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PROOF OF THE DETERMINANTAL FORM OF THE SPONTANEOUS MAGNETIZATION OF THE SUPERINTEGRABLE CHIRAL POTTS MODEL

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Abstract

The superintegrable chiral Potts model has many resemblances to the Ising model, so it is natural to look for algebraic properties similar to those found for the Ising model by Onsager, Kaufman and Yang. The spontaneous magnetization $\mathcal{M}_r$ can be written in terms of a sum over the elements of a matrix S_r. The author conjectured the form of the elements, and this conjecture has been verified by Iorgov *et al.* The author also conjectured in 2008 that this sum could be expressed as a determinant, and has recently evaluated the determinant to obtain the known result for $\mathcal{M}_r$. Here we prove that the sum and the determinant are indeed identical expressions. Since the order parameters of the superintegrable chiral Potts model are also those of the more general solvable chiral Potts model, this completes the algebraic calculation of $\mathcal{M}_r$ for the general model.

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1. Introduction

The first exact solution of a two-dimensional statistical mechanical model of an interacting system was Onsager's derivation [18] of the free energy of the Ising model in 1944. This is a model of spins living on a square lattice, each having two possible states ("up" or "down"), and interacting ferromagnetically with their neighbours. The model has a phase transition: there is a critical temperature T_c such that the system is ferromagnetically ordered at temperatures $T < T_c$, disordered for $T \geq T_c$.

For $T < T_c$, there is a nonzero spontaneous magnetization $\mathcal{M}_1$ (also known as the order parameter). The result for this was announced by Onsager in 1949 [19], and a derivation published by Yang in 1952 [20]. This and subsequent derivations were explicit algebraic calculations, applied to a system of finite size. Only at the end of the calculation was the desired thermodynamic limit taken of an infinite lattice.

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Since then many other two-dimensional models have been solved that also have order–disorder transitions. Notable amongst these is the solvable chiral Potts model, which can be regarded as an N-state generalization of the two-state Ising model. Its free energy was calculated in 1988 [5, 6] and in 1989 it was conjectured [1] that the order parameters $\mathcal{M}_1, \ldots, \mathcal{M}_{N-1}$ are given by

$$\mathcal{M}_r = (1 - k'^2)^{r(N-r)/2N^2} \tag{1.1}$$

for $r = 1, \ldots, N-1$ and $0 < k' < 1$. Here k' is a temperature-like parameter, increasing from 0 to ∞ as T increases from 0 to ∞, and having value $k' = 1$ at $T = T_c$.

It was not until 2005 [7, 8] that a derivation was obtained of that conjecture. The method used was analytic: one obtained equations for a generalization of $\mathcal{M}_r$ in the infinite lattice limit, and solved them. This necessarily involves plausible assumptions that the desired functions are analytic in certain domains.

However, the order parameters $\mathcal{M}_r$ are known to depend only on N, r and k', and not on other "rapidity" parameters in the model. There are special values of these extra parameters where the model becomes "superintegrable", and it has recently been shown that the model is then amenable to explicit algebraic calculations that parallel those of the Ising model.

By considering the square lattice of finite width L, the author [9–11] showed that $\mathcal{M}_r$ could be expressed in terms of a sum over the elements of a 2^m-by-$2^{m'}$ matrix S_r. In [11] he conjectured a formula for these elements as simple products. He also conjectured in [10, 11] that the sum could be written as the determinant D_{PQ} of an m-dimensional (or m'-dimensional) matrix. These P, Q, r are related by $Q = P + r$ (mod N), and (for $0 \le P, Q < N$)

$$m = \left[\frac{(N-1)L - P}{N} \right], \quad m' = \left[\frac{(N-1)L - Q}{N} \right] \tag{1.2}$$

where $[x]$ is the integer part of x. Note that these definitions imply

$$|m - m'| \le 1. \tag{1.3}$$

In a recent paper [12], the author showed that this determinant can be written as the product (or ratio) of Cauchy-like determinants, so is also a simple product. Taking the limit $L \to \infty$, one does indeed regain the known result (1.1).

