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C^*-ALGEBRAS FROM K GROUP REPRESENTATIONS

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Abstract

We introduce certain C^*-algebras and k-graphs associated to k finite-dimensional unitary representations $\rho_1, \ldots, \rho_k$ of a compact group G. We define a higher rank Doplicher-Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$, constructed from intertwiners of tensor powers of these representations. Under certain conditions, we show that this C^*-algebra is isomorphic to a corner in the C^*-algebra of a row-finite rank k graph Λ with no sources. For G finite and ρ_i faithful of dimension at least two, this graph is irreducible, it has vertices $\hat{G}$ and the edges are determined by k commuting matrices obtained from the character table of the group. We illustrate this with some examples when $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is simple and purely infinite, and with some K-theory computations.

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1. Introduction

The study of graph C^*-algebras was motivated, among other reasons, by the Doplicher–Roberts algebra $\mathcal{O}_\rho$ associated to a group representation ρ (see [19, 22]). It is natural to imagine that a rank k graph is related to a fixed set of k representations $\rho_1, \ldots, \rho_k$ satisfying certain properties.

Given a compact group G and k finite-dimensional unitary representations ρ_i on Hilbert spaces $\mathcal{H}_i$ of dimensions d_i for $i = 1, \ldots, k$, we first construct a product system $\mathcal{E}$ indexed by the semigroup $(\mathbb{N}^k, +)$ with fibers $\mathcal{E}_n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$ for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$. Using the representations ρ_i, the group G acts on each fiber of $\mathcal{E}$ in a compatible way, so we obtain an action of G on the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{E})$. This action determines the crossed product $\mathcal{O}(\mathcal{E}) \rtimes G$ and the fixed point algebra $\mathcal{O}(\mathcal{E})^G$.

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Inspired by Section 7 of [19] and Section 3.3 of [1], we define a higher rank Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\dots,\rho_k}$ associated to the representations $\rho_1, \dots, \rho_k$. This algebra is constructed from intertwiners $Hom(\rho^n, \rho^m)$, where $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$ is acting on $\mathcal{H}^n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$ for $n = (n_1, \dots, n_k) \in \mathbb{N}^k$. We show that $\mathcal{O}_{\rho_1,\dots,\rho_k}$ is isomorphic to $\mathcal{O}(\mathcal{E})^G$.

If the representations $\rho_1, \dots, \rho_k$ satisfy some mild conditions, we construct a k-colored graph Λ with vertex space $\Lambda^0 = \hat{G}$, and with edges Λ^{ε_i} given by some matrices M_i indexed by $\hat{G}$. Here $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^k$ with 1 in position i are the canonical generators. For $v, w \in \hat{G}$, the matrices M_i have entries

$$M_i(w, v) = |\{e \in \Lambda^{\varepsilon_i} : s(e) = v, r(e) = w\}| = \dim Hom(v, w \otimes \rho_i),$$

which is the multiplicity of v in $w \otimes \rho_i$ for $i = 1, \dots, k$. Note that the matrices M_i commute because $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ for all $i, j = 1, \dots, k$ and therefore

$$\dim Hom(v, w \otimes \rho_i \otimes \rho_j) = \dim Hom(v, w \otimes \rho_j \otimes \rho_i).$$

By a particular choice of isometric intertwiners in $Hom(v, w \otimes \rho_i)$ for each $v, w \in \hat{G}$ and for each i, we can choose bijections

$$\lambda_{ij} : \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \to \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i},$$

obtaining a set of commuting squares for Λ. For $k \geq 3$, we need to check the associativity of the commuting squares, that is,

$$(id_\ell \times \lambda_{ij})(\lambda_{i\ell} \times id_j)(id_i \times \lambda_{j\ell}) = (\lambda_{j\ell} \times id_i)(id_j \times \lambda_{i\ell})(\lambda_{ij} \times id_\ell)$$

as bijections from $\Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_\ell}$ to $\Lambda^{\varepsilon_\ell} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i}$ for all $i < j < \ell$ (see [14]). If these conditions are satisfied, we obtain a rank k graph Λ, which is row-finite with no sources but, in general, is not unique.

In many situations, Λ is cofinal and it satisfies the aperiodicity condition, so $C^*(\Lambda)$ is simple. For $k = 2$, the C^*-algebra $C^*(\Lambda)$ is unique when it is simple and purely infinite, because its K-theory depends only on the matrices M_1, M_2. It is an open question what happens for $k \geq 3$.

Assuming that the representations $\rho_1, \dots, \rho_k$ determine a rank k graph Λ, we prove that the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\dots,\rho_k}$ is isomorphic to a corner of $C^*(\Lambda)$, so if $C^*(\Lambda)$ is simple, then $\mathcal{O}_{\rho_1,\dots,\rho_k}$ is Morita equivalent to $C^*(\Lambda)$. In particular cases, we can compute its K-theory using results from [11].

2. The product system

Product systems over arbitrary semigroups were introduced by Fowler [13], inspired by work of Arveson, and studied by several authors (see [1, 4, 26]). In this paper, we are mostly interested in product systems $\mathcal{E}$ indexed by $(\mathbb{N}^k, +)$, associated to some representations $\rho_1, \dots, \rho_k$ of a compact group G. We remind the reader of some general definitions and constructions with product systems, but we restrict our attention to the

Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{E})$ and we mention some properties in particular cases only (see Example 2.3 for $P = \mathbb{N}^k$).

DEFINITION 2.1. Let $(P, \cdot)$ be a discrete semigroup with identity e and let A be a C^*-algebra. A *product system* of C^*-correspondences over A indexed by P is a semigroup $\mathcal{E} = \bigsqcup_{p\in P} \mathcal{E}_p$ and a map $\mathcal{E} \to P$ such that:

- for each $p \in P$, the fiber $\mathcal{E}_p \subset \mathcal{E}$ is a C^*-correspondence over A with inner product $\langle\cdot,\cdot\rangle_p$;
- the identity fiber $\mathcal{E}_e$ is A viewed as a C^*-correspondence over itself;
- for $p, q \in P \setminus \{e\}$, the multiplication map

  $$\mathcal{M}_{p,q} : \mathcal{E}_p \times \mathcal{E}_q \to \mathcal{E}_{pq}, \quad \mathcal{M}_{p,q}(x, y) = xy$$

  induces an isomorphism $\mathcal{M}_{p,q} : \mathcal{E}_p \otimes_A \mathcal{E}_q \to \mathcal{E}_{pq}$; and
- multiplication in $\mathcal{E}$ by elements of $\mathcal{E}_e = A$ implements the right and left actions of A on each $\mathcal{E}_p$. In particular, $\mathcal{M}_{p,e}$ is an isomorphism.

Let $\phi_p : A \to \mathcal{L}(\mathcal{E}_p)$ be the homomorphism implementing the left action. The product system $\mathcal{E}$ is said to be *essential* if each $\mathcal{E}_p$ is an essential correspondence, that is, if the span of $\phi_p(A)\mathcal{E}_p$ is dense in $\mathcal{E}_p$ for all $p \in P$. In this case, the map $\mathcal{M}_{e,p}$ is also an isomorphism.

If the maps ϕ_p take values in $\mathcal{K}(\mathcal{E}_p)$, then the product system is called *row-finite* or *proper*. If all maps ϕ_p are injective, then $\mathcal{E}$ is called *faithful*.

DEFINITION 2.2. Given a product system $\mathcal{E} \to P$ over A and a C^*-algebra B, a map $\psi : \mathcal{E} \to B$ is called a *Toeplitz representation* of $\mathcal{E}$ if:

- denoting $\psi_p := \psi|_{\mathcal{E}_p}$, each $\psi_p : \mathcal{E}_p \to B$ is linear, $\psi_e : A \to B$ is a $*$-homomorphism, and

  $$\psi_e(\langle x, y\rangle_p) = \psi_p(x)^*\psi_p(y)$$

  for all $x, y \in \mathcal{E}_p$; and
- $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for all $p, q \in P, x \in \mathcal{E}_p, y \in \mathcal{E}_q$.

For each $p \in P$, we write $\psi^{(p)}$ for the homomorphism $\mathcal{K}(\mathcal{E}_p) \to B$ obtained by extending the map $\theta_{\xi,\eta} \mapsto \psi_p(\xi)\psi_p(\eta)^*$, where

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle.$$

The Toeplitz representation $\psi : \mathcal{E} \to B$ is *Cuntz–Pimsner covariant* if $\psi^{(p)}(\phi_p(a)) = \psi_e(a)$ for all $p \in P$ and all $a \in A$ such that $\phi_p(a) \in \mathcal{K}(\mathcal{E}_p)$.

