# The Relation between the Distances of a Point from Three Vertices of a Regular Polygon.

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§1. The following notes owe their origin to a consideration of the problem stated at the beginning of §2. The solution there given was shown to me first by Mr J. T. Mossop, a student of mine, and the result appeared capable of extension. The relations obtained are shown below.

§2. *Given the distance of a point inside an equilateral triangle from each of the three vertices to find the side.*

Analyse the problem.

O is a point inside the equilateral triangle ABC, and its distances from A, B, C respectively are $\alpha$, $\beta$, $\gamma$ (Fig. 1.)

On $\beta$ describe an equilateral triangle OBX, so that OBC is part of one of its angles.

Then

$\triangle AOB = \triangle CBX$ (I. 4),

so that

$CX = AO$ and the $\triangle COX$

has its sides equal to $\alpha$, $\beta$, $\gamma$.

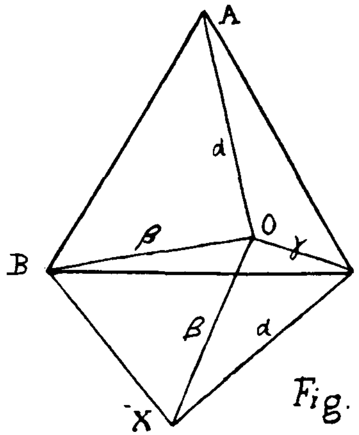


Fig. 1.

Hence we have the construction.

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Describe a triangle OXC whose sides are equal to $\alpha$, $\beta$, $\gamma$ respectively.

On OX describe an equilateral triangle XBO on the side opposite to C.

Then BC is a side of the equilateral triangle sought.

The solution is unique, as can be seen by comparing two equilateral triangles which have the same $\alpha\beta\gamma$; this uniqueness gives rise to the proposition—"*If on the sides of any triangle equilateral triangles be described outwards, the lines joining each vertex of the original triangle to the opposing vertex of an equilateral triangle are all equal.*"

The length of the side BC can be got in terms of $\alpha$, $\beta$, $\gamma$ as follows :—

$$BC^2 = \beta^2 + \gamma^2 - 2\beta\gamma\cos(60^\circ + XOC) \text{ and } \cos XOC = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$$

whence ultimately

$$BC^2 = \tfrac{1}{2}\left[\alpha^2 + \beta^2 + \gamma^2 + \sqrt{3(\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(\alpha-\beta+\gamma)(\alpha+\beta-\gamma)}\right].$$

The sign of the radical is here positive, arising as a factor in the sine of an angle of a triangle.

The function on the right hand side is constant for all points inside the equilateral triangle ABC.

§3. In the case where the point is well outside the triangle the previous analysis holds, as shown in Fig. 2.

The corresponding change in the synthesis is that the equilateral triangle on OX is described on the same side as C.

The change in construction is, however, not defined by the fact that O is outside the triangle.

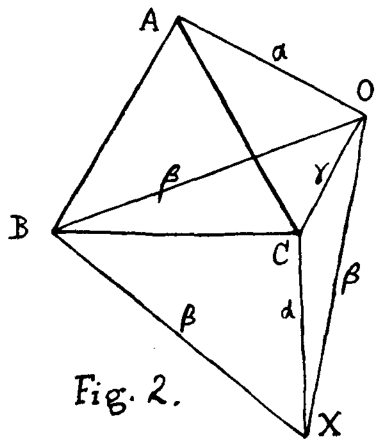


Fig. 2.



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In Fig. 3 the triangle OCX and OBX are on the opposite sides of OX. Cases on the border line between one construction and the other are such that OCX is a straight line. In that case $\angle$ BOC is 60°, and B, A, O, C are concyclic.

We have simultaneously $\alpha+\gamma=\beta$, which gives the proposition: "*For any point on the circumscribing circle of an equilateral triangle the sum of its distances from the two adjacent vertices equals its distance from the third vertex.*" The only case of failure of the construction is when C and B coincide, *i.e.*, $\alpha$, $\beta$ and $\gamma$ equal. The equilateral triangle becomes a point.