This still left open the two conjectures, the first being the product expression for the elements of S_r. The matrix S_r satisfies two commutation relations. In unpublished work, the author was able to prove that the conjectured form did in fact satisfy these relations, and from numerical studies for small values of N, L it appeared that there was only one such solution to these linear equations, but this fell short of a proof. The problem of calculating such matrix elements has been studied more directly by Au-Yang and Perk [2–4]. Iorgov *et al.* [14] have now given a proof, and have gone on to calculate $\mathcal{M}_r$ directly.

The second conjecture was that the sum over the elements of S_r was the determinant D_{PQ}. This is the problem we address here. It is a very self-contained problem, being just an algebraic identity between rational functions of many variables. It completes the algebraic proof of the formula (1.1) via the determinant D_{PQ}.

We shall refer to papers [9–12] as papers I, II, III, IV, respectively, and quote their equations accordingly.

2. Definitions

Let $c_1, \ldots, c_m, y_1, \ldots, y_m$ and $c'_1, \ldots, c'_n, y'_1, \ldots, y'_n$ be sets of variables, where $n = m'$ and m, m' are the integers mentioned above. In this paper we do *not* use the above definitions (1.2), nor the integers N, L, P, Q. We can take m, m' to both be arbitrary positive integers, and the c_i, y_i, c'_i, y'_i to be arbitrary variables. In this paper we can and do allow m and m' to be arbitrary. We *ignore* the restriction (1.3).

Let $s = \{s_1, \ldots, s_m\}$ be a set of m integers with values

$$s_i = 0 \text{ or } 1 \quad \text{for } 1 \leq i \leq m.$$

Similarly, let $s' = \{s'_1, \ldots, s'_n\}$, where each $s'_i = 0$ or 1. Set

$$\kappa_s = s_1 + \cdots + s_m, \quad \kappa_{s'} = s'_1 + \cdots + s'_n. \tag{2.1}$$

For a given set s, let V be the set of integers i such that $s_i = 0$ and W the set such that $s_i = 1$. Hence, from (2.1), V has $m - \kappa_s$ elements, while W has κ_s. Define V', W' similarly for the set s', so V' has $n - \kappa_{s'}$ elements, while W' has $\kappa_{s'}$.

Define

$$\begin{aligned} A_{s,s'} &= \prod_{i \in W} \prod_{j \in V'} (c_i - c'_j), \quad & B_{s,s'} &= \prod_{i \in V} \prod_{j \in W'} (c_i - c'_j), \\ C_s &= \prod_{i \in W} \prod_{j \in V} (c_j - c_i), \quad & D_{s'} &= \prod_{i \in V'} \prod_{j \in W'} (c'_j - c'_i). \end{aligned} \tag{2.2}$$

Then the afore-mentioned matrix S_r has elements $(S_r)_{s,s'}$ which are proportional to $A_{s,s'} B_{s,s'} / (C_s D_{s'})$ and the sum over its elements is given in (III.3.48) as

$$\mathcal{R} = \sum_s \sum_{s'} y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m} \left(\frac{A_{s,s'} B_{s,s'}}{C_s D_{s'}} \right) y_1'^{s'_1} y_2'^{s'_2} \cdots y_n'^{s'_n}, \tag{2.3}$$

the sum being restricted to s, s' such that

$$\kappa_s = \kappa_{s'}. \tag{2.4}$$

Now define $a_i, \ldots, a_m, b_1, \ldots b_m$ and $a'_i, \ldots, a'_n, b'_1, \ldots b'_n$ by

$$a_i = \prod_{j=1}^{n} (c_i - c'_j), \quad a'_i = \prod_{j=1}^{m} (c'_i - c_j), \tag{2.5}$$

$$b_i = \prod_{j=1, j \neq i}^{m} (c_i - c_j), \quad b'_i = \prod_{j=1, j \neq i}^{n} (c'_i - c'_j), \tag{2.6}$$

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and let $\mathcal{B}$ be an m-by-n matrix, and $\mathcal{B}'$ an n-by-m matrix, with elements

$$\mathcal{B}_{ij} = \frac{a_i}{b_i(c_i - c'_j)}, \quad \mathcal{B}'_{ij} = \frac{a'_i}{b'_i(c'_i - c_j)}.$$

Also, define an m-by-m diagonal matrix Y and an n-by-n diagonal matrix Y' by

$$Y_{i,j} = y_i \delta_{ij}, \quad Y'_{i,j} = y'_i \delta_{ij}.$$

Then the determinant mentioned above is

$$\mathcal{D} = D_{PQ} = \det[I_m + Y\mathcal{B}Y'\mathcal{B}'], \tag{2.7}$$

where I_m is the identity matrix of dimension m.