There is a C^*-algebra $\mathcal{T}_A(\mathcal{E})$ called the Toeplitz algebra of $\mathcal{E}$ and a representation $i_{\mathcal{E}} : \mathcal{E} \to \mathcal{T}_A(\mathcal{E})$ which is universal in the following sense: $\mathcal{T}_A(\mathcal{E})$ is generated by $i_{\mathcal{E}}(\mathcal{E})$ and, for any representation $\psi : \mathcal{E} \to B$, there is a homomorphism $\psi_* : \mathcal{T}_A(\mathcal{E}) \to B$ such that $\psi_* \circ i_{\mathcal{E}} = \psi$.

The Cuntz–Pimsner algebra $\mathcal{O}_A(\mathcal{E})$ of a product system $\mathcal{E} \to P$ is universal for Cuntz–Pimsner covariant representations.

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There are various extra conditions on a product system $\mathcal{E} \to P$ and several other notions of covariance besides the Cuntz–Pimsner covariance from Definition 2.2, which allow one to define the Cuntz–Pimsner algebra $\mathcal{O}_A(\mathcal{E})$ or the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_A(\mathcal{E})$ satisfying certain properties (see [1, 4, 10, 13, 26], among others). We mention that $\mathcal{O}_A(\mathcal{E})$ (or $\mathcal{NO}_A(\mathcal{E})$) comes with a covariant representation $j_{\mathcal{E}} : \mathcal{E} \to \mathcal{O}_A(\mathcal{E})$ and is universal in the following sense: $\mathcal{O}_A(\mathcal{E})$ is generated by $j_{\mathcal{E}}(\mathcal{E})$ and, for any covariant representation $\psi : \mathcal{E} \to B$, there is a homomorphism $\psi_* : \mathcal{O}_A(\mathcal{E}) \to B$ such that $\psi_* \circ j_{\mathcal{E}} = \psi$. Under certain conditions, $\mathcal{O}_A(\mathcal{E})$ satisfies a gauge invariant uniqueness theorem.

EXAMPLE 2.3. For a product system $\mathcal{E} \to P$ with fibers $\mathcal{E}_p$ that are nonzero finite-dimensional Hilbert spaces, and, in particular, $A = \mathcal{E}_e = \mathbb{C}$, let us fix an orthonormal basis $\mathcal{B}_p$ in $\mathcal{E}_p$. Then a Toeplitz representation $\psi : \mathcal{E} \to B$ gives rise to a family of isometries $\{\psi(\xi) : \xi \in \mathcal{B}_p\}_{p\in P}$ with mutually orthogonal range projections. In this case, $\mathcal{T}(\mathcal{E}) = \mathcal{T}_{\mathbb{C}}(\mathcal{E})$ is generated by a collection of Cuntz–Toeplitz algebras which interact according to the multiplication maps $\mathcal{M}_{p,q}$ in $\mathcal{E}$.

A representation $\psi : \mathcal{E} \to B$ is Cuntz–Pimsner covariant if

$$\sum_{\xi\in\mathcal{B}_p} \psi(\xi)\psi(\xi)^* = \psi(1)$$

for all $p \in P$. The Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{E}) = \mathcal{O}_{\mathbb{C}}(\mathcal{E})$ is generated by a collection of Cuntz algebras, so it could be thought of as a multidimensional Cuntz algebra. Fowler proved in [12] that if the function $p \mapsto \dim \mathcal{E}_p$ is injective, then the algebra $\mathcal{O}(\mathcal{E})$ is simple and purely infinite. For other examples of multidimensional Cuntz algebras, see [3].

EXAMPLE 2.4. A row-finite k-graph with no sources Λ (see [18]) determines a product system $\mathcal{E} \to \mathbb{N}^k$ with $\mathcal{E}_0 = A = C_0(\Lambda^0)$ and $\mathcal{E}_n = \overline{C_c(\Lambda^n)}$ for $n \neq 0$ such that we have a $\mathbb{T}^k$-equivariant isomorphism $\mathcal{O}_A(\mathcal{E}) \cong C^*(\Lambda)$. Recall that, for product systems indexed by $\mathbb{N}^k$, the universal property induces a gauge action on $\mathcal{O}_A(\mathcal{E})$ defined by $\gamma_z(j_{\mathcal{E}}(\xi)) = z^n j_{\mathcal{E}}(\xi)$ for $z \in \mathbb{T}^k$ and $\xi \in \mathcal{E}_n$.

The following two definitions and two results are taken from [7]; see also [15, 17].

DEFINITION 2.5. An action β of a locally compact group G on a product system $\mathcal{E} \to P$ over A is a family $(\beta^p)_{p\in P}$ such that β^p is an action of G on each fiber $\mathcal{E}_p$ compatible with the action $\alpha = \beta^e$ on A, and, furthermore, the actions $(\beta^p)_{p\in P}$ are compatible with the multiplication maps $\mathcal{M}_{p,q}$ in the sense that

$$\beta_g^{pq}(\mathcal{M}_{p,q}(x \otimes y)) = \mathcal{M}_{p,q}(\beta_g^p(x) \otimes \beta_g^q(y))$$

for all $g \in G$, $x \in \mathcal{E}_p$ and $y \in \mathcal{E}_q$.

DEFINITION 2.6. If β is an action of G on the product system $\mathcal{E} \to P$, we define the crossed product $\mathcal{E} \rtimes_\beta G$ as the product system indexed by P with fibers $\mathcal{E}_p \rtimes_{\beta^p} G$, which are C^*-correspondences over $A \rtimes_\alpha G$. For $\zeta \in C_c(G, \mathcal{E}_p)$ and $\eta \in C_c(G, \mathcal{E}_q)$, the product

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$\zeta\eta \in C_c(G, \mathcal{E}_{pq})$ is defined by

$$(\zeta\eta)(s) = \int_G \mathcal{M}_{p,q}(\zeta(t) \otimes \beta_t^q(\eta(t^{-1}s)))\, dt.$$

PROPOSITION 2.7. *The set $\mathcal{E} \rtimes_\beta G = \bigsqcup_{p\in P} \mathcal{E}_p \rtimes_{\beta^p} G$ with the above multiplication satisfies all the properties of a product system of C^*-correspondences over $A \rtimes_\alpha G$.*

PROPOSITION 2.8. *Suppose that a locally compact group G acts on a row-finite and faithful product system $\mathcal{E}$ indexed by $P = (\mathbb{N}^k, +)$ via automorphisms β_g^p. Then G acts on the Cuntz–Pimsner algebra $\mathcal{O}_A(\mathcal{E})$ via automorphisms denoted by γ_g. Moreover, if G is amenable, then $\mathcal{E} \rtimes_\beta G$ is row-finite and faithful, and*

$$\mathcal{O}_A(\mathcal{E}) \rtimes_\gamma G \cong \mathcal{O}_{A\rtimes_\alpha G}(\mathcal{E} \rtimes_\beta G).$$

Now we define the product system associated to k representations of a compact group G. We limit ourselves to finite-dimensional unitary representations, even though the definition makes sense in greater generality.

DEFINITION 2.9. Given a compact group G and k finite-dimensional unitary representations ρ_i of G on Hilbert spaces $\mathcal{H}_i$ for $i = 1, \ldots, k$, we construct the product system $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ indexed by the commutative monoid $(\mathbb{N}^k, +)$, with fibers

$$\mathcal{E}_n = \mathcal{H}^n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$$

for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$; in particular, $A = \mathcal{E}_0 = \mathbb{C}$. The multiplication maps

$$\mathcal{M}_{n,m} : \mathcal{E}_n \times \mathcal{E}_m \to \mathcal{E}_{n+m}$$

in $\mathcal{E}$ are defined by using the standard isomorphisms $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ for all $i < j$. The associativity in $\mathcal{E}$ follows from the fact that

$$\mathcal{M}_{n+m,p} \circ (\mathcal{M}_{n,m} \times id) = \mathcal{M}_{n,m+p} \circ (id \times \mathcal{M}_{m,p})$$

as maps from $\mathcal{E}_n \times \mathcal{E}_m \times \mathcal{E}_p$ to $\mathcal{E}_{n+m+p}$. Then $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ is called the product system of the representations $\rho_1, \ldots, \rho_k$.

REMARK 2.10. Similarly, a semigroup P of unitary representations of a group G determines a product system $\mathcal{E} \to P$.

PROPOSITION 2.11. *With notation as in Definition 2.9, assume that $d_i = \dim \mathcal{H}_i \geq 2$. Then the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{E})$ associated to the product system $\mathcal{E} \to \mathbb{N}^k$ described above is isomorphic with the C^*-algebra of a rank k graph Γ with a single vertex and with $|\Gamma^{\varepsilon_i}| = d_i$. This isomorphism is equivariant for the gauge action. Moreover,*

$$\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k},$$

where $\mathcal{O}_n$ is the Cuntz algebra.

