ON QUOTIENT-LIKE EXTENSIONS OF $C(X)$

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The maximal ring of quotients (in the sense of Johnson [4] and Utumi [5]) of the ring $C(X)$ of real valued continuous functions on a completely regular Hausdorff space X has been studied in [1] and [2]. The aim of the present paper is to provide some additional results to those, and to study the relevant extensions on the absence of the real maximal ideals.

In the first part of this paper, it will be shown that, for the set $\mathbb{Q}$ of rational numbers, $C(\mathbb{Q})$ is not a ring of quotients of $C(\mathbb{R})$ with respect to the natural restriction homomorphism. In the second part of this paper, it is shown that, for a separable metric space X without isolated points, the maximal ring of quotients of $C(X)$ is totally unreal; i.e., it does not have any real maximal ideal.

1. Direct limit. Let $\mathfrak{D}$ be a filter base of dense subsets of X. For each $D \in \mathfrak{D}$, let $C_D(X)$ denote the ring of all real valued functions f on X which have continuous restriction $f|D$ to D, and $Z_D(X)$ the subset of $C_D(X)$ consisting of f with $f|D = 0$. Put $C_{\mathfrak{D}}(X) = \bigcup \{C_D(X) : D \in \mathfrak{D}\}$ and $Z_{\mathfrak{D}}(X) = \bigcup \{Z_D(X) : D \in \mathfrak{D}\}$. Evidently $C_{\mathfrak{D}}(X)$ is a ring of functions on X containing $Z_{\mathfrak{D}}(X)$ as an ideal. Denote $Q_{\mathfrak{D}}(X) \equiv C_{\mathfrak{D}}(X)/Z_{\mathfrak{D}}(X)$, and, by a straightforward checking, one proves the following lemma.

LEMMA 1. 1) $C_D(X) \cap Z_{\mathfrak{D}}(X) = Z_D(X)$ for each $D \in \mathfrak{D}$.

2) The restriction ρ_D, defined by $\rho_D(f) = f|D$, induces an isomorphism $C_D(X)/Z_D(X) \to C(D)$ for each $D \in \mathfrak{D}$.

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3) *The natural homomorphism* $\nu : C_{\mathcal{D}}(X) \to Q_{\mathcal{D}}(X)$ *determines, for each* $D \in \mathcal{D}$, *an embedding* $j_D : C(D) \to Q_{\mathcal{D}}(X)$ *such that* $\nu(f) = j_D(f|D)$ *for each* $f \in C_D(X)$.

4) *For any* $D, E \in \mathcal{D}$ *with* $D \subseteq E$, $j_D(f|D) = j_E(f)$ *for all* $f \in C(E)$.

5) $C(X) \cap Z_{\mathcal{D}}(X) = 0$.

The following is the immediate consequence of the above lemma.

PROPOSITION 2. $Q_{\mathcal{D}}(X)$ *is the direct limit of the direct system* $(C(D))(D \in \mathcal{D})$, *with respect to the restriction homomorphisms* $f \to f|D$, $f \in C(E)$, $D \subseteq E$ *in* $\mathcal{D}$, *with* $(j_D)(D \in \mathcal{D})$ *as a family of the limit homomorphisms, in the category of all rings with unit and unitary ring homomorphisms*.

Now suppose a ring A is a subring of a ring B, then we call B *a ring of quotients* of A provided that for b, $0 \neq b' \in B$, there exists $a \in A$ such that $ba \in A$ and $b'a \neq 0$. To say that a ring B is *a ring of quotients of* A *with respect to an embedding* $e : A \to B$ means that the ring B is a ring of quotients of the subring $e(A)$ of B. It is evident that $Q_{\mathcal{D}}(X)$ is a ring of quotients of $C(X)$ with respect to the embedding ν (in virtue of (5) of Lemma 1) if and only if each $C(D)$, $D \in \mathcal{D}$, is a ring of quotients of $C(X)$ with respect to the embedding given by the restriction $f \to f|D$. By making use of [2, Theorem 1.5], we prove the following:

PROPOSITION 3. *A necessary and sufficient condition for* $Q_{\mathcal{D}}(X)$ *to be a ring of quotients of* $C(X)$ *with respect to* ν *is that: for each* $D \in \mathcal{D}$, *for any* $f \in C(D)$ *and open subset* U *of* D, *there exists an open subset* V *of* X *such that* $V \cap D \subseteq U$ *and* $f|V \cap D$ *has a continuous extension to* V.