(The definition (2.7) is the same as (II.7.2), (III.4.9), (IV.1.18). If f, f'_i are defined by (IV.2.31) and B_{PQ} by (IV.2.29), and F, F' are the diagonal matrices with elements $F_{i,i} = f_i$, $F'_{i,i} = f'_i$, then $\mathcal{B} = \epsilon F B_{PQ} F'^{-1}$, $\mathcal{B}' = -\epsilon F' B_{QP} F^{-1}$ and $\epsilon^2 = 1$.)

We can also write (2.7) as

$$\mathcal{D} = \det[I_n + Y'\mathcal{B}'Y\mathcal{B}],$$

so both $\mathcal{R}$ and $\mathcal{D}$ are unaltered by simultaneously interchanging m with n, the c_i with the c'_i and the y_i with the y'_i. It follows that without loss of generality, we can choose

$$n \geq m. \tag{2.8}$$

The expressions $\mathcal{R}$, $\mathcal{D}$ are functions of $m, n, c_1, \ldots c_m, y_1, \ldots, y_m, c'_1, \ldots c'_n, y'_1, \ldots, y'_n$. We shall write them as $\mathcal{R}_{mn}$, $\mathcal{D}_{mn}$.

3. Proof that $\mathcal{R}_{mn} = \mathcal{D}_{mn}$

Both $\mathcal{R}_{mn}$ and $\mathcal{D}_{mn}$ are rational functions of $c_1, \ldots c_m, c'_1, \ldots c'_n$. They are symmetric, being unchanged by simultaneously permuting the c_i and the y_i, as well as by simultaneously permuting the c'_i and the y'_i. We find that they are identical, for all c_i, y_i, c'_i, y'_i. The proof proceeds by recurrence, in the following four steps.

3.1. The case $m = 1$

Calculation of $\mathcal{R}_{1n}$ If $m = 1$ then $s = \{s_1\}$ and either $s_1 = 0$ or $s_1 = 1$. In the first case, since $\kappa_s = \kappa_{s'}$, all the s'_i must be zero and the sets W, W' are both empty, so we get a contribution to (2.3) of unity.

In the second case, $s' = \{0, \ldots, 0, 1, 0, \ldots, 0\}$, with the 1 in position r, for $r = 1, \ldots, n$. Then V is empty, so $B_{s,s'} = C_s = 1$, while

$$A_{s,s'} = \prod_{j=1, j\neq r}^{n} (c_1 - c'_j), \quad D_{s'} = \prod_{j=1, j\neq r}^{n} (c'_r - c'_j).$$

From (2.3) it follows that

$$\mathcal{R}_{1n} = 1 + \sum_{r=1}^{n} y_1 y'_r \prod_{j=1, j\neq r}^{n} \frac{c_1 - c'_j}{c'_r - c'_j}.$$

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Calculation of $\mathcal{D}_{1n}$ The right-hand side of (2.7) is a determinant of dimension one, so

$$\begin{aligned}\mathcal{D}_{1n} &= 1 + Y_{1,1} \sum_{r=1}^{n} \mathcal{B}_{1,r} Y'_{r,r} \mathcal{B}'_{r,1} \\ &= 1 - \frac{a_1 y_1}{b_1} \sum_{r=1}^{n} \frac{a'_r y'_r}{b'_r (c_1 - c'_r)^2},\end{aligned} \tag{3.1}$$

where a_i, a'_i, b_i, b'_i are defined by (2.5), (2.6). Note that here $b_1 = 1$.