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PROOF. Indeed, by choosing a basis in each $\mathcal{H}_i$, we get the edges Γ^{ε_i} in a k-colored graph Γ with a single vertex. The isomorphisms $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ determine the factorization rules of the form $ef = fe$ for $e \in \Gamma^{\varepsilon_i}$ and $f \in \Gamma^{\varepsilon_j}$, which obviously satisfy the associativity condition. In particular, the corresponding isometries in $C^*(\Gamma)$ commute and determine, by the universal property, a surjective homomorphism φ onto $\mathcal{O}(\mathcal{E})$, preserving the gauge action. Using the gauge invariant uniqueness theorem for k-graph algebras, the map φ is an isomorphism. In particular, $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k}$. □

REMARK 2.12. For $d_i \geq 2$, the C^*-algebra $\mathcal{O}(\mathcal{E}) \cong C^*(\Gamma)$ is always simple and purely infinite since it is a tensor product of simple and purely infinite C^*-algebras. If $d_i = 1$ for some i, then the isomorphism in Proposition 2.11 still holds, but $C^*(\Gamma) \cong \mathcal{O}(\mathcal{E})$ contains a copy of $C(\mathbb{T})$, so it is not simple. Of course, if $d_i = 1$ for all i, then $\mathcal{O}(\mathcal{E}) \cong C(\mathbb{T}^k)$. For more on single vertex rank k graphs, see [5, 6].

PROPOSITION 2.13. *The compact group G acts on each fiber $\mathcal{E}_n$ of the product system $\mathcal{E}$ via the representation $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$. This action is compatible with the multiplication maps and commutes with the gauge action of $\mathbb{T}^k$. The crossed product $\mathcal{E} \rtimes G$ becomes a row-finite and faithful product system indexed by $\mathbb{N}^k$ over the group C^*-algebra $C^*(G)$. Moreover,*

$$\mathcal{O}(\mathcal{E}) \rtimes G \cong \mathcal{O}_{C^*(G)}(\mathcal{E} \rtimes G).$$

PROOF. Indeed, for $g \in G$ and $\xi \in \mathcal{E}_n = \mathcal{H}^n$, we define $g \cdot \xi = \rho^n(g)(\xi)$, and since $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$, we have $g \cdot (\xi \otimes \eta) = g \cdot \xi \otimes g \cdot \eta$ for $\xi \in \mathcal{E}_n, \eta \in \mathcal{E}_m$. Clearly,

$$g \cdot \gamma_z(\xi) = g \cdot (z^n \xi) = z^n (g \cdot \xi) = \gamma_z(g \cdot \xi),$$

so the action of G commutes with the gauge action. Using Proposition 2.7, $\mathcal{E} \rtimes G$ becomes a product system indexed by $\mathbb{N}^k$ over $C^*(G) \cong \mathbb{C} \rtimes G$ with fibers $\mathcal{E}_n \rtimes G$. The isomorphism $\mathcal{O}(\mathcal{E}) \rtimes G \cong \mathcal{O}_{C^*(G)}(\mathcal{E} \rtimes G)$ follows from Proposition 2.8. □

COROLLARY 2.14. *Since the action of G commutes with the gauge action, the group G acts on the core algebra $\mathcal{F} = \mathcal{O}(\mathcal{E})^{\mathbb{T}^k}$.*

REMARK 2.15. In some cases, $\mathcal{O}(\mathcal{E}) \rtimes G$ is isomorphic to the self-similar k-graph C^*-algebras $\mathcal{O}_{G,\Lambda}$ introduced in [21]. Moreover, for a self-similar k-graph (G, Λ) with $|\Lambda^0| = 1$, we have $\mathcal{O}_{G,\Lambda} \cong \mathcal{Q}(\Lambda \bowtie G)$, where $\Lambda \bowtie G$ is a Zappa–Szép product and $\mathcal{Q}(\Lambda \bowtie G)$ is its boundary quotient C^*-algebra (see Example 3.10(4) in [21] and Theorem 3.3 in [20]). I thank the referee for bringing this relationship to my attention.

3. The Doplicher–Roberts algebra

The Doplicher–Roberts algebras $\mathcal{O}_\rho$, denoted by $\mathcal{O}_G$ in [8], were introduced to construct a new duality theory for compact Lie groups G that strengthens the Tannaka–Krein duality. Here ρ is the n-dimensional representation of G defined by the inclusion $G \subseteq U(n)$ in some unitary group $U(n)$. Let $\mathcal{T}_G$ denote the representation

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category whose objects are tensor powers $\rho^p = \rho^{\otimes p}$ for $p \geq 0$, and whose arrows are the intertwiners $Hom(\rho^p, \rho^q)$. The group G acts via ρ on the Cuntz algebra $\mathcal{O}_n$ and $\mathcal{O}_G = \mathcal{O}_\rho$ is identified in [8] with the fixed point algebra $\mathcal{O}_n^G$. If σ denotes the restriction to $\mathcal{O}_\rho$ of the canonical endomorphism of $\mathcal{O}_n$, then $\mathcal{T}_G$ can be reconstructed from the pair $(\mathcal{O}_\rho, \sigma)$. Subsequently, Doplicher–Roberts algebras were associated to any object ρ in a strict tensor C^*-category (see [9]).

Given finite-dimensional unitary representations $\rho_1, \ldots, \rho_k$ of a compact group G on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$, we construct a Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ from intertwiners

$$Hom(\rho^n, \rho^m) = \{T \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m) \mid T\rho^n(g) = \rho^m(g)T \text{ for all } g \in G\},$$

where, for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, the representation $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$ acts on $\mathcal{H}^n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$. Note that $\rho^0 = \iota$ is the trivial representation of G, acting on $\mathcal{H}^0 = \mathbb{C}$. This Doplicher–Roberts algebra is a subalgebra of $\mathcal{O}(\mathcal{E})$ for the product system $\mathcal{E}$, as in Definition 2.9.

LEMMA 3.1. *Consider*

$$\mathcal{A}_0 = \bigcup_{m,n \in \mathbb{N}^k} \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m).$$

Then the linear span of $\mathcal{A}_0$ becomes a $*$-algebra $\mathcal{A}$ with appropriate multiplication and involution. This algebra has a natural $\mathbb{Z}^k$-grading coming from a gauge action of $\mathbb{T}^k$. Moreover, the Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{E})$ of the product system $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ is equivariantly isomorphic to the C^*-closure of $\mathcal{A}$ in the unique C^*-norm for which the gauge action is isometric.

PROOF. Recall that the Cuntz algebra $\mathcal{O}_n$ contains a canonical Hilbert space $\mathcal{H}$ of dimension n and it can be constructed as the closure of the linear span of $\bigcup_{p,q \in \mathbb{N}} \mathcal{L}(\mathcal{H}^p, \mathcal{H}^q)$ using embeddings

$$\mathcal{L}(\mathcal{H}^p, \mathcal{H}^q) \subseteq \mathcal{L}(\mathcal{H}^{p+1}, \mathcal{H}^{q+1}), \quad T \mapsto T \otimes I,$$

where $\mathcal{H}^p = \mathcal{H}^{\otimes p}$ and $I : \mathcal{H} \to \mathcal{H}$ is the identity map. This linear span becomes a $*$-algebra with a multiplication given by composition and an involution (see [8] and Proposition 2.5 in [16]).

Similarly, for all $r \in \mathbb{N}^k$, we consider embeddings $\mathcal{L}(\mathcal{H}^n, \mathcal{H}^m) \subseteq \mathcal{L}(\mathcal{H}^{n+r}, \mathcal{H}^{m+r})$ given by $T \mapsto T \otimes I_r$, where $I_r : \mathcal{H}^r \to \mathcal{H}^r$ is the identity map, and we endow $\mathcal{A}$ with a multiplication given by composition and an involution. More precisely, if $S \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m)$ and $T \in \mathcal{L}(\mathcal{H}^q, \mathcal{H}^p)$, then the product ST is

$$(S \otimes I_{p \vee n-n}) \circ (T \otimes I_{p \vee n-p}) \in \mathcal{L}(\mathcal{H}^{q+p \vee n-p}, \mathcal{H}^{m+p \vee n-n}),$$

where we write $p \vee n$ for the coordinatewise maximum. This multiplication is well defined in $\mathcal{A}$ and is associative. The adjoint of $T \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m)$ is $T^* \in \mathcal{L}(\mathcal{H}^m, \mathcal{H}^n)$.