Proof. Take any $D \in \mathcal{D}$. Put $C(X)|D = \{f|D : f \in C(X)\}$. It suffices to show that $C(D)$ is a ring of quotients of $C(X)|D$. Let $0 \neq f \in C(D)$. Put $U = \{x \in D : f(x) \neq 0\}$. Then there is an open subset V of X such that $V \cap D \subseteq U$, and $f|V \cap D$ has a continuous extension $\tilde{f}$ to V. Now for an element $c \in V \cap D$ we find an $h \in C(X)$ such that $h(c) \neq 0$ and $h|X - V' = 0$ for some neighborhood V' of c whose closure is in V. Define a function u on X by $u(x) = \tilde{f}(x) \cdot h(x)$ for $x \in V$ and $u(x) = 0$ for $x \in X - V$. Then $u \in C(X)$, and clearly $0 \neq f \cdot h|D = u|D \in C(X)|D$.

For the proof of the converse, let $f \in C(D)$ and U be a non-void open subset of D. Then there is an open subset W of X with $U = W \cap D$. We may assume $f|U \neq 0$, for otherwise it is trivial.

Now we show that f is continuous at each point $q \in \mathbf{Q}$. Clearly,

$$\lim_{x \to q^-} f(x) = f(q) ;$$

on the other hand

$$\lim_{x \to q^+} f(x) = \inf_{x > q} \sum_{a < x} 1/\lambda^2(a)$$

$$= \inf_{x > q} \left\{ \sum_{a < q} 1/\lambda^2(a) + \sum_{q < a < x} 1/\lambda^2(a) \right\}$$

$$= f(q) + \inf_{x > q} \left\{ \sum_{q < a < x} 1/\lambda^2(a) \right\}.$$

Let $n_0 (\geq 1)$ be a given natural number, and take a point $a_{n_0} \in A \cap [q, q + \frac{1}{n_0}]$ such that $\lambda(a_{n_0}) \leq \lambda(a)$ for all $a \in A \cap [q, q + \frac{1}{n_0}]$; next, take $n_1 (> n_0)$ such that $a_{n_0} \notin A \cap [q, q + \frac{1}{n_1}]$, and pick a point $a_{n_1} \in A \cap [q, q + \frac{1}{n_1}]$ such that $\lambda(a_{n_1}) \leq \lambda(a)$ for all $a \in A \cap [q, q + \frac{1}{n_1}]$. Now, inductively, take a natural number $n_k (> n_{k-1})$ such that $a_{n_{k-1}} \notin A \cap [q, q + \frac{1}{n_k}]$ and $a_{n_k} \in A \cap [q, q + \frac{1}{n_k}]$ such that $\lambda(a_{n_k}) \leq \lambda(a)$ for all $a \in A \cap [q, q + \frac{1}{n_k}]$. Let P be the set of all n_k $(k = 0, 1\ 2, \ldots)$ defined by the above process. Then it is clear that $\lambda(a_{n_k}) < \lambda(a_{n_{k'}})$ for any n_k and $n_{k'}$ belonging to P with $n_k < n_{k'}$. Clearly

$$\inf_{n \geq 1} \left\{ \sum_{\substack{n_k \geq n \\ n_k \in P}} 1/\lambda^2(a_{n_k}) \right\} = 0 .$$

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Let $a \in U$ with $f(a) \neq 0$. Then there exists $h \in C(X)$ such that $h(a) \neq 0$ and $h|X - W = 0$. Clearly, $0 \neq f \cdot h|D \in C(D)$. Then, by the assumption, there exists $h' \in C(X)$ such that $f \cdot (h \cdot h')|D = u|D \neq 0$ for some $u \in C(X)$. Note that if $f(c) \cdot h(c) \cdot h'(c) \neq 0$, then $c \in W$. Hence there exists a neighborhood V of c in W such that $hh'|V \neq 0$, and $f|V \cap D = \dfrac{u|V \cap D}{(hh')|V \cap D}$. Consequently the function $\dfrac{u|V}{(hh')|V}$ is the desired extension of $f|V \cap D$ to V.

Remark. It is known [1; 2] that if $\mathfrak{D}$ is the set of all dense open subsets of X, then $Q_{\mathfrak{D}}(X)$ is the maximal ring of quotients of $C(X)$ with respect to ν. Evidently the condition in Proposition 3 holds for every dense open subset D of X.