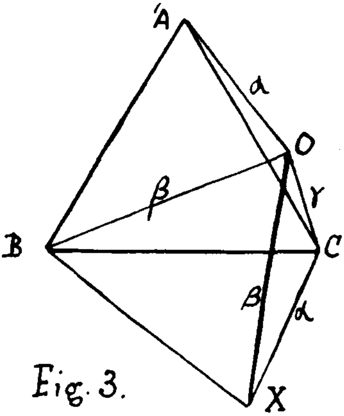


Fig. 3.

If no restriction is made as to whether the point is to be inside or outside the equilateral triangle two solutions are, in general, possible. For certain cases neither point is *inside* the equilateral triangle, the fact being that of the two points one is inside and the other outside the circumscribing circle of the triangle.

For a point on a side of the equilateral triangle a relation obtains of the type $\gamma=\sqrt{\alpha^2+\alpha\beta+\beta^2}$; and for a point in the region between a side and the adjacent arc of the circumscribing circle $\alpha+\beta>\gamma>\sqrt{\alpha^2+\alpha\beta+\beta^2}$; hence the result; *If a relation of the type* $\alpha+\beta>\gamma>\sqrt{\alpha^2+\alpha\beta+\beta^2}$ *holds there is no equilateral triangle whose vertices are distant* $\alpha$, $\beta$, $\gamma$ *from a point inside the triangle.*

§4. Certain inequality theorems may be stated.

(1) Of the three distances of a point from the vertices of an equilateral triangle the sum of any two is not less than the third.

(2) If $a$, $b$, $c$ be the sides of a triangle

$$(a^2+b^2+c^2)^2>3(a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

§5. Consider a similar problem with regard to a square, viz.: *Given the distances of a point from three vertices of a square, to construct the square.*

Proceed again by analysis.

ABCD is the square; $\alpha$, $\beta$, $\gamma$ the distances of O from the points A, B, C.

Draw BX $\perp$ to BO and cut off BX = BO.

Then $\triangle$CBX = $\triangle$OBA and CX = AO.

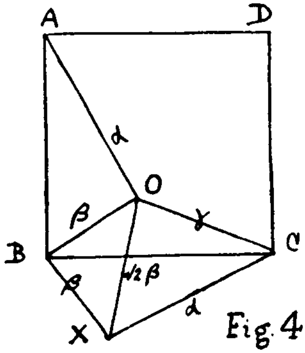


Fig. 4.

Thus $\triangle$XOC has sides $\alpha$, $\beta\sqrt{2}$, $\gamma$.

Hence the construction:—Make a triangle COX, whose sides are $\alpha$, $\beta\sqrt{2}$, $\gamma$, OX corresponding to $\beta\sqrt{2}$, and on OX as base describe an isosceles right angled triangle OBX. Then CB is a side of the square.

Considerations similar to those in §3 will affect the uniqueness of the solution; but in this case we must have the additional information—which of the distances corresponds to a vertex lying between the vertices corresponding to the other two distances.

The length of BC determined as in §2 is given by

$$BC^2 = \tfrac{1}{2}(\alpha^2+\gamma^2) + \tfrac{1}{2}\sqrt{(\alpha+\beta\sqrt{2}+\gamma)(-\alpha+\beta\sqrt{2}+\gamma)(\alpha-\beta\sqrt{2}+\gamma)(\alpha+\beta\sqrt{2}-\gamma}$$
$$= \tfrac{1}{2}(\alpha^2+\gamma^2) + \tfrac{1}{2}\triangle_\beta, \text{ say.}$$

Hence if $\delta$ be the distance of the fourth vertex D from O, we have

$$\alpha^2+\gamma^2+\triangle_\beta = \beta^2+\delta^2+\triangle_\gamma = \gamma^2+\alpha^2+\triangle_\delta = \delta^2+\beta^2+\triangle_\alpha,$$

whence $\qquad \triangle_\beta = \triangle_\delta$ and $\triangle_\alpha = \triangle_\gamma$.