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There is a natural $\mathbb{Z}^k$-grading on $\mathcal{A}$ given by the gauge action γ of $\mathbb{T}^k$, where, for $z = (z_1, \ldots, z_k) \in \mathbb{T}^k$ and $T \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m)$, we define

$$\gamma_z(T)(\xi) = z_1^{m_1-n_1} \cdots z_k^{m_k-n_k} T(\xi).$$

Adapting the argument in Theorem 4.2 in [9] for $\mathbb{Z}^k$-graded C^*-algebras, the C^*-closure of $\mathcal{A}$ in the unique C^*-norm for which γ_z is isometric is well defined. The map

$$(T_1, \ldots, T_k) \mapsto T_1 \otimes \cdots \otimes T_k,$$

where

$$T_1 \otimes \cdots \otimes T_k : \mathcal{H}^n \to \mathcal{H}^m, \ (T_1 \otimes \cdots \otimes T_k)(\xi_1 \otimes \cdots \otimes \xi_k) = T_1(\xi_1) \otimes \cdots \otimes T_k(\xi_k)$$

for $T_i \in \mathcal{L}(\mathcal{H}_i^{n_i}, \mathcal{H}_i^{m_i})$ for $i = 1, \ldots, k$ preserves the gauge action and it can be extended to an equivariant isomorphism from $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k}$ to the C^*-closure of $\mathcal{A}$. Note that the closure of $\bigcup_{n \in \mathbb{N}^k} \mathcal{L}(\mathcal{H}^n, \mathcal{H}^n)$ is isomorphic to the core $\mathcal{F} = \mathcal{O}(\mathcal{E})^{\mathbb{T}^k}$, that is the fixed point algebra under the gauge action, which is a UHF-algebra. □

To define the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$, we again identify $Hom(\rho^n, \rho^m)$ with a subset of $Hom(\rho^{n+r}, \rho^{m+r})$ for each $r \in \mathbb{N}^k$, via $T \mapsto T \otimes I_r$. After this identification, it follows that the linear span ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k}$ of $\bigcup_{m,n \in \mathbb{N}^k} Hom(\rho^n, \rho^m) \subseteq \mathcal{A}_0$ has a natural multiplication and involution inherited from $\mathcal{A}$. Indeed, a computation shows that if $S \in Hom(\rho^n, \rho^m)$ and $T \in Hom(\rho^q, \rho^p)$, then $S^* \in Hom(\rho^m, \rho^n)$ and

$$\begin{aligned}((S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p}))\rho^{q+p\vee n-p}(g) \\ = \rho^{m+p\vee n-n}(g)((S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p})),\end{aligned}$$

so $(S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p}) \in Hom(\rho^{q+p\vee n-p}, \rho^{m+p\vee n-n})$ and ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is closed under these operations. Since the action of G commutes with the gauge action, there is a natural $\mathbb{Z}^k$-grading of ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k}$ given by the gauge action γ of $\mathbb{T}^k$ on $\mathcal{A}$.

It follows that the closure $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ of ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k}$ in $\mathcal{O}(\mathcal{E})$ is well defined, obtaining the Doplicher–Roberts algebra associated to the representations $\rho_1, \ldots, \rho_k$. This C^*-algebra also has a $\mathbb{Z}^k$-grading and a gauge action of $\mathbb{T}^k$. By construction, $\mathcal{O}_{\rho_1,\ldots,\rho_k} \subseteq \mathcal{O}(\mathcal{E})$.

REMARK 3.2. For a compact Lie group G, our Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is Morita equivalent with the higher rank Doplicher–Roberts algebra $\mathcal{D}$ defined in [1]. It is also the section C^*-algebra of a Fell bundle over $\mathbb{Z}^k$.

THEOREM 3.3. *Let ρ_i be finite-dimensional unitary representations of a compact group G on Hilbert spaces $\mathcal{H}_i$ of dimensions $d_i \geq 2$ for $i = 1, \ldots, k$. Then the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is isomorphic to the fixed point algebra $\mathcal{O}(\mathcal{E})^G \cong (\mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k})^G$, where $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ is the product system described in Definition 2.9.*

PROOF. We know from Lemma 3.1 that $\mathcal{O}(\mathcal{E})$ is isomorphic to the C^*-algebra generated by the linear span of $\mathcal{A}_0 = \bigcup_{m,n \in \mathbb{N}^k} \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m)$. The group G acts on

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$\mathcal{L}(\mathcal{H}^n, \mathcal{H}^m)$ by

$$(g \cdot T)(\xi) = \rho^m(g)T(\rho^n(g^{-1})\xi)$$

and the fixed point set is $Hom(\rho^n, \rho^m)$. Indeed, we have $g \cdot T = T$ if and only if $T\rho^n(g) = \rho^m(g)T$. This action is compatible with the embeddings and the operations, so it extends to the $*$-algebra $\mathcal{A}$ and the fixed point algebra is the linear span of $\bigcup_{m,n\in\mathbb{N}^k} Hom(\rho^n, \rho^m)$.

It follows that ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k} \subseteq \mathcal{O}(\mathcal{E})^G$ and therefore its closure $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is isomorphic to a subalgebra of $\mathcal{O}(\mathcal{E})^G$. For the other inclusion, any element in $\mathcal{O}(\mathcal{E})^G$ can be approximated with an element from ${}^0\mathcal{O}_{\rho_1,\ldots,\rho_k}$, and hence $\mathcal{O}_{\rho_1,\ldots,\rho_k} = \mathcal{O}(\mathcal{E})^G$. □

REMARK 3.4. By left tensoring with I_r for $r \in \mathbb{N}^k$, we obtain some canonical unital endomorphisms σ_r of $\mathcal{O}_{\rho_1,\ldots,\rho_k}$.

In the next section, we show that, in many cases, $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is isomorphic to a corner of $C^*(\Lambda)$ for a rank k graph Λ, so, in some cases, we can compute its K-theory. It would be nice to express the K-theory of $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ in terms of the maps $\pi \mapsto \pi \otimes \rho_i$ defined on the representation ring $\mathcal{R}(G)$.

4. The rank *k* graphs

For convenience, we first collect some facts about higher rank graphs, introduced in [18]. A rank k graph or k-graph (Λ, d) consists of a countable small category Λ with range and source maps r and s together with a functor $d : \Lambda \to \mathbb{N}^k$ called the degree map, satisfying the factorization property: for every $\lambda \in \Lambda$ and all $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbb{N}^k$, we write $\Lambda^n := d^{-1}(n)$ and call it the set of paths of degree n. For $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ with 1 in position i, the elements in Λ^{ε_i} are called edges and the elements in Λ^0 are called vertices.

A k-graph Λ can be constructed from Λ^0 and from its k-colored skeleton $\Lambda^{\varepsilon_1} \cup \cdots \cup \Lambda^{\varepsilon_k}$ using a complete and associative collection of commuting squares or factorization rules (see [25]).

The k-graph Λ is *row-finite* if, for all $n \in \mathbb{N}^k$ and all $v \in \Lambda^0$, the set $v\Lambda^n := \{\lambda \in \Lambda^n : r(\lambda) = v\}$ is finite. It has no sources if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. A k-graph Λ is said to be *irreducible* (or *strongly connected*) if, for every $u, v \in \Lambda^0$, there is $\lambda \in \Lambda$ such that $u = r(\lambda)$ and $v = s(\lambda)$.

Recall that $C^*(\Lambda)$ is the universal C^*-algebra generated by a family $\{S_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

- $\{S_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
- $S_{\lambda\mu} = S_\lambda S_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$;
- $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

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- for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*.$$

A k-graph Λ is said to satisfy the *aperiodicity condition* if, for every vertex $v \in \Lambda^0$, there is an infinite path $x \in v\Lambda^\infty$ such that $\sigma^m x \neq \sigma^n x$ for all $m \neq n$ in $\mathbb{N}^k$, where $\sigma^m : \Lambda^\infty \to \Lambda^\infty$ are the shift maps. We say that Λ is *cofinal* if, for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$, there is $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n)$ and $r(\lambda) = v$.

Assume that Λ is row-finite with no sources and that it satisfies the aperiodicity condition. Then $C^*(\Lambda)$ is simple if and only if Λ is cofinal (see Proposition 4.8 in [18] and Theorem 3.4 in [23]).

We say that a path $\mu \in \Lambda$ is a loop with an entrance if $s(\mu) = r(\mu)$, and there exists $\alpha \in s(\mu)\Lambda$ such that $d(\mu) \geq d(\alpha)$ and there is no $\beta \in \Lambda$ with $\mu = \alpha\beta$. We say that every vertex *connects to a loop with an entrance* if, for every $v \in \Lambda^0$, there is a loop with an entrance $\mu \in \Lambda$, and a path $\lambda \in \Lambda$ with $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$. If Λ satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, then $C^*(\Lambda)$ is purely infinite (see Proposition 4.9 in [18] and Proposition 8.8 in [24]).