We therefore obtain

$$\mathcal{D}_{1n} = 1 + \sum_{r=1}^{n} y_1 y'_r \prod_{j=1, j\neq r}^{n} \frac{c_1 - c'_j}{c'_r - c'_j}$$

and we see explicitly that

$$\mathcal{R}_{1n} = \mathcal{D}_{1n}.$$

3.2. Degree of the numerator polynomials Consider $\mathcal{R}_{mn}$ and $\mathcal{D}_{mn}$ as functions of c_m. They are both rational functions. We show here that they are both of the form

$$\frac{\text{polynomial of degree } (n-1)}{b_m}. \tag{3.2}$$

Degree for $\mathcal{R}_{mn}$ First consider the sum $\mathcal{R}_{mn}$ in (2.3) as a function of c_m. Each term is plainly a polynomial divided by b_m. If $m \in W$, then $s_m = 1$ and the numerator is proportional to $A_{s,s'}$. The degree of the numerator is the number of elements of V'. The condition (2.4) means that W' must have at least one element, so V' must have at most $n-1$. The degree of the numerator is therefore at most $n-1$.

If $m \in V$, then the numerator is proportional to $B_{s,s'}$ and the degree of the numerator is the number of elements of W', which from (2.4) is the same as the number of elements of W. Since V has at least one element, W can have at most $m-1$. From (2.8), this is at most $n-1$.

The sum of all the terms in (2.3) is therefore a polynomial in c_m of degree at most $n-1$, divided by b_m, as in (3.2).

Degree for $\mathcal{D}_{mn}$ Now consider the determinant $\mathcal{D}_{mn}$ in (2.7) as a function of c_m. At first sight there appear to be poles at $c_m = c'_j$, coming from $\mathcal{B}_{mj}$. However, they are cancelled by the factor a_m. Similarly, the ones in the element $\mathcal{B}'_{jm}$ of the matrix $\mathcal{B}'$ are cancelled by the factor a'_j. So there are no poles at $c_m = c'_j$, for any j.

There are poles at $c_m = c_i$ (for $1 \leq i < m$) coming from the b_i, b_m factors in $\mathcal{B}_{ij}$, $\mathcal{B}_{mj}$, respectively, so there are simple poles in each of the rows i and m. This threatens to create a double pole in the determinant $\mathcal{D}_{mn}$. However, if $c_m = c_i$, the rows i and m of the matrix $(c_m - c_i)\mathcal{B}$ are equal and opposite. By replacing row i by the sum of the two rows (corresponding to pre-multiplying $\mathcal{B}$ by an elementary matrix), we can eliminate the poles in row i. Hence there is only a single pole at $c_m = c_i$. The determinant is therefore a polynomial in c_m, divided by b_m.

To determine the degree of this polynomial, consider the behaviour of $\mathcal{D}_{mn}$ when $c_m \to \infty$. Then, writing c_m simply as c,

$$\mathcal{B}_{ij} \sim c^{-1} \text{ if } i < m, \quad \mathcal{B}_{ij} \sim c^{n-m} \text{ if } i = m,$$
$$\mathcal{B}'_{ij} \sim c \text{ if } j < m, \quad \mathcal{B}'_{i,j} \sim 1 \text{ if } j = m$$

and hence the orders of the elements of the matrix product in (2.7) are given by

$$Y\mathcal{B}Y'\mathcal{B}' \sim \begin{pmatrix} 1 & 1 & \cdots & 1 & c^{-1} \\ 1 & 1 & \cdots & 1 & c^{-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & c^{-1} \\ c^{n-m+1} & c^{n-m+1} & \cdots & c^{n-m+1} & c^{n-m} \end{pmatrix}.$$

Since $n \geq m$, it follows that $\mathcal{D}_{mn}$ grows at most as

$$\mathcal{D}_{mn} \sim c^{n-m}$$

as $c \to \infty$. The numerator polynomial in (3.2) is therefore of degree at most $n - 1$. This completes the proof of (3.2).

3.3. The case $c_m = c'_n$

The sum $\mathcal{R}_{mn}$ Consider the case when $c_m = c'_n$. If $m \in W$ and $n \in V'$, then from (2.2) $A_{s,s'} = 0$. Similarly, if $m \in V$ and $n \in W'$, then $B_{s,s'} = 0$. So the summand in (2.3) is zero unless either $m \in V$, $n \in V'$, or $m \in W$, $n \in W'$.