LEMMA 4. *Let A be a countable dense subset of irrational numbers and let $\lambda : A \to \{1, 2, 3, \ldots\}$ be a one-to-one mapping. For each $x \in \mathbb{R}$, define a function f by*

$$f(x) = \sum_{\substack{a < x \\ a \in A}} 1/\lambda^2(a);$$

then f is continuous at each point of $\mathbb{Q}$ but $f(a^+) > f(a)$ for each $a \in A$.

Proof. First we show $f(a^+) > f(a)$ for each $a \in A$. Take $a_0 \in A$, then

$$\lim_{x \to a_0^-} f(x) = \lim_{x \to a_0^-} \sum_{a < x} 1/\lambda^2(a)$$

$$= \sum_{a < a_0} 1/\lambda^2(a) = f(a_0);$$

on the other hand

$$\lim_{x \to a_0^+} f(x) = \lim_{x \to a_0^+} \sum_{a < x} 1/\lambda^2(a) = \sum_{a < a_0} 1/\lambda^2(a) +$$

$$\inf_{x > a_0} \left\{ \sum_{a_0 \leq a < x} 1/\lambda^2(a) \right\} > f(a_0).$$

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Hence $\inf_{x > q} \left\{ \sum_{q < a < x} 1/\lambda^2(a) \right\} = 0$. This implies that the function f is continuous at the point q.

COROLLARY. $C(\mathbb{Q})$ *is not a ring of quotients of* $C(\mathbb{R})$ *with respect to the embedding* $f \to f|\mathbb{Q}$.

Proof. If f is the function defined in Lemma 4, then $g \equiv f|\mathbb{Q} \in C(\mathbb{Q})$. Since every open set V in $\mathbb{R}$ contains a point of A, the function $g|V \cap \mathbb{Q}$ cannot be continuously extended to V. By the previous proposition, $C(\mathbb{Q})$ is not a ring of quotients of $C(\mathbb{R})$ with respect to the restriction homomorphism.

2. Totally unreal rings.

Definition. A maximal ideal M in a ring A is said to be *real* iff its quotient field A/M is isomorphic with $\mathbb{R}$. A ring is said to be *totally unreal* iff it does not have any real ideal.

It is of natural interest to know whether the maximal ring of quotients of $C(X)$ has real maximal ideals. We shall provide a sufficient condition for the maximal ring of quotients of $C(X)$ to be totally unreal. In the following, $\Delta(Q_{\mathfrak{D}}(X))$ denotes the set of unitary ring homomorphisms from $Q_{\mathfrak{D}}(X)$ into $\mathbb{R}$. Also note that each $\phi \in \Delta(Q_{\mathfrak{D}}(X))$ is onto. We prove the following result.

THEOREM 5. *If* $x \in \bigcap \mathfrak{D}$, *then there exists a unique* $\phi_x \in \Delta(Q_{\mathfrak{D}}(X))$ *such that* $(\phi_x \circ j_D)(f) = f(x)$ *for all* $f \in C(D)$ *and for each* $D \in \mathfrak{D}$, *and* $x \to \phi_x$ *is a one-to-one correspondence from* $\bigcap \mathfrak{D}$ *into* $\Delta(Q_{\mathfrak{D}}(X))$. *Moreover, if each* $D \in \mathfrak{D}$ *is realcompact, then* $x \to \phi_x$ *is onto*.

Proof. Let $x \in \bigcap \{D : D \in \mathfrak{D}\}$ be a fixed point. For each $D \in \mathfrak{D}$, define a mapping $\phi_D : C(D) \to \mathbb{R}$ by $\phi_D(f) = f(x)$, then ϕ_D is a unitary ring homomorphism, and, for each pair D, E in $\mathfrak{D}$ with $D \subseteq E$ and $f \in C(E)$, we have $\phi_E(f) = f(x) = (f|D)(x) = \phi_D(f|D)$. This implies that the family $(\phi_D)(D \in \mathfrak{D})$ is compatible with respect to the direct system $(C(D))(D \in \mathfrak{D})$. Hence there exists a unique ring homomorphism $\phi_x : Q_{\mathfrak{D}}(X) \to \mathbb{R}$ such that $\phi_x \circ j_D = \phi_D$ for each $D \in \mathfrak{D}$ (see Prop. 2). Since, for each $D \in \mathfrak{D}$, $C(D)$ separates the points of D, $\phi_x = \phi_y$ implies $x = y$; i.e., $x \to \phi_x$ is one-to-one. Finally, let each $D \in \mathfrak{D}$ be real-compact, and $\phi \in \Delta(Q_{\mathfrak{D}}(X))$. Then clearly $\phi \circ j_D$ is a unitary ring homomorphism from $C(D)$ into $\mathbb{R}$ for each $D \in \mathfrak{D}$. Since each D is realcompact, to the homomorphism $\phi \circ j_D$, there corresponds a point x_ϕ of D such that