Given finite-dimensional unitary representations ρ_i of a compact group G on Hilbert spaces $\mathcal{H}_i$ for $i = 1, \ldots, k$, we want to construct a rank k graph $\Lambda = \Lambda(\rho_1, \ldots, \rho_k)$. Let R be the set of equivalence classes of irreducible summands $\pi : G \to U(\mathcal{H}_\pi)$ which appear in the tensor powers $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$ for $n \in \mathbb{N}^k$, as in [22]. Take $\Lambda^0 = R$ and, for each $i = 1, \ldots, k$, consider the set of edges Λ^{ε_i} which are uniquely determined by the matrices M_i with entries

$$M_i(w, v) = |\{e \in \Lambda^{\varepsilon_i} : s(e) = v, r(e) = w\}| = \dim Hom(v, w \otimes \rho_i),$$

where $v, w \in R$. The matrices M_i commute since $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ and therefore

$$\dim Hom(v, w \otimes \rho_i \otimes \rho_j) = \dim Hom(v, w \otimes \rho_j \otimes \rho_i)$$

for all $i < j$. This allows us to fix some bijections

$$\lambda_{ij} : \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \to \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i}$$

for all $1 \leq i < j \leq k$, which determine the commuting squares of Λ. As usual,

$$\Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} = \{(e, f) \in \Lambda^{\varepsilon_i} \times \Lambda^{\varepsilon_j} : s(e) = r(f)\}.$$

For $k \geq 3$, we also need to verify that λ_{ij} can be chosen to satisfy the associativity condition, that is,

$$(id_\ell \times \lambda_{ij})(\lambda_{i\ell} \times id_j)(id_i \times \lambda_{j\ell}) = (\lambda_{j\ell} \times id_i)(id_j \times \lambda_{i\ell})(\lambda_{ij} \times id_\ell)$$

as bijections from $\Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_\ell}$ to $\Lambda^{\varepsilon_\ell} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i}$ for all $i < j < \ell$.

REMARK 4.1. Many times $R = \hat{G}$, so $\Lambda^0 = \hat{G}$, for example, if ρ_i are faithful and $\rho_i(G) \subseteq SU(\mathcal{H}_i)$ or if G is finite, ρ_i are faithful and $\dim \rho_i \geq 2$ for all $i = 1, \ldots, k$ (see Lemma 7.2 and Remark 7.4 in [19]).

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PROPOSITION 4.2. *Given representations $\rho_1, \ldots, \rho_k$ as above, assume that ρ_i are faithful and that $R = \hat{G}$. Then each choice of bijections λ_{ij} satisfying the associativity condition determines a rank k graph Λ which is cofinal and locally finite with no sources.*

PROOF. Indeed, the sets Λ^{ε_i} are uniquely determined and the choice of bijections λ_{ij} satisfying the associativity condition is enough to determine Λ. Since the entries of the matrices M_i are finite and there are no zero rows, the graph is locally finite with no sources. To prove that Λ is cofinal, fix a vertex $v \in \Lambda^0$ and an infinite path $x \in \Lambda^\infty$. Arguing as in Lemma 7.2 in [19], any $w \in \Lambda^0$, in particular, $w = x(n)$ for a fixed n, can be joined by a path to v, so there is $\lambda \in \Lambda$ with $s(\lambda) = x(n)$ and $r(\lambda) = v$. See also Lemma 3.1 in [22]. □

REMARK 4.3. Note that the entry $M_i(w, v)$ is just the multiplicity of the irreducible representation v in $w \otimes \rho_i$ for $i = 1, \ldots, k$. If $\rho_i^* = \rho_i$, then the matrices M_i are symmetric since

$$\dim Hom(v, w \otimes \rho_i) = \dim Hom(\rho_i^* \otimes v, w)$$

which implies $M_i(w; v) = M_i(v; w)$. Here ρ_i^* denotes the dual representation defined by $\rho_i^*(g) = \rho_i(g^{-1})^t$ and equal, in our case, to the conjugate representation $\bar{\rho}_i$.

For G finite, these matrices are finite, and the entries $M_i(w, v)$ can be computed using the character table of G. For G infinite, the Clebsch–Gordan relations can be used to determine the numbers $M_i(w, v)$. Since the bijections λ_{ij} are, in general, not unique, the rank k graph Λ is not unique, as illustrated in some examples. It is an open question how the C^*-algebra $C^*(\Lambda)$ depends, in general, on the factorization rules.

To relate the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ to a rank k graph Λ, we mimic the construction in [22]. For each edge $e \in \Lambda^{\varepsilon_i}$, choose an isometric intertwiner

$$T_e : \mathcal{H}_{s(e)} \to \mathcal{H}_{r(e)} \otimes \mathcal{H}_i$$

in such a way that

$$\mathcal{H}_\pi \otimes \mathcal{H}_i = \bigoplus_{e \in \pi\Lambda^{\varepsilon_i}} T_e T_e^* (\mathcal{H}_\pi \otimes \mathcal{H}_i)$$

for all $\pi \in \Lambda^0$, that is, the edges in Λ^{ε_i} ending at π give a specific decomposition of $\mathcal{H}_\pi \otimes \mathcal{H}_i$ into irreducibles. When $\dim Hom(s(e), r(e) \otimes \rho_i) \geq 2$, we must choose a basis of isometric intertwiners with orthogonal ranges, so, in general, T_e is not unique. In fact, specific choices for the isometric intertwiners T_e determine the factorization rules in Λ and whether or not they satisfy the associativity condition.

Given $e \in \Lambda^{\varepsilon_i}$ and $f \in \Lambda^{\varepsilon_j}$ with $r(f) = s(e)$, we know how to multiply $T_e \in Hom(s(e), r(e) \otimes \rho_i)$ with $T_f \in Hom(s(f), r(f) \otimes \rho_j)$ in the algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$, by viewing $Hom(s(e), r(e) \otimes \rho_i)$ as a subspace of $Hom(\rho^n, \rho^m)$ for some m, n, and similarly for $Hom(s(f), r(f) \otimes \rho_j)$. We choose edges $e' \in \Lambda^{\varepsilon_i}, f' \in \Lambda^{\varepsilon_j}$ with $s(f) = s(e'), r(e) = r(f'), r(e') = s(f')$ such that $T_e T_f = T_{f'} T_{e'}$, where $T_{f'} \in Hom(s(f'), r(f') \otimes \rho_j)$ and

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$T_{e'} \in Hom(s(e'), r(e') \otimes \rho_i)$. This is possible since

$$T_e T_f = (T_e \otimes I_j) \circ T_f \in Hom(s(f), r(e) \otimes \rho_i \otimes \rho_j),$$
$$T_{f'} T_{e'} = (T_{f'} \otimes I_i) \circ T_{e'} \in Hom(s(e'), r(f') \otimes \rho_j \otimes \rho_i),$$

and $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$. In this case, we declare that $ef = f'e'$. Repeating this process, we obtain bijections $\lambda_{ij} : \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \to \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i}$. Assuming that the associativity conditions are satisfied, we obtain a k-graph Λ.

We write $T_{ef} = T_e T_f = T_{f'} T_{e'} = T_{f'e'}$. A finite path $\lambda \in \Lambda^n$ is a concatenation of edges and determines by composition a unique intertwiner

$$T_\lambda : \mathcal{H}_{s(\lambda)} \to \mathcal{H}_{r(\lambda)} \otimes \mathcal{H}^n.$$

Moreover, the paths $\lambda \in \Lambda^n$ with $r(\lambda) = \iota$, the trivial representation, provide an explicit decomposition of $\mathcal{H}^n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$ into irreducibles, and hence

$$\mathcal{H}^n = \bigoplus_{\lambda \in \iota\Lambda^n} T_\lambda T_\lambda^*(\mathcal{H}^n).$$

PROPOSITION 4.4. *Assuming that the choices of isometric intertwiners T_e, as above, determine a k-graph Λ, the family*

$$\{T_\lambda T_\mu^* : \lambda \in \Lambda^m, \mu \in \Lambda^n, r(\lambda) = r(\mu) = \iota, s(\lambda) = s(\mu)\}$$

is a basis for $Hom(\rho^n, \rho^m)$ and each $T_\lambda T_\mu^*$ is a partial isometry.