In the first instance, $s_m = s'_n = 0$. The AB/CD factor in (2.3) is the same as if we replace m, n by $m - 1, n - 1$, respectively, except for a factor

$$\prod_{i \in W} \frac{c_i - c'_n}{c_m - c_i} \prod_{j \in W'} \frac{c_m - c'_j}{c'_j - c'_n}.$$

Since $c_m = c'_n$, the factors in the product cancel, except for a sign. From (2.4), there are as many elements in W as in W', so the sign products also cancel, leaving unity. Thus this contribution to (2.3) is exactly that obtained by replacing m, n by $m - 1, n - 1$.

In the second instance, $s_m = s'_n = 1$. This time AB/CD has an extra factor

$$\prod_{j \in V'} \frac{c_m - c'_j}{c'_n - c'_j} \prod_{i \in V} \frac{c_i - c'_n}{c_i - c_m} = 1,$$

so this contribution to (2.3) is again that obtained by replacing m, n by $m - 1, n - 1$, except that now there is an extra factor $y_m y'_n$. Adding the two contributions, we see that

$$\mathcal{R}_{mn} = (1 + y_m y'_n)\mathcal{R}_{m-1,n-1}. \tag{3.3}$$

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The determinant $\mathcal{D}_{mn}$ Now look at the determinant (2.7) when $c_m = c'_n$. Since a_m and a'_n both contain the factor $c_m - c'_n$, the mth row of $\mathcal{B}$ vanishes except for the element m, n, which is

$$\mathcal{B}_{m,n} = \frac{\prod_{j=1}^{n-1}(c_m - c'_j)}{\prod_{j=1}^{m-1}(c_m - c_j)}.$$

Similarly, the nth row of $\mathcal{B}'$ vanishes except for

$$\mathcal{B}'_{n,m} = \frac{\prod_{j=1}^{m-1}(c'_n - c_j)}{\prod_{j=1}^{n-1}(c'_n - c'_j)}.$$

Since $c_m = c'_n$, we see that $\mathcal{B}_{m,n}\mathcal{B}'_{n,m} = 1$.

It follows that the matrix in (2.7) has a block-triangular structure:

$$I_m + Y\mathcal{B}Y'\mathcal{B}' = \begin{pmatrix} \mathbf{1} + \mathbf{yby'b'} & \cdots \\ \mathbf{0} & 1 + y_m y'_n \end{pmatrix},$$

where $\mathbf{1}$, $\mathbf{y}$, $\mathbf{b}$, $\mathbf{y}'$, $\mathbf{b}'$ are the matrices I_m, Y, $\mathcal{B}$, Y', $\mathcal{B}'$ with their last rows and columns omitted. Hence

$$\mathcal{D}_{m,n} = (1 + y_m y'_n)\mathcal{D}_{m-1,n-1}. \tag{3.4}$$

3.4. Proof by recurrence The proof now proceeds by recurrence. Suppose that $\mathcal{D}(m-1, n-1) = \mathcal{R}(m-1, n-1)$ for all c_i, c'_i. Then from (3.3) and (3.4) it is true that $\mathcal{D}_{m,n} = \mathcal{R}_{m,n}$ when $c_m = c'_n$. By symmetry it is also true for $c_m = c'_j$ for $j = 1, \ldots, n$. Thus $\mathcal{D}_{m,n} - \mathcal{R}_{m,n}$ is zero for all these n values. However, from Subsection 3.2 above, this difference (times b_m) is a polynomial in c_m of degree $n - 1$. The polynomial must therefore vanish identically, so $\mathcal{R}_{m,n} = \mathcal{D}_{m,n}$ for *arbitrary* values of c_m.

Since it is true for $m = 1$, it follows that

$$\mathcal{R}_{m,n} = \mathcal{D}_{m,n}$$

for *all* m, n.

This proves the second conjecture of [11].

4. Summary

We have proved that the sum $\mathcal{R}$ over the elements of the matrix S_r is identical to the determinant $\mathcal{D}$. In [12] we calculated $\mathcal{D}$ and hence obtained the spontaneous magnetization $\mathcal{M}_r$.

The recent work by Iorgov *et al.* [14] proves that the elements of the matrix S_r are indeed given by (III.3.45), being proportional to $A_{s,s'}B_{s,s'}/(C_s D_{s'})$, so the algebraic calculation of $\mathcal{M}_r$ is now complete.

