ANALYTICAL THEORY OF A TRAPPING IN A TWO-BODY PROBLEM OF VARIABLE MASS.

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SUMMARY.

The new nonstationary model problem is considered. Its solution generalizes by form the known particular Mestschersky-Vinti solution in a two-body problem of variable mass. The equations of the corresponding perturbed motion are deduced. In the case of a two-body problem of variable mass μ the perturbing force is proportional to second temporal derivative from the value μ^{-1}. It is possible to describe with a good approximation such qualitative effects in this problem as a trapping and disintegration on a basis of properties of the model problem. Let us consider the example of a trapping.

INTRODUCTION.

Let us consider the two-body problem of variable mass $\mu(t)$

$$\frac{d^2\vec{z}}{dt^2} = -\mu(t)\frac{\vec{z}}{z^3} \tag{1}$$

Well known is the Mestschersky-Vinti solution (Mestschersky 1893, Vinti 1974) for the following particular form of a mass function

$$\mu^* = \frac{1}{\alpha + \beta t} \tag{2}$$

where α, β - constants. In polar coordinates z, ϑ in this case we have:

$$z = \frac{p}{\mu^*(1 + e\cos\varphi)}, \quad z^2\frac{d\vartheta}{dt} = \sqrt{p}, \quad \varphi = \vartheta - \omega,$$

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where ρ , e , ω - constants, accordingly

$$\vec{r} = \frac{\rho}{\mu^* (1 + e \cos\varphi)} \vec{e}_r$$
$$\frac{d\vec{r}}{dt} = \frac{-1}{\mu^*} \frac{d\mu^*}{dt} r \vec{e}_r + \frac{\mu^*}{\sqrt{\rho}} e \sin\varphi \vec{e}_r + \frac{\mu^*}{\sqrt{\rho}} (1 + e \cos\varphi) \vec{e}_n \tag{3}$$

where $\vec{e}_r$ - single radius-vector, $\vec{e}_n$ - transversal single vector.

If we expand the inverse value of arbitrary mass $\mu(t)$ into Tailor series by the time and restrict it by the linear part of this expansion,

$$\frac{1}{\mu} = \left(\frac{1}{\mu}\right)_{t=t_0} + \left[\frac{d}{dt}\left(\frac{1}{\mu}\right)\right]_{t=t_0} (t - t_0) , \tag{4}$$

then we'll have the law (2) for μ . Let us generalize the solution (3) by form for the case of arbitrary mass $\mu(t)$:

$$\vec{r} = \frac{\rho}{\mu (1 + e \cos\varphi)} \vec{e}_r$$
$$\frac{d\vec{r}}{dt} = -\frac{1}{\mu} \frac{d\mu}{dt} r \vec{e}_r + \frac{\mu}{\sqrt{\rho}} e \sin\varphi \vec{e}_r + \frac{\mu}{\sqrt{\rho}} (1 + e \cos\varphi) \vec{e}_n \tag{5}$$

It is possible to expect that there will be a solution of the equation of the following appearance:

$$\frac{d^2\vec{r}}{dt^2} = -\mu \frac{\vec{r}}{r^3} + \vec{f} \tag{6}$$

where the vector-function $\vec{f}$ is proportional to second derivative of the value μ^{-1}.

MODEL PROBLEM.

Let us consider the equation:

$$\frac{d^2\vec{r}}{dt^2} = -\mu \frac{\vec{r}}{r^3} + \mu \vec{r} \frac{d^2}{dt^2}\left(\frac{1}{\mu}\right) \tag{7}$$

It is easy to find its integrals:

$$\left[\vec{r} \times \frac{d\vec{r}}{dt}\right] = \overrightarrow{const} = \vec{K} \tag{8}$$

$$\frac{1}{\mu^4}\left[\left(\frac{d}{dt}\mu\vec{r}\right)^2 - 2\frac{\mu^3}{r}\right] = const = h\ , \tag{9}$$

$$\frac{1}{\mu^4}\left[\frac{d}{dt}\mu\vec{r}\times\left[\mu\vec{r}\times\frac{d}{dt}\mu\vec{r}\right]\right] - \frac{\vec{r}}{r} = \overrightarrow{const} = \vec{q} \tag{10}$$

Let us rewrite the equation (7) in the following way:

$$\frac{d^2}{dt^2}\mu\vec{r} = -\mu^4\frac{\mu\vec{r}}{(\mu r)^3} + \frac{1}{2\mu^4}\cdot\frac{d\mu^4}{dt}\cdot\frac{d}{dt}\mu\vec{r} \tag{11}$$

In polar coordinates r , ϑ equations (8) and (11) have the form:

$$\mu^2 r^2\frac{d\vartheta}{dt} = \mu^2 K\ , \tag{12}$$

$$\frac{d^2}{dt^2}\mu r - \mu^2 r^2\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{\mu^4}{(\mu r)^2} + \frac{1}{2\mu^4}\cdot\frac{d\mu^4}{dt}\cdot\frac{d}{dt}\mu r \tag{13}$$