PROOF. Each pair of paths λ, μ with $d(\lambda) = m, d(\mu) = n$ and $r(\lambda) = r(\mu) = \iota$ determines a pair of irreducible summands $T_\lambda(\mathcal{H}_{s(\lambda)}), T_\mu(\mathcal{H}_{s(\mu)})$ of $\mathcal{H}^m$ and $\mathcal{H}^n$, respectively. By Schur's lemma, the space of intertwiners of these representations is trivial unless $s(\lambda) = s(\mu)$, in which case it is the one-dimensional space spanned by $T_\lambda T_\mu^*$. It follows that any element of $Hom(\rho^n, \rho^m)$ can be uniquely represented as a linear combination of elements $T_\lambda T_\mu^*$, where $s(\lambda) = s(\mu)$. Since T_μ is isometric, T_μ^* is a partial isometry with range $\mathcal{H}_{s(\mu)}$ and hence $T_\lambda T_\mu^*$ is also a partial isometry whenever $s(\lambda) = s(\mu)$. □

THEOREM 4.5. *Consider $\rho_1, \ldots, \rho_k$ finite-dimensional unitary representations of a compact group G and let Λ be the k-colored graph with $\Lambda^0 = R \subseteq \hat{G}$ and edges Λ^{ε_i} determined by the incidence matrices M_i defined above. Assume that the factorization rules determined by the choices of $T_e \in Hom(s(e), r(e) \otimes \rho_i)$ for all edges $e \in \Lambda^{\varepsilon_i}$ satisfy the associativity condition, so Λ becomes a rank k graph. If we consider $P \in C^*(\Lambda)$,*

$$P = \sum_{\lambda \in \iota\Lambda^{(1,\ldots,1)}} S_\lambda S_\lambda^*,$$

where ι is the trivial representation, then there is a $*$-isomorphism of the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ onto the corner $PC^*(\Lambda)P$.

PROOF. Since $C^*(\Lambda)$ is generated by linear combinations of $S_\lambda S_\mu^*$ with $s(\lambda) = s(\mu)$ (see Lemma 3.1 in [18]), we first define the maps

$$\phi_{n,m} : Hom(\rho^n, \rho^m) \to C^*(\Lambda), \quad \phi_{n,m}(T_\lambda T_\mu^*) = S_\lambda S_\mu^*,$$

where $s(\lambda) = s(\mu)$ and $r(\lambda) = r(\mu) = \iota$. Since $S_\lambda S_\mu^* = PS_\lambda S_\mu^* P$, the maps $\phi_{n,m}$ take values in $PC^*(\Lambda)P$. We claim that, for any $r \in \mathbb{N}^k$,

$$\phi_{n+r,m+r}(T_\lambda T_\mu^* \otimes I_r) = \phi_{n,m}(T_\lambda T_\mu^*).$$

This is because

$$\mathcal{H}_{s(\lambda)} \otimes \mathcal{H}^r = \bigoplus_{\nu \in s(\lambda)\Lambda^r} T_\nu T_\nu^*(\mathcal{H}_{s(\lambda)} \otimes \mathcal{H}^r),$$

so that

$$T_\lambda T_\mu^* \otimes I_r = \sum_{\nu \in s(\lambda)\Lambda^r} (T_\lambda \otimes I_r)(T_\nu T_\nu^*)(T_\mu^* \otimes I_r) = \sum_{\nu \in s(\lambda)\Lambda^r} T_{\lambda\nu} T_{\mu\nu}^*$$

and

$$S_\lambda S_\mu^* = \sum_{\nu \in s(\lambda)\Lambda^r} S_\lambda (S_\nu S_\nu^*) S_\mu^* = \sum_{\nu \in s(\lambda)\Lambda^r} S_{\lambda\nu} S_{\mu\nu}^*.$$

The maps $\phi_{n,m}$ determine a map $\phi : {}^0\mathcal{O}_{\rho_1,\ldots,\rho_k} \to PC^*(\Lambda)P$ which is linear, $*$-preserving and multiplicative. Indeed,

$$\phi_{n,m}(T_\lambda T_\mu^*)^* = (S_\lambda S_\mu^*)^* = S_\mu S_\lambda^* = \phi_{m,n}(T_\mu T_\lambda^*).$$

Consider now $T_\lambda T_\mu^* \in Hom(\rho^n, \rho^m)$, $T_\nu T_\omega^* \in Hom(\rho^q, \rho^p)$ with $s(\lambda) = s(\mu), s(\nu) = s(\omega), r(\lambda) = r(\mu) = r(\nu) = r(\omega) = \iota$. Since, for all $n \in \mathbb{N}^k$,

$$\sum_{\lambda \in \iota\Lambda^n} T_\lambda T_\lambda^* = I_n,$$

we get

$$T_\mu^* T_\nu = \begin{cases} T_\beta^* & \text{if } \mu = \nu\beta, \\ T_\alpha & \text{if } \nu = \mu\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\phi((T_\lambda T_\mu^*)(T_\nu T_\omega^*)) = \begin{cases} \phi(T_\lambda T_{\omega\beta}^*) = S_\lambda S_{\omega\beta}^* & \text{if } \mu = \nu\beta, \\ \phi(T_{\lambda\alpha} T_\omega^*) = S_{\lambda\alpha} S_\omega^* & \text{if } \nu = \mu\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, from Lemma 3.1 in [18],

$$S_\lambda S_\mu^* S_\nu S_\omega^* = \begin{cases} S_\lambda S_{\omega\beta}^* & \text{if } \mu = \nu\beta, \\ S_{\lambda\alpha} S_\omega^* & \text{if } \nu = \mu\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\phi((T_\lambda T_\mu^*)(T_\nu T_\omega^*)) = \phi(T_\lambda T_\mu^*)\phi(T_\nu T_\omega^*).$$

Since $PS_\lambda S_\mu^* P = \phi_{n,m}(T_\lambda T_\mu^*)$ if $r(\lambda) = r(\mu) = \iota$ and $s(\lambda) = s(\mu)$, it follows that ϕ is surjective. Injectivity follows from the fact that ϕ is equivariant for the gauge action. □

COROLLARY 4.6. *If the k-graph Λ associated to $\rho_1, \ldots, \rho_k$ is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, then the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ is simple and purely infinite, and is Morita equivalent with $C^*(\Lambda)$.*

PROOF. This follows from the fact that $C^*(\Lambda)$ is simple and purely infinite and because $PC^*(\Lambda)P$ is a full corner. □

REMARK 4.7. There is a groupoid $\mathcal{G}_\Lambda$ associated to a row-finite rank k graph Λ with no sources (see [18]). By taking the pointed groupoid $\mathcal{G}_\Lambda(\iota)$, the reduction to the set of infinite paths with range ι, under the same conditions as in Theorem 4.5, we get an isomorphism of the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ onto $C^*(\mathcal{G}_\Lambda(\iota))$.

5. Examples

EXAMPLE 5.1. Let $G = S_3$ be the symmetric group with $\hat{G} = \{\iota, \epsilon, \sigma\}$ and character table

	(1)	(12)	(123)
ι	1	1	1
ϵ	1	−1	1
σ	2	0	−1

Here ι denotes the trivial representation, ϵ is the sign representation and σ is an irreducible 2-dimensional representation, for example,

$$\sigma((12)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma((123)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

By choosing $\rho_1 = \sigma$ on $\mathcal{H}_1 = \mathbb{C}^2$ and $\rho_2 = \iota + \sigma$ on $\mathcal{H}_2 = \mathbb{C}^3$, we get a product system $\mathcal{E} \to \mathbb{N}^2$ and an action of S_3 on $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_3$ with fixed point algebra $\mathcal{O}(\mathcal{E})^{S_3} \cong \mathcal{O}_{\rho_1,\rho_2}$ isomorphic to a corner of the C^*-algebra of a rank two graph Λ. The

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set of vertices is $\Lambda^0 = \{\iota, \epsilon, \sigma\}$ and the edges are given by the incidence matrices

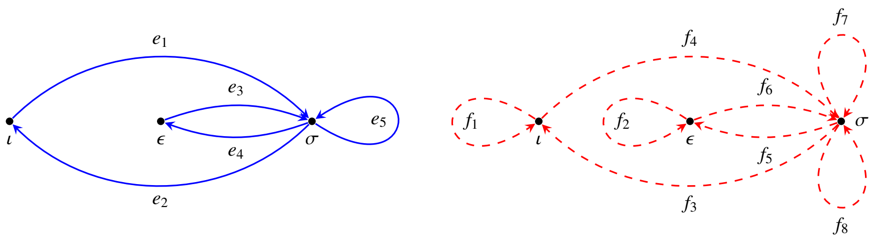
$$M_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

This is because

$$\iota \otimes \rho_1 = \sigma,\ \epsilon \otimes \rho_1 = \sigma,\ \sigma \otimes \rho_1 = \iota + \epsilon + \sigma,$$

$$\iota \otimes \rho_2 = \iota + \sigma,\ \epsilon \otimes \rho_2 = \epsilon + \sigma,\ \sigma \otimes \rho_2 = \iota + \epsilon + 2\sigma.$$

We label the blue (solid) edges by $e_1, \ldots, e_5$ and the red (dashed) edges by $f_1, \ldots, f_8$ as in the figure below.



