HEATH $g$-FUNCTIONS AND METRIZATION

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Abstract

In this paper, we present some new metrization theorems in terms of Heath $g$-functions.


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1. Introduction

In this short note we characterize metrizability in terms of Heath $g$-functions.

Heath in [3] introduced a method of describing a generalized metric property of a topological space $(X, \tau)$ by means of a function $g : \mathbb{N} \times X \to \tau$. Hodel, Fletcher, Lindgren and Nagata have modified this method to obtain important new classes of spaces.

A Heath $g$-function [$\text{COC-map (= countable open covering map)}$] for a topological space $X$ is a function $g$ from $\mathbb{N} \times X$ into the topology of $X$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n + 1, x) \subseteq g(n, x)$.

It is well known that many important classes of generalized metrizable spaces can be characterized in terms of a Heath $g$-function. In particular, $X$ is developable [3] ($\omega \Delta$-space) if and only if $X$ has a Heath $g$-function $g$ such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all $n$, then $p$ is a cluster point of the sequence $\langle x_n \rangle$ (then $\langle x_n \rangle$ has a cluster point).

A space $X$ is called a $\omega M$-space [4] if and only if $X$ has a Heath $g$-function $g$ such that if $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all $n$ then $\langle x_n \rangle$ has a cluster point. Let $\mathcal{G}$ be a collection of sets. We define $st(x, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : x \in G\}$ and $st^2(x, \mathcal{G}) = \bigcup_{y \in st(x, \mathcal{G})} st(y, \mathcal{G})$.

In this paper all spaces will be $T_0$, unless we state otherwise.

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2. Main results

First we consider what additional conditions need to be attached to a space already known to be stratifiable (conditions (1)–(3) in the next theorem are the axioms of stratifiability [2, Theorem 5.8]) to make it metrizable. To do this, the function \( g : \omega \times X \to \mathcal{I} \) can be strengthened to give it some sort of symmetry as shown by the next theorem.

**Theorem 2.1.** A space \( X \) is metrizable if and only if there exists a function \( g : \omega \times X \to \mathcal{I} \) such that

1. \( \{x\} = \bigcap_{n \in \omega} g(n, x) \);
2. if \( y \in g(n, x_n) \) for all \( n \) then \( x_n \to y \);
3. for any \( y \notin H \) closed, \( y \notin \bigcup \{g(x, n) : x \in H\} \) for some \( n \in \omega \);
4. if \( y \in g(n, x) \) then \( x \in g(n, y) \).

**Proof.** For any metric space we can define \( g \) to satisfy the axioms of stratifiability as given in [2, Theorem 5.8]. The fourth condition of the theorem holds because of the symmetry of a metric.

To prove the converse, we assume that without loss of generality,

\[ g(n, x) \subseteq g(n + 1, x) \text{ for any } x \in X. \]

If this was not the case, we define the function \( g^* : \omega \times X \to \mathcal{I} \) by \( g^*(n, x) = \bigcap_{k \leq n} g(k, x) \). Certainly each \( g^*(n, x) \) is open as the finite intersection of open sets and the axioms for stratifiability still hold since \( g^*(n, x) \subseteq g(n, x) \) for each \( n \in \omega \) and \( x \in X \). Notice also that condition (4) remains true when considering these new open sets.

\( X \) can be shown to be a \( T_1 \) space by showing \( \{x\} \) is closed for each \( x \in X \). Suppose \( y \notin \{x\} \); that is, \( y \neq x \). Then we must have some \( n \in \omega \) such that \( x \notin g(n, y) \), otherwise \( x \in \bigcap_{n \in \omega} g(n, y) = \{y\} \) and so the points are not distinct. Hence there is an open neighbourhood of \( y \) which does not meet \( \{x\} \) and so \( \{x\} \) is closed.

For each \( n \in \omega \), we define an open cover \( \mathcal{G}_n = \{g(n, x) : x \in X\} \). Suppose that \( x \) is in some open set \( U \). If we can show that there exists some \( n \in \omega \) such that \( st^2(x, \mathcal{G}_n) \subseteq U \) then since \( X \) is \( T_0 \), the space will be metrizable by the Moore Metrization theorem [1].