From here the equation follows:

$$\frac{d^2}{d\vartheta^2}\left(\frac{1}{\mu r}\right) + \frac{1}{\mu r} = \frac{1}{K^2} \tag{14}$$

We can write the general solution of this equation:

$$\mu r = \frac{p}{1 + e\cos(\vartheta - \omega)}\ , \qquad p = K^2 \tag{15}$$

It is easy to determine the connection between constants of integration e and ω and values (9) and (10). In particular, for the constant e we have:

$$e = q\ , \quad e^2 - 1 = hq \tag{16}$$

Let us introduce the designations :

$$V_r = \frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt}\ , \quad V_n = r\frac{d\varphi}{dt}\ , \quad \varphi = \vartheta - \omega \tag{17}$$

From the equation of orbit (15) and integral (12) we find:

$$V_r = -\frac{1}{\mu}\frac{d\mu}{dt}r + \frac{\mu}{\sqrt{p}}e\sin\varphi\ , \quad V_n = \frac{\mu}{\sqrt{p}}(1 + e\cos\varphi) \tag{18}$$

Hence, the equation (7) possesses the general solution of the form (5).

PERTURBED MOTION.

Let us consider now the perturbed motion

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} + \mu\vec{r}\,\frac{d^2}{dt^2}\left(\frac{1}{\mu}\right) + \vec{F} \tag{19}$$

where vector-function $\vec{F}$ in a general case depends on $\vec{r}$, $d\vec{r}/dt$ and t.

Changing constants in a solution of the form (5), we have:

$$\frac{dp}{dt} = \frac{2p^{3/2}}{\mu(1+e\cos\varphi)}\,F_n\,, \tag{20}$$

$$\frac{de}{dt} = \frac{\sqrt{p}}{\mu}\sin\varphi\,F_r + \frac{\sqrt{p}}{\mu}\left(\cos\varphi + \frac{e+\cos\varphi}{1+e\cos\varphi}\right)F_n\,, \tag{21}$$

$$\frac{d\omega}{dt} = -\frac{\sqrt{p}}{\mu e}\cos\varphi\,F_r + \frac{\sqrt{p}}{\mu e}\left(\sin\varphi + \frac{\sin\varphi}{1+e\cos\varphi}\right)F_n - \frac{\sqrt{p}}{\mu}\,\frac{\sin\vartheta\,\mathrm{Ctg}\,i}{1+e\cos\varphi}\,F_3\,, \tag{22}$$

$$\frac{d\varphi}{dt} = \frac{\mu^2}{p^{3/2}}(1+e\cos\varphi)^2 + \frac{\sqrt{p}}{\mu e}\cos\varphi\,F_r - \frac{\sqrt{p}}{\mu e}\left(\sin\varphi + \frac{\sin\varphi}{1+e\cos\varphi}\right)F_n \tag{23}$$

where F_r, F_n, F_3 - projections of $\vec{F}$ in a mobile orthogonal trihedron $\vec{e}_r$, $\vec{e}_n$, $\vec{e}_3 = \vec{e}_r \times \vec{e}_n$. Let us add here equations describing the variation of an ascending node (Ω) and inclination (i) of orbit:

$$\frac{d\Omega}{dt} = \frac{\sqrt{p}}{\mu}\,\frac{\sin\vartheta}{1+e\cos\varphi}\,F_3\,, \tag{24}$$

$$\frac{di}{dt} = \frac{\sqrt{p}}{\mu}\,\frac{\cos\vartheta}{1+e\cos\varphi}\,F_3 \tag{25}$$

For the equation (1) of the two-body problem of variable mass the perturbing force has the appearance:

$$\vec{F} = -\mu\vec{r}\,\frac{d^2}{dt^2}\left(\frac{1}{\mu}\right) \tag{26}$$

Accordingly, we have:

$$\frac{dp}{dt} = 0\,, \tag{27}$$

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$$\frac{de}{dt} = -\frac{\sin\varphi}{1+e\cos\varphi}\cdot\frac{p^{3/2}}{\mu}\,\frac{d^2}{dt^2}\left(\frac{1}{\mu}\right), \tag{28}$$

$$\frac{d\omega}{dt} = \frac{\cos\varphi}{e(1+e\cos\varphi)}\cdot\frac{p^{3/2}}{\mu}\cdot\frac{d^2}{dt^2}\left(\frac{1}{\mu}\right), \tag{29}$$

$$\frac{d\varphi}{dt} = \frac{\mu^2}{p^{3/2}}(1+e\cos\varphi)^2 - \frac{d\omega}{dt}\,. \tag{30}$$

ON THE TRAPPING IN A TWO-BODY PROBLEM OF VARIABLE MASS.