The isometric intertwiners are

$$T_{e_1} : \mathcal{H}_\iota \to \mathcal{H}_\sigma \otimes \mathcal{H}_1,\ T_{e_2} : \mathcal{H}_\sigma \to \mathcal{H}_\iota \otimes \mathcal{H}_1,\ T_{e_3} : \mathcal{H}_\epsilon \to \mathcal{H}_\sigma \otimes \mathcal{H}_1,$$

$$T_{e_4} : \mathcal{H}_\sigma \to \mathcal{H}_\epsilon \otimes \mathcal{H}_1,\ T_{e_5} : \mathcal{H}_\sigma \to \mathcal{H}_\sigma \otimes \mathcal{H}_1,$$

$$T_{f_1} : \mathcal{H}_\iota \to \mathcal{H}_\iota \otimes \mathcal{H}_2,\ T_{f_2} : \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon \otimes \mathcal{H}_2,\ T_{f_3} : \mathcal{H}_\sigma \to \mathcal{H}_\iota \otimes \mathcal{H}_2,$$

$$T_{f_4} : \mathcal{H}_\iota \to \mathcal{H}_\sigma \otimes \mathcal{H}_2,\ T_{f_5} : \mathcal{H}_\sigma \to \mathcal{H}_\epsilon \otimes \mathcal{H}_2,\ T_{f_6} : \mathcal{H}_\epsilon \to \mathcal{H}_\sigma \otimes \mathcal{H}_2,$$

$$T_{f_7}, T_{f_8} : \mathcal{H}_\sigma \to \mathcal{H}_\sigma \otimes \mathcal{H}_2$$

such that

$$T_{e_1}T_{e_1}^* + T_{e_3}T_{e_3}^* + T_{e_5}T_{e_5}^* = I_\sigma \otimes I_1,\ T_{e_2}T_{e_2}^* = I_\iota \otimes I_1,\ T_{e_4}T_{e_4}^* = I_\epsilon \otimes I_1,$$

$$T_{f_1}T_{f_1}^* + T_{f_3}T_{f_3}^* = I_\iota \otimes I_2,\ T_{f_2}T_{f_2}^* + T_{f_5}T_{f_5}^* = I_\epsilon \otimes I_2,$$

$$T_{f_4}T_{f_4}^* + T_{f_6}T_{f_6}^* + T_{f_7}T_{f_7}^* + T_{f_8}T_{f_8}^* = I_\sigma \otimes I_2.$$

Here I_π is the identity of $\mathcal{H}_\pi$ for $\pi \in \hat{G}$ and I_i is the identity of $\mathcal{H}_i$ for $i = 1, 2$. Since

$$M_1M_2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

and

$$T_{e_2}T_{f_4}, T_{f_3}T_{e_1} \in Hom(\iota, \iota \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_2}T_{f_6}, T_{f_3}T_{e_3} \in Hom(\epsilon, \iota \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_2}T_{f_7}, T_{e_2}T_{f_8}, T_{f_1}T_{e_2}, T_{f_3}T_{e_5} \in Hom(\sigma, \iota \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_4}T_{f_4}, T_{f_5}T_{e_1} \in Hom(\iota, \epsilon \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_4}T_{f_6}, T_{f_5}T_{e_3} \in Hom(\epsilon, \epsilon \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_4}T_{f_7}, T_{e_4}T_{f_8}, T_{f_2}T_{e_4}, T_{f_5}T_{e_5} \in Hom(\sigma, \epsilon \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_1}T_{f_1}, T_{e_5}T_{f_4}, T_{f_7}T_{e_1}, T_{f_8}T_{e_1} \in Hom(\iota, \sigma \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_3}T_{f_2}, T_{e_5}T_{f_6}, T_{f_7}T_{e_3}, T_{f_8}T_{e_3} \in Hom(\epsilon, \sigma \otimes \rho_1 \otimes \rho_2),$$
$$T_{e_5}T_{f_7}, T_{e_5}T_{f_8}, T_{e_3}T_{f_5}, T_{e_1}T_{f_3}, T_{f_6}T_{e_4}, T_{f_4}T_{e_2}, T_{f_7}T_{e_5}, T_{f_8}T_{e_5} \in Hom(\sigma, \sigma \otimes \rho_1 \otimes \rho_2),$$

a possible choice of commuting squares is

$$e_2f_4 = f_3e_1,\ e_2f_6 = f_3e_3,\ e_2f_7 = f_1e_2,\ e_2f_8 = f_3e_5,\ e_4f_4 = f_5e_1,\ e_4f_6 = f_5e_3,$$
$$e_4f_7 = f_2e_4,\ e_4f_8 = f_5e_5,\ e_1f_1 = f_7e_1,\ e_5f_4 = f_8e_1,\ e_3f_2 = f_7e_3,\ e_5f_6 = f_8e_3,$$
$$e_5f_7 = f_6e_4,\ e_5f_8 = f_4e_2,\ e_3f_5 = f_7e_5,\ e_1f_3 = f_8e_5.$$

This data is enough to determine a rank two graph Λ associated to ρ_1, ρ_2. But this is not the only choice, since, for example, we could have taken

$$e_2f_4 = f_3e_1,\ e_2f_6 = f_3e_3,\ e_2f_8 = f_1e_2,\ e_2f_7 = f_3e_5,\ e_4f_4 = f_5e_1,\ e_4f_6 = f_5e_3,$$
$$e_4f_8 = f_2e_4,\ e_4f_7 = f_5e_5,\ e_1f_1 = f_7e_1,\ e_5f_4 = f_8e_1,\ e_3f_2 = f_8e_3,\ e_5f_6 = f_7e_3,$$
$$e_5f_7 = f_6e_4,\ e_5f_8 = f_4e_2,\ e_3f_5 = f_7e_5,\ e_1f_3 = f_8e_5,$$

which determines a different 2-graph.

A direct analysis using the definitions shows that, in each case, the 2-graph Λ is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance. It follows that $C^*(\Lambda)$ is simple and purely infinite and the Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\rho_2}$ is Morita equivalent with $C^*(\Lambda)$.

The K-theory of $C^*(\Lambda)$ can be computed using Proposition 3.16 in [11] and it does not depend on the choice of factorization rules. We have

$$K_0(C^*(\Lambda)) \cong \operatorname{coker}[I - M_1^t \;\; I - M_2^t] \oplus \ker \begin{bmatrix} M_2^t - I \\ I - M_1^t \end{bmatrix} \cong \mathbb{Z}/2\mathbb{Z},$$

$$K_1(C^*(\Lambda)) \cong \ker[I - M_1^t \;\; I - M_2^t]/\operatorname{im} \begin{bmatrix} M_2^t - I \\ I - M_1^t \end{bmatrix} \cong 0.$$

In particular, $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}_3$.

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On the other hand, since ρ_1, ρ_2 are faithful, both Doplicher–Roberts algebras $\mathcal{O}_{\rho_1}, \mathcal{O}_{\rho_2}$ are simple and purely infinite with

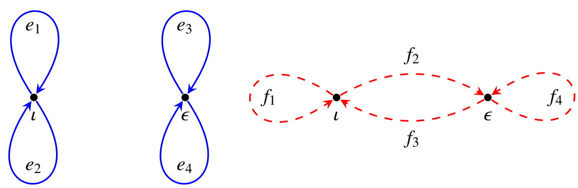
$$K_0(\mathcal{O}_{\rho_1}) \cong \mathbb{Z}/2\mathbb{Z},\ K_1(\mathcal{O}_{\rho_1}) \cong 0,\ K_0(\mathcal{O}_{\rho_2}) \cong \mathbb{Z},\ K_1(\mathcal{O}_{\rho_2}) \cong \mathbb{Z},$$

so $\mathcal{O}_{\rho_1,\rho_2} \not\cong \mathcal{O}_{\rho_1} \otimes \mathcal{O}_{\rho_2}$.

EXAMPLE 5.2. With $G = S_3$ and $\rho_1 = 2\iota, \rho_2 = \iota + \epsilon$, then $R = \{\iota, \epsilon\}$, so Λ has two vertices and incidence matrices

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which give



Again, a corresponding choice of isometric intertwiners determines some factorization rules, for example,

$$e_1f_1 = f_1e_2,\ e_2f_1 = f_1e_1,\ e_1f_3 = f_3e_3,\ e_2f_3 = f_3e_4,$$

$$e_3f_2 = f_2e_1,\ e_4f_2 = f_2e_2,\ e_3f_4 = f_4e_4,\ e_4f_4 = f_4e_3.$$

Even though ρ_1, ρ_2 are not faithful, the obtained 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so $\mathcal{O}_{\rho_1,\rho_2}$ is simple and purely infinite with trivial K-theory. In particular, $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}_2$.

