1$, Du being the gradient of u in M and $|Du|_M = g_M(Du, Du)^{1/2}$.

It is interesting to observe that, in contrast to the case of graphs in a Riemannian product space, an entire spacelike graph $S(u)$ in a Lorentzian product space $-I \times M$ is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete on M. For instance, Albujer [1, Section 3] obtained explicit examples of noncomplete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^2$. However, it follows from [2, Lemma 17] that when the fibre M is complete and $|Du|_M \leq c$, for a certain constant c with $0 < c < 1$, then $S(u)$ must also be complete. On an entire graph $S(u)$, the existence of such a constant c prevents the tangent vector field to a divergent curve in $S(u)$ from asymptotically approaching a lightlike direction in the ambient space.

In this setting, we get the following nonparametric version of Theorem 1.2.

Corollary 1.3. *Let M be an n-dimensional complete noncompact manifold with nonnegative sectional curvatures. The only maximal entire functions into $-I \times M$ such that $|Du|_M \leq c$, for some constant $c \in (0, 1)$, are the constant ones.*

2. Key lemmas

In this section, we present two key lemmas used in the proof of Theorem 1.1. Before stating the first one, we recall some important facts.

Let $(\Sigma, g, d\mu = e^{-f}\, dV)$ be an n-dimensional weighted complete Riemannian manifold. Take any point $x \in \Sigma$ and denote the volume form in geodesic polar coordinates centred at x by

$$dV|_{\exp_x(r\xi)} = J(x, r, \xi)\, dr\, d\xi,$$

where $r > 0$ and $\xi \in S_x\Sigma$ is a unitary tangent vector at x. It is well known that if $y \in \Sigma$ is any point such that $y = \exp_x(r\xi)$, then

$$\Delta_f\, d(x, y) = \frac{J_f'(x, r, \xi)}{J_f(x, r, \xi)},$$

where $J_f(x, r, \xi) := e^{-f} J(x, r, \xi)$ is the f-volume form in geodesic polar coordinates. For a fixed point $p \in \Sigma$ and $R > 0$, define

$$A(R) := \sup_{B_p(3R)} |f(x)|. \tag{2.1}$$

For a subset $\Omega \subseteq \Sigma$, we will denote by $V(\Omega)$ the volume of Ω with respect to the usual volume form dV, and by $V_f(\Omega)$ the f-volume of Ω, $V_f(\Omega) = \int_\Omega e^{-f}\, dV$. If Σ has nonnegative Bakry–Émery–Ricci curvature, then along any minimising geodesic starting from $x \in B_p(R)$,

$$\frac{J_f(x, r_2, \xi)}{J_f(x, r_1, \xi)} \le e^{4A} \Big(\frac{r_2}{r_1}\Big)^{n-1} \tag{2.2}$$

for any r_1, r_2 with $0 < r_1 < r_2 < R$; in particular, for $0 < r_1 < r_2 < R$,

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \le e^{4A} \Big(\frac{r_2}{r_1}\Big)^{n}, \tag{2.3}$$

where $A = A(R)$ is defined in (2.1) (see [15, Lemma 2.1]). If Σ is noncompact, the comparison inequality (2.2) guarantees that there exist constants $\nu > 2$, C_1 and C_2, depending only on n, such that for all $\phi \in C_0^\infty(B_x(r))$ the following local Sobolev inequality holds:

$$\Big(\int_{B_x(r)} |\phi|^{2\nu/(\nu-2)}\, d\mu\Big)^{(\nu-2)/\nu} \le C_1 \frac{e^{C_2 A} r^2}{V_f(B_x(r))^{2/\nu}} \int_{B_x(r)} (|\nabla\phi|^2 + r^{-2}|\phi|^2)\, d\mu, \tag{2.4}$$

where $x \in B_p(R)$ and $0 < r < R$ (see [7, Lemma 2.3]). Such a family of Sobolev inequalities can be used to obtain a mean-value inequality for subsolutions to the f-heat equation as in [18, Theorem 5.2.9] (see [7, Lemma 2.5]).