There is known the following system of elements of osculating conic section in a two-body problem of variable mass (Hadjidemetriou, 1967):

$$\frac{d\tilde{p}}{dt} = -\frac{\tilde{p}}{\mu}\cdot\frac{d\mu}{dt}, \tag{31}$$

$$\frac{d\tilde{e}}{dt} = -(\tilde{e} + \cos\tilde{\varphi})\cdot\frac{1}{\mu}\cdot\frac{d\mu}{dt}, \tag{32}$$

$$\frac{d\tilde{\omega}}{dt} = -\frac{\sin\tilde{\varphi}}{\tilde{e}}\,\frac{1}{\mu}\cdot\frac{d\mu}{dt}, \tag{33}$$

$$\frac{d\tilde{\varphi}}{dt} = \sqrt{\frac{\mu}{\tilde{p}}}\cdot\frac{(1+\tilde{e}\cos\tilde{\varphi})^2}{\tilde{p}} - \frac{d\tilde{\omega}}{dt} \tag{34}$$

Model problem here is aperiodical motion along a conic section

$$\vec{r} = \frac{\tilde{p}}{1+\tilde{e}\cos\tilde{\varphi}}\,\vec{e}_r \tag{35}$$

$$\vec{V} = \sqrt{\frac{\mu}{\tilde{p}}}\left[\tilde{e}\sin\tilde{\varphi}\,\vec{e}_r + (1+\tilde{e}\cos\tilde{\varphi})\,\vec{e}_n\right] \tag{36}$$

which is described by the equation of the form:

$$\frac{d^2\vec{r}}{dt^2} = -\mu\,\frac{\vec{r}}{r^3} + \frac{1}{2\mu}\cdot\frac{d\mu}{dt}\cdot\frac{d\vec{r}}{dt} \tag{37}$$

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The corresponding perturbed force has the following structure:

$$\vec{F} = -\frac{1}{2\mu} \cdot \frac{d\mu}{dt} \frac{d\vec{r}}{dt} \tag{38}$$

This interpretation of the two-body problem of variable mass has been made simultaneously and independently by Hadjidemetriou (1963) and Omarov (1963).

The parameter $\tilde{p}$ of osculating conic section changes inverse proportionally to mass μ. As it is clear from the formula (27), the value p in the two-body problem of variable mass is constant . Besides that, values de/dt and $d\omega/dt$ in this problem have a higher order of small magnitude comparatively with velocities of variation of the osculating eccentricity $\tilde{e}$ of the element $\tilde{\omega}$. In consequence of that fact unperturbed motion (5) and its integrals (8)-(10) can be considered as good approximate correlations of the two-body problem of variable mass when describing such qualitative effects, as trapping and disintegration.

Let us consider examples of trapping for the case when in the solution (5) for the increasing mass the constant value $e = 0$, and, accordingly, we have:

$$h = -\frac{1}{p}, \quad r = \frac{p}{\mu}, \quad p = \mu_0 r_0 \tag{39}$$

Let us rewrite the integral (9) in the following form:

$$V^2 - 2\frac{\mu}{r} = h\mu^2 - 2\frac{\dot{\mu}}{\mu}\dot{r}r - \frac{\dot{\mu}^2}{\mu^2} r^2 \tag{40}$$

where the point designates a differentiation by time. For our case we have:

$$V^2 - 2\frac{\mu}{r} = -\frac{\mu^2}{p}\left(1 - p^3\frac{\dot{\mu}^2}{\mu^6}\right) \tag{41}$$

Let, in a general initial moment of time t_0, the energy of the corresponding binary system of increasing mass be a positive one

$$V_0^2 - 2\frac{\mu_0}{r_0} > 0\,. \tag{42}$$

With due regard for $e = 0$, simultaneously, the inequality must be carried out that

$$\frac{\dot{\mu}_0^2}{\mu_0^4} > \frac{1}{r_0^3} \tag{43}$$

As it is clear from the formula (41), for the next trapping

$$V^2 - 2\frac{\mu}{r} < 0 \qquad (44)$$

accomplishment of the following condition is necessary

$$\frac{d}{dt}\left(\frac{\dot{\mu}^2}{\mu^6}\right) < 0 . \qquad (45)$$

If the mass μ increases according to Eddington-Jean's law

$$\dot{\mu} = \alpha \mu^n , \quad \alpha > 0 \qquad (46)$$

then the necessary condition of a trapping (45) is fulfilled for meaning $n < 3$. The following expression results from the law (46):

$$\mu(t) = \left[\alpha(1-n)(t-t_0) + \mu_0^{1-n}\right]^{\frac{1}{1-n}} , \quad n \neq 1 \qquad (47)$$

and, hence, for the case $n < 1$, a trapping really takes place in the time interval

$$\Delta t = \frac{\left(\alpha\, r_0^{3/2} \mu_0^{3/2}\right)^{\frac{n-1}{n-3}} - \mu_0^{1-n}}{\alpha(1-n)} \qquad (48)$$

When $n = 1$, for the time of trapping we have:

$$\Delta t = \frac{1}{2\alpha} \ln\left(\frac{\mu_0^{1/2}}{\alpha\, r_0^{3/2}}\right) \qquad (49)$$

Finally, the restriction (43), connected with the condition $e = 0$, excluded a trapping for cases, when $1 < n < 3$.

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