Note that, since ρ_1, ρ_2 have kernel $N = \langle(123)\rangle \cong \mathbb{Z}/3\mathbb{Z}$, we could replace G by $G/N \cong \mathbb{Z}/2\mathbb{Z}$ and consider ρ_1, ρ_2 as representations of $\mathbb{Z}/2\mathbb{Z}$.

EXAMPLE 5.3. Consider $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with $\hat{G} = \{\iota, \chi\}$ and character table

	0	1
ι	1	1
χ	1	−1

Choose the 2-dimensional representations

$$\rho_1 = \iota + \chi,\ \rho_2 = 2\iota,\ \rho_3 = 2\chi,$$

which determine a product system $\mathcal{E}$ such that $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathcal{O}_2$ and a Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\rho_2,\rho_3} \cong \mathcal{O}(\mathcal{E})^{\mathbb{Z}/2\mathbb{Z}}$.

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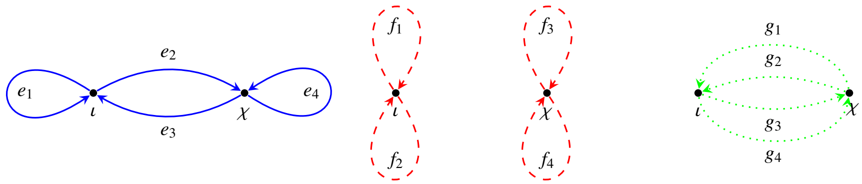
An easy computation shows that the incidence matrices of the blue (solid), red (dashed) and green (dotted) graphs are

$$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$



With labels as in the figure, we choose the following factorization rules.

$$e_1f_1 = f_2e_1,\ e_1f_2 = f_1e_1,\ e_2f_1 = f_4e_2,\ e_2f_2 = f_3e_2,$$

$$e_3f_3 = f_2e_3,\ e_3f_4 = f_1e_3,\ e_4f_4 = f_3e_4,\ e_4f_3 = f_4e_4,$$

$$f_1g_1 = g_2f_3,\ f_1g_2 = g_1f_3,\ f_2g_1 = g_2f_4,\ f_2g_2 = g_1f_4,$$

$$f_3g_3 = g_4f_1,\ f_3g_4 = g_3f_1,\ f_4g_3 = g_4f_2,\ f_4g_4 = g_3f_2,$$

$$e_1g_1 = g_2e_4,\ e_1g_2 = g_1e_4,\ e_2g_1 = g_3e_3,\ e_2g_2 = g_4e_3,$$

$$e_3g_3 = g_1e_2,\ e_3g_4 = g_2e_2,\ e_4g_3 = g_4e_1,\ e_4g_4 = g_3e_1.$$

A tedious verification shows that all the following paths are well defined.

$$e_1f_1g_1,\ e_1f_1g_2,\ e_1f_2g_1,\ e_1f_2g_2,\ e_2f_1g_1,\ e_2f_1g_2,\ e_2f_2g_1,\ e_2f_2g_2,$$

$$e_3f_3g_3,\ e_3f_3g_4,\ e_3f_4g_3,\ e_3f_4g_4,\ e_4f_3g_3,\ e_4f_3g_4,\ e_4f_4g_3,\ e_4f_4g_4,$$

so the associativity property is satisfied and we get a rank three graph Λ with two vertices. It is not difficult to check that Λ is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so $C^*(\Lambda)$ is simple and purely infinite.

Since $\partial_1 = [I - M_1^t\ I - M_2^t\ I - M_3^t] : \mathbb{Z}^6 \to \mathbb{Z}^2$ is surjective, using Corollary 3.18 in [11], we obtain

$$K_0(C^*(\Lambda)) \cong \ker \partial_2/\operatorname{im} \partial_3 \cong 0,\ K_1(C^*(\Lambda)) \cong \ker \partial_1/\operatorname{im} \partial_2 \oplus \ker \partial_3 \cong 0,$$

where

$$\partial_2 = \begin{bmatrix} M_2^t - I & M_3^t - I & 0 \\ I - M_1^t & 0 & M_3^t - I \\ 0 & I - M_1^t & I - M_2^t \end{bmatrix}, \quad \partial_3 = \begin{bmatrix} I - M_3^t \\ M_2^t - I \\ I - M_1^t \end{bmatrix},$$

and, in particular, $\mathcal{O}_{\rho_1,\rho_2,\rho_3} \cong \mathcal{O}_2$.

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EXAMPLE 5.4. Let $G = \mathbb{T}$. We have $\hat{G} = \{\chi_k : k \in \mathbb{Z}\}$, where $\chi_k(z) = z^k$ and $\chi_k \otimes \chi_\ell = \chi_{k+\ell}$. The faithful representations

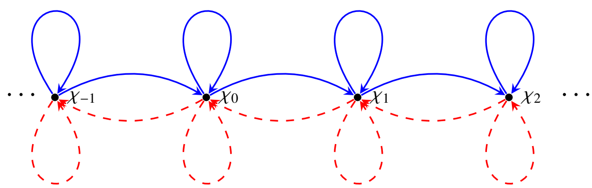
$$\rho_1 = \chi_{-1} + \chi_0, \ \rho_2 = \chi_0 + \chi_1$$

of $\mathbb{T}$ determine a product system $\mathcal{E}$ with $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ and a Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}(\mathcal{E})^{\mathbb{T}}$ isomorphic to a corner in the C^*-algebra of a rank 2 graph Λ with $\Lambda^0 = \hat{G}$ and infinite incidence matrices, where

$$M_1(\chi_k, \chi_\ell) = \begin{cases} 1 & \text{if } \ell = k \text{ or } \ell = k-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_2(\chi_k, \chi_\ell) = \begin{cases} 1 & \text{if } \ell = k \text{ or } \ell = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

The skeleton of Λ looks like



and this 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so $C^*(\Lambda)$ is simple and purely infinite.

EXAMPLE 5.5. Let $G = SU(2)$. It is known (see page 84 in [2]) that the elements in $\hat{G}$ are labeled by V_n for $n \geq 0$, where $V_0 = \iota$ is the trivial representation on $\mathbb{C}$, V_1 is the standard representation of $SU(2)$ on $\mathbb{C}^2$, and, for $n \geq 2$, $V_n = S^n V_1$, the nth symmetric power. In fact, $\dim V_n = n + 1$ and V_n can be taken as the representation of $SU(2)$ on the space of homogeneous polynomials p of degree n in variables z_1, z_2, where, for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SU(2)$,

$$(g \cdot p)(z) = p(az_1 + cz_2, bz_1 + dz_2).$$

The irreducible representations V_n satisfy the Clebsch–Gordan formula

$$V_k \otimes V_\ell = \bigoplus_{j=0}^{q} V_{k+\ell-2j}, \ q = \min\{k, \ell\}.$$

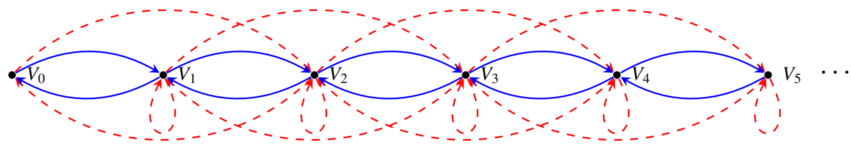
If we choose $\rho_1 = V_1, \rho_2 = V_2$, then we get a product system $\mathcal{E}$ with $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_3$ and a Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}(\mathcal{E})^{SU(2)}$ isomorphic to a corner in the

C^*-algebra of a rank two graph with $\Lambda^0 = \hat{G}$ and edges given by the matrices

$$M_1(V_k, V_\ell) = \begin{cases} 1 & \text{if } k = 0 \text{ and } \ell = 1, \\ 1 & \text{if } k \geq 1 \text{ and } \ell \in \{k-1, k+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_2(V_k, V_\ell) = \begin{cases} 1 & \text{if } k = 0 \text{ and } \ell = 2, \\ 1 & \text{if } k = 1 \text{ and } \ell \in \{1, 3\}, \\ 1 & \text{if } k \geq 2 \text{ and } \ell \in \{k-2, k, k+2\}, \\ 0 & \text{otherwise.} \end{cases}$$

The skeleton looks like



and this 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance; in particular, $\mathcal{O}_{\rho_1,\rho_2}$ is simple and purely infinite.

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