Further, Iorgov *et al.* [14] go on to calculate $\mathcal{R}$, and hence $\mathcal{M}_r$, directly, thereby avoiding the determinantal formulation altogether. While this last step is efficient, it bypasses the author's original motivation for this work, which was to obtain a derivation of $\mathcal{M}_r$ that more closely resembled the algebraic and combinatorial determinantal calculations for the Ising model of Yang [20], Kac and Ward [15], Hurst and Green [13], and Montroll *et al.* [17] all of whom write the partition function in terms of a determinant or Pfaffian (the square root of an antisymmetric determinant). Indeed, the $\mathcal{D} = D_{PQ}$ of this paper is the immediate generalization of the Ising model determinant, as formulated in (I.7.7) in the first paper of this series.

In fact, it would still be illuminating to obtain a simple and direct derivation of D_{PQ} parallelling Kaufman's spinor operators (Clifford algebra) method for the Ising model [16].

References

[1] G. Albertini, B. M. McCoy, J. H. H. Perk and S. Tang, "Excitation spectrum and order parameter for the integrable N-state chiral Potts model", *Nuclear Phys. B* **314** (1989) 741–763.

[2] H. Au-Yang and J. H. H. Perk, "Quantum loop subalgebra and eigenvectors of the superintegrable chiral Potts transfer matrices", 2009, arXiv0907.0362.

[3] H. Au-Yang and J. H. H. Perk, "Spontaneous magnetization of the integrable chiral Potts model", 2010, arXiv1003.4805.

[4] H. Au-Yang and J. H. H. Perk, "Identities in the superintegrable chiral Potts model", *J. Phys. A: Math. Theor.* **43** (2010) 025203 (10pp).

[5] R. J. Baxter, "Free energy of the solvable chiral Potts model", *J. Stat. Phys.* **52** (1988) 639–667.

[6] R. J. Baxter, "Chiral Potts model: eigenvalues of the transfer matrix", *Phys. Lett. A* **146** (1990) 110–114.

[7] R. J. Baxter, "Derivation of the order parameter of the chiral Potts model", *Phys. Rev. Lett.* **94** (2005) 130602 (3pp).

[8] R. J. Baxter, "The order parameter of the chiral Potts model", *J. Stat. Phys.* **120** (2005) 1–36.

[9] R. J. Baxter, "Algebraic reduction of the Ising model", *J. Stat. Phys.* **132** (2008) 959–982.

[10] R. J. Baxter, "A conjecture for the superintegrable chiral Potts model", *J. Stat. Phys.* **132** (2008) 983–1000.

[11] R. J. Baxter, "Some remarks on a generalization of the superintegrable chiral Potts model", *J. Stat. Phys.* **137** (2009) 798–813.

[12] R. J. Baxter, "Spontaneous magnetization of the superintegrable chiral Potts model: calculation of the determinant D_{PQ}", *J. Phys. A: Math. Theor.* **43** (2010) 145002 (16pp).

[13] C. A. Hurst and H. S. Green, "New solution of the Ising problem for a rectangular lattice", *J. Chem. Phys.* **33** (1960) 1059–1062.

[14] N. Iorgov, V. Shadura, Yu. Tykhyy, S. Pakuliak and G. von Gehlen, "Spin operator matrix elements in the superintegrable chiral Potts quantum chain", *J. Stat. Phys.* **139** (2010) 743–768.

[15] M. Kac and J. C. Ward, "A combinatorial solution of the the two-dimensional Ising model", *Phys. Rev.* **88** (1952) 1332–1337.

[16] B. Kaufman, "Crystal statistics. II. Partition function evaluated by spinor analysis", *Phys. Rev.* **76** (1949) 1232–1243.

[17] E. W. Montroll, R. B. Potts and J. C. Ward, "Correlations and spontaneous magnetization of the two-dimensional Ising model", *J. Math. Phys.* **4** (1963) 308–322.

[18] L. Onsager, "Crystal statistics. I. A two-dimensional model with an order-disorder transition", *Phys. Rev.* **65** (1944) 117–149.

[19] L. Onsager, "'Discussione e observazioni', Proceedings of the IUPAP conference on statistical mechanics", *Nuovo Cimento, Suppl, IX Ser.* **6** (1949) 261.

[20] C. N. Yang, "The spontaneous magnetization of a two-dimensional Ising model", *Phys. Rev.* **85** (1952) 808–816.