Lemma 2.1. *Let $(\Sigma, g, d\mu = e^{-f} dV)$ be an n-dimensional weighted complete Riemannian manifold which satisfies the local Sobolev inequality (2.4) for all functions $\phi \in C_0^\infty(B_o(\rho))$ and $0 < \rho \le R$, where $o \in \Sigma$ is a fixed origin. Fix $q \in (0, +\infty)$ and let u be a positive subsolution of the f-heat equation, that is,*

$$Lu := \Delta_f u + \frac{\partial u}{\partial t} \ge 0,$$

in the cylinder $Q = B_o(r) \times (s - r^2, s)$ for some $s \in \mathbb{R}$ and $0 < r < R$. Then, for $0 < \delta < \delta' \le 1$, there exist constants $C_3 = C_3(n, \sup|f|, \nu, q)$ and $C_4 = C_4(n, \nu, q)$ such that

$$\sup_{Q_\delta} u^q \le C_3 \frac{e^{C_4 A(R)}}{(\delta' - \delta)^{\nu+2} r^2 V_f(B_o(r))} \int_{Q_{\delta'}} u^q\, d\mu\, dt,$$

where $Q_z = B_o(zr) \times (s - zr^2, s)$.

Our second key lemma gives sufficient conditions to guarantee that the Bakry–Émery–Ricci curvature of an f-maximal spacelike hypersurface immersed in a weighted Lorentzian product is nonnegative.

https://doi.org/10.1017/S0004972717000296 Published online by Cambridge University Press

Lemma 2.2. *Let $\overline{M}_f = -I \times M_f$ be a Lorentzian weighted product whose fibre M has nonnegative sectional curvatures and with convex weight function. Let $\iota : S \to \overline{M}_f$ be an f-maximal spacelike hypersurface. Then the Bakry–Émery–Ricci curvature* Ric_f *of S is nonnegative.*

Proof. The curvature tensor R of a spacelike hypersurface S immersed in $\overline{M}$ can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of $\overline{M}$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^{\top} - g_S(AX,Z)AY + g_S(AY,Z)AX \tag{2.5}$$

for all tangent vector fields $X, Y, Z \in \mathfrak{X}(S)$. The curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\cdot\,,\cdot]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(S)$ (see [16]).

Consider $X \in \mathfrak{X}(S)$ and a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(S)$. It follows from the Gauss equation (2.5) that

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \overline{g}(\overline{R}(X,E_i)X,E_i) + nHg_S(AX,X) + g_S(AX,AX). \tag{2.6}$$

Moreover,

$$\overline{R}(X,Y)Z = R^M(X^M, Y^M)Z^M,$$

R^M and $(\cdot)^M$ being, respectively, the curvature tensor and the projection of a vector field onto the fibre M (see [16, Proposition 7.42] for details). Hence,

$$\overline{g}(\overline{R}(X,E_i)X,E_i) = K^M(X^M, E_i^M)(g_M(X^M,X^M)g_M(E_i^M,E_i^M) - g_M(X^M,E_i^M)).$$

Since M is nonnegatively curved, substituting this equation into (2.6) gives

$$\mathrm{Ric}(X,X) \geq nHg_S(AX,X) + g_S(AX,AX). \tag{2.7}$$

Since f is convex,

$$\mathrm{Hess}\, f(X,X) = \overline{\mathrm{Hess}}\, f(X,X) - \overline{g}(\overline{\nabla} f, N)g_S(AX,X) \geq -\overline{g}(\overline{\nabla} f, N)g_S(AX,X). \tag{2.8}$$

Therefore, from (1.1), (1.3), (2.7) and (2.8),

$$\mathrm{Ric}_f(X,X) \geq nH_f g_S(AX,X) + g_S(AX,AX). \tag{2.9}$$

The result follows from (2.9) taking into account the hypothesis that S is f-maximal. □

https://doi.org/10.1017/S0004972717000296 Published online by Cambridge University Press

3. Proof of Theorem 1.1

Let S be a complete noncompact spacelike f-maximal hypersurface immersed in $\overline{M}_f = -I \times M_f$. The height function, h, of S is an f-harmonic function. In fact, the Laplacian of h on S is $\Delta h = -nH\overline{g}(N, \partial_t)$ (see, for instance, [3, Lemma 4.1]). In view of (1.3), the f-Laplacian of h on S is $\Delta_f h = -nH_f\overline{g}(N, \partial_t)$, and the f-harmonicity of h follows from the hypothesis that S is f-maximal.