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$(\phi \circ j_D)(f) = f(x_\phi)$ for all $f \in C(D)$ [3]. We claim that x_ϕ belongs to each member of $\mathfrak{D}$. Let E be a member of $\mathfrak{D}$. Then there exists a member D' in $\mathfrak{D}$ such that $D' \subseteq D \cap E$; and hence $j_D(f) = j_{D'}(f|D')$ for all $f \in C(D)$. Similarly, $\phi \circ j_{D'}$ is a unitary ring homomorphism from $C(D')$ into $\mathbf{R}$. Hence there exists $y \in D'$ such that $(\phi \circ j_{D'})(f') = f'(y)$ for all $f' \in C(D')$. In particular $(\phi \circ j_{D'})(f|D')$ $= (f|D')(y) = f(y)$ for all $f \in C(D)$; i.e., $f(x_\phi) = (\phi \circ j_D)(f)$ $= \phi(j_D(f)) = \phi(j_{D'}(f|D')) = (\phi \circ j_{D'})(f|D') = f(y)$ for all $f \in C(D)$. This implies that $x_\phi = y \in E$; i.e., $x_\phi \in \bigcap \{D : D \in \mathfrak{D}\}$. Since, for each $D \in \mathfrak{D}$, $\phi_{x_\phi} \circ j_D = \phi \circ j_D$ $(\equiv \phi'_D)$ and the family $(\phi'_D)(D \in \mathfrak{D})$ is compatible with respect to the direct system $(C(D))(D \in \mathfrak{D})$, and ϕ_{x_ϕ} is unique, hence $\phi_{x_\phi} = \phi$. Q. E. D.

COROLLARY 1. If $Q_{\mathfrak{D}}(X)$ is totally unreal, then $\bigcap \mathfrak{D} = \phi$, and the converse holds, provided each member of $\mathfrak{D}$ is realcompact.

COROLLARY 2. Let X be a separable realcompact space without isolated points such that every closed subset is a G_δ-set; then the maximal ring of quotients of $C(X)$ is totally unreal.

Proof. Note that if X is realcompact, and each point of X is a G_δ, then every subspace of X is realcompact [3]. Let A be a countable dense subset of X, say $A = \bigcup_{i \in I} \{a_i\}$ where the index set $I = \{1, 2, 3, \ldots\}$. For each $i \in I$, let J_i be a countable index set; then $\{a_i\} = \bigcap_{j \in J_i} V_{i,j}$, where $V_{i,j}$ is an open set containing a_i for each j. Then $A = \bigcup_{i \in I} \left(\bigcap_{j \in J_i} V_{i,j} \right) = \bigcap_{\phi \in \Phi} \left(\bigcup_{i \in I} V_{i,\phi(i)} \right)$ where Φ is the set of all functions ϕ with domain I such that $\phi(i) \in J_i$ for each $i \in I$. Hence A itself is an intersection of dense open sets. On the other hand, the set $X - \{a_1, a_2, \ldots, a_n\}$, $a_i \in A (i = 1, 2, \ldots, n)$ is a dense open subset of X. $\bigcap_{i=1}^{\infty} (X - \{a_1, \ldots, a_i\}) = X - \bigcup_{i=1}^{\infty} \{a_1, \ldots, a_i\}$ $= X - A$; thus $X - A$ is an intersection of dense open subsets of X. Let $\mathfrak{D}$ be the set of all dense open subsets of X. Then clearly $\bigcap \mathfrak{D} = \phi$. Hence the maximal ring of quotients of $C(X)$ is totally unreal.

COROLLARY 3. For a separable metric space X without isolated points, the maximal ring of quotients of $C(X)$ is totally unreal.

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