But further $\qquad \alpha^2+\gamma^2 = \beta^2+\delta^2$

$$\therefore\ \triangle_\alpha = \triangle_\beta = \triangle_\gamma = \triangle_\delta$$

giving the curious proposition:— *If lines be drawn from a point to the vertices of a square in order, then the triangles whose sides are any two non consecutive lines of the above and $\sqrt{2}$ times either of the others are equal in area.*

The other properties of the figure follow so closely those of the equilateral triangle that it is not thought necessary to reproduce them here.

§6. An exactly similar problem can be worked out for a regular $n$-gon; it will be sufficient to write down the value obtained for twice the square of the side, viz. :—

$$d_{r-1}^2 - 2d_r^2 \cdot \cos\frac{2\pi}{n} + d_{r+1}^2$$

$$+ \tan\frac{\pi}{n}\sqrt{(d_{r-1} + d_r' + d_{r+1})(-d_{r-1} + d_r' + d_{r+1})(d_{r-1} - d_r' + d_{r+1})(d_{r-1} + d_r' - d_{r+1})}$$

where $d_{r-1}$, $d_r$, $d_{r+1}$ are the distances of the point from three successive vertices, and $d_r' = 2d_r\cos\frac{\pi}{n}$. This gives an invariant for three such vertices.

§7. Examine finally the general case of a regular $n$-gon where the data refer to vertices not necessarily consecutive. Let the vertices be numbered in order from a particular one, and let $d_r$, $d_s$, $d_t$ be the distances of a point from the $r^{\text{th}}$, $s^{\text{th}}$, and $t^{\text{th}}$ vertices respectively.

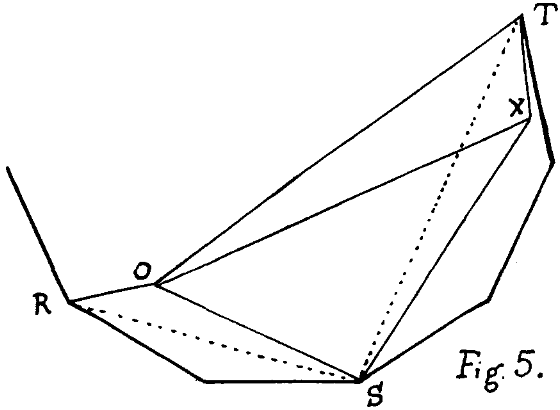

Fig. 5.

At S in OS make $\angle$ OSX equal to $\angle$ RST, and cut off a length SX such that $\frac{SX}{OS} = \frac{ST}{RS}$.

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Then $\triangle TXS$ is similar to $\triangle ORS$ $\therefore \frac{TX}{OR} = \frac{TS}{RS}$

*i.e.*, in the $\triangle OTX$, $OT = d_t$, $OX = d_s . \frac{RT}{RS}$, $TX = d_r . \frac{ST}{RS}$.

Thus from $d_r$, $d_s$ and $d_t$ the triangle OTX and thence the whole figure can be constructed (provided it be possible to draw a regular $n$-gon).

The value for the square of the side has been worked out as in the earlier cases, there being here, however, a greater degree of complication which it is unnecessary to reproduce; the result is

$$\frac{-\sin^2\frac{\pi}{n}}{2\sin(s-r)\frac{\pi}{n}\sin(t-s)\frac{\pi}{n}\sin(r-t)\frac{\pi}{n}}$$
$$\times\left\{\Sigma d_r^2\sin(t-s)\frac{2\pi}{n} + \sqrt{\Pi\left[\pm d_r\sin(t-s)\frac{\pi}{n} \pm d_s\sin(r-t)\frac{\pi}{n} \pm d_t\sin(s-r)\frac{\pi}{n}\right]}\right\}$$

giving an invariant for the distances of any point from the stated vertices of a regular $n$-gon.