We claim that h must be bounded from above. We will argue as in the proof of [13, Theorem 0.13]. Suppose otherwise. Then $S \cap S_t \neq 0$ for all $t \in I$. For a fixed $t \in I$, let $\Sigma_t := \{p \in S : h(p) \geq t\}$. By Sard's theorem, we can suppose that t is a regular value of $h|_{\operatorname{int} S}$, so that Σ_t is a smooth complete manifold with boundary $\partial\Sigma_t = \{p \in S : h(p) = t\}$ and exterior unit normal $\nu_t = -\nabla h/|\nabla h|$. For any $\epsilon > 0$, define h_ϵ on Σ_t by

$$h_\epsilon = \max\{h, t + \epsilon\}.$$

Then h_ϵ is f-harmonic on Σ_t. Indeed, set

$$\Sigma_1 = \{p \in \Sigma_t : h(p) > t + \epsilon\},$$
$$\Sigma_2 = \{p \in \Sigma_t : h(p) = t + \epsilon\},$$
$$\Sigma_3 = \{p \in \Sigma_t : t < h(p) < t + \epsilon\}.$$

Then $h_\epsilon = h$ on Σ_1 and h_ϵ is constant (equal to $t + \epsilon$) on Σ_3, so $\Delta_f h_\epsilon = 0$ on both Σ_1 and Σ_3. The tranversality of S and ∂_t, by which we mean the fact that the function $\overline{g}(N, \partial_t)$ is negatively signed globally on S, implies a certain kind of monotonic behaviour of the height function h, which in turn guarantees that h_ϵ is smooth on Σ_2 and on $\partial\Sigma_t$. So, we also have $\Delta_f h_\epsilon = 0$ on both Σ_2 and $\partial\Sigma_t$ by continuity. On noting that $h_\epsilon \equiv t + \epsilon$ on $\partial\Sigma_t$, by the maximum principle for the f-Laplacian, we see that $t \leq h \leq t + \epsilon$ on Σ_t. Since this holds for every $\epsilon > 0$, we conclude that $h \equiv t$ on Σ_t, contradicting the assumption of h being unbounded from above. The same reasoning also proves that h is bounded from below. We now observe that h has sublinear growth, that is,

$$\lim_{p \to \infty} \frac{|h|(p)}{r(p)} = 0, \tag{3.1}$$

where $r(p) := d(p, p_0)$ is the distance from a fixed point $p_0 \in S$. This follows from the noncompactness of S and from the fact just established that $\alpha := \sup_S |h| < +\infty$, so that

$$0 \leq \frac{|h|(p)}{r(p)} \leq \frac{\alpha}{r(p)} \to 0 \quad \text{as } p \to \infty.$$

From $\Delta_f h = 0$, Lemma 2.2 and the weighted Bochner formula (1.2), we conclude that $|\nabla h|^2$ is f-subharmonic. By setting $q = 1$ in Lemma 2.1 and observing that an f-subharmonic function is also a subsolution to the f-heat equation, we get the following mean-value inequality:

$$\sup_{B_p(R/2)} |\nabla h|^2 \leq \frac{\beta}{R^2 V_f(B_p(R))} \int_{B_p(R)} |\nabla h|^2 \, d\mu, \tag{3.2}$$

where β is a constant depending only on n and $\sup_S |f|$.

https://doi.org/10.1017/S0004972717000296 Published online by Cambridge University Press

We now follow the proof of [15, Theorem 3.2] and apply a standard cut-off argument. Choose a cut-off function ϕ such that $\phi = 1$ on $B_p(R)$, $\phi = 0$ on $S^n \backslash B_p(2R)$ and $|\nabla\phi| \le \beta/R$. Integrating by parts and using $\Delta_f^S h = 0$ yields

$$\begin{aligned}\int_S |\nabla h|^2\phi^2\, d\mu &= -2\int_S h\phi\langle\nabla h, \nabla\phi\rangle\, d\mu \\ &\le 2\int_S |h|\phi|\langle\nabla h, \nabla\phi\rangle|\, d\mu \\ &\le 2\Big(\int_S |\nabla h|^2\phi^2\, d\mu\Big)^{1/2}\Big(\int_S h^2|\nabla\phi|^2\, d\mu\Big)^{1/2}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_S |\nabla h|^2\phi^2\, d\mu \le 4\int_S h^2|\nabla\phi|^2\, d\mu &\le \frac{\beta^2}{R^2}\int_{B_p(2R)\backslash B_p(R)} h^2\, d\mu \\ &\le \frac{\beta^2}{R^2}\Big(\sup_{B_p(2R)} h^2\Big)V_f(B_p(2R)) \le \frac{\gamma}{R^2}\Big(\sup_{B_p(2R)} h^2\Big)V_f(B_p(R))\end{aligned}$$

for a positive constant γ, where in the last inequality we used (2.3).

Taking into account (3.1),

$$\lim_{R\to\infty}\frac{\beta}{R^2V_f(B_p(R))}\int_{B_p(R)} |\nabla h|^2\, d\mu = 0,$$

so that, by (3.2), $|\nabla h| = 0$ on S, that is, S is a slice of $\overline{M}_f$. But, since we are assuming that the fibre M is noncomplete, this cannot occur.

Remark 3.1. Munteanu and Wang [15] already established that an f-harmonic function of sublinear growth in a complete noncompact weighted manifold with bounded weight function f must be constant. They used a somewhat different mean-value inequality (see [15, (3.14)]).

Acknowledgement

The authors would like to thank the referee for his/her valuable comments.

References

[1] A. L. Albujer, 'New examples of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$', *Differ. Geom. Appl.* **26** (2008), 456–462.

[2] J. A. Aledo, A. Romero and R. M. Rubio, 'Constant mean curvature spacelike hypersurfaces in Lorentzian warped products and Calabi–Bernstein type problems', *Nonlinear Anal.* **106** (2014), 57–69.

[3] L. J. Alías and A. G. Colares, 'Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson–Walker spacetimes', *Math. Proc. Cambridge Philos. Soc.* **143** (2007), 703–729.

[4] D. Bakry and M. Émery, 'Diffusions hypercontractives', in: *Séminaire de Probabilités, XIX, 1983/84*, Lecture Notes in Mathematics, 1123 (Springer, Berlin, 1985), 177–206.

[5] E. Calabi, 'Examples of Bernstein problems for some nonlinear equations', *Proc. Sympos. Pure Math.* **15** (1970), 223–230.

[6] J. S. Case, 'Singularity theorems and the Lorentzian splitting theorem for the Bakry–Émery–Ricci tensor', *J. Geom. Phys.* **60** (2010), 477–490.

[7] N. Charalambous and Z. Lu, 'The L^1 Liouville property on weighted manifolds', in: *Complex Analysis and Dynamical Systems VI: Part 1: PDE, Differential Geometry, Radon Transform*, Contemporary Mathematics, 653 (American Mathematical Society, Providence, RI, 2015), 65–80.

[8] S. Y. Cheng and S. T. Yau, 'Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces', *Ann. of Math.* **104** (1976), 407–419.

[9] G. J. Galloway, 'The Lorentzian splitting theorem without the completeness assumption', *J. Differential Geom.* **29** (1989), 373–387.

[10] G. J. Galloway and E. Woolgar, 'Cosmological singularities in Bakry–Émery spacetimes', *J. Geom. Phys.* **86** (2014), 359–369.

[11] M. Gromov, 'Isoperimetry of waists and concentration of maps', *Geom. Funct. Anal.* **13** (2003), 178–215.

[12] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, London–New York, 1973).

[13] D. Impera, S. Pigola and A. G. Setti, 'Potential theory for manifolds with boundary and applications to controlled mean curvature graphs', *J. reine angew. Math.*, to appear, doi:10.1515/crelle-2014-0137.

[14] A. Lichnerowicz, 'Variétés Riemanniennes à tenseur C non négatif', *C. R. Acad. Sci. Paris A–B* **271** (1970), A650–A653.

[15] O. Munteanu and J. Wang, 'Smooth metric measure spaces with non-negative curvature', *Comm. Anal. Geom.* **19** (2011), 451–486.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic Press, New York, 1983).

[17] M. Rupert and E. Woolgar, 'Bakry–Émery black holes', *Classical Quantum Gravity* **31** (2014), Article ID 025008, 16 pages.

[18] L. Saloff-Coste, *Aspects of Sobolev-type Inequalities*, London Mathematical Society Lecture Note Series, 289 (Cambridge University Press, Cambridge, 2002).

[19] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds* (Springer, Netherlands, 1994).

[20] E. Woolgar, 'Scalar–tensor gravitation and the Bakry–Émery–Ricci tensor', *Classical Quantum Gravity* **30** (2013), Article ID 085007, 8 pages.

ARLANDSON M. S. OLIVEIRA, Departamento de Matemática,
Universidade Federal de Campina Grande,
58429-970 Campina Grande, Paraíba, Brazil
e-mail: arlandsonm@gmail.com

HENRIQUE F. DE LIMA, Departamento de Matemática,
Universidade Federal de Campina Grande,
58429-970 Campina Grande, Paraíba, Brazil
e-mail: henrique@mat.ufcg.edu.br