Firstly we notice that there must exist some \( n_0 \in \omega \) such that \( g(n_0, x) \subseteq U \), otherwise we can define a sequence of points \( x_n \) such that \( x_n \in g(n, x) \setminus U \) for each \( n \in \omega \). Then by our new symmetry condition, \( x \in g(n, x_n) \) for each \( n \in \omega \), hence \( x_n \to x \), contradicting the fact that \( x \in U \) since the points \( x_n \) all lie in the closed set \( X \setminus U \) and so their limit must also lie in \( X \setminus U \). Define \( U_1 = g(n_0, x) \) and notice that

\[ x \notin X \setminus U_1 \Rightarrow x \notin \bigcup \{g(n_1, y) : y \in X \setminus U_1\} = X \setminus U_2, \text{ some } n_1 \in \omega, \]
Let $n = \max\{n_0, n_1, n_2, n_3, n_4\}$. We now show that $st(x, G_n) \subseteq U$. If $x_2$ is any point in $st(x, G_n)$ then there is some $x_1$ such that $x \in g(n, x_1)$ and $x_2 \in g(n, x_1)$, hence $x_1 \in g(n, x_2)$ and $x_1 \in g(n_2, x_2)$. If we assume that $x_2 \notin U_3$, then $x_2 \in X \setminus U_3$, some $n_3 \in \omega$, $x \notin X \setminus U_2 \Rightarrow x \notin \bigcup\{g(n_2, y) : y \in X \setminus U_2\} = X \setminus U_3$, some $n_2 \in \omega$.

This means that $x_2 \in U_2$ and so $st(x, G_n) \subseteq U_2$.

The final stage of the proof is to show that $st^2(x, G_n) \subseteq U$ by showing that $st^2(x, G_n) \subseteq U_1$. Consider $x_4 \in st^2(x, G_n)$. This means we have some point $x_3$ such that $x_2 \in g(n, x_3)$ and $x_4 \in g(n, x_3)$ (for some $x_2 \in st(x, G_n)$), hence $x_3 \in g(n, x_4)$ and $x_3 \in g(n_1, x_4)$. If we assume that $x_4 \notin U_1$, then $x_3 \in \bigcup\{g(n_1, y) : y \in X \setminus U_1\} = X \setminus U_2$. Similarly, since we have $x_3 \notin U_2$ and $x_2 \in g(n, x_3)$ (hence $x_2 \in g(n_2, x_3)$), then $x_2 \in \bigcup\{g(n_2, y) : y \in X \setminus U_2\} = X \setminus U_3$ which contradicts the fact that $x_2 \in U_3$. This means that $x_4 \in U_1$ and so $st^2(x, G_n) \subseteq U_1 \subseteq U$.

We now consider some similar results where, instead of requiring convergence of sequences, we only require clustering.

**Theorem 2.2.** A space $X$ is metrizable if and only if there is a Heath $g$-function $g$ such that

1. if $x \in g(n, y)$ then $y \in g(n, x)$;
2. if $\{x, x_n\} \subseteq g(n, y_n)$ for all $n$ then $x$ is a cluster point of the sequence $(x_n)$.

**Proof.** Necessity is clear. For sufficiency: since the condition (2) gives developability to the space $X$, we need only to prove that $X$ is a regular and $wM$-space (every regular, developable, $wM$-space is metrizable [5]). We first prove $X$ is regular. Let $x \in U$ be open in $X$. Suppose $x_n \in g(n, x) - U$ for all $n \in \mathbb{N}$. Then $y_n \in g(n, x) \cap g(n, x_n)$ for each $n$. So $x \in g(n, y_n)$ and $x_n \in g(n, y_n)$. Therefore, we have $\{x, x_n\} \subseteq g(n, y_n)$, so $x$ is a cluster point of the sequence $(x_n)$. But $x \in U$ is open and $x_n \notin U$ for each $n$, which contradicts that $x$ is a cluster point of the sequence $(x_n)$. Therefore, $g(n, x) \subseteq U$ for some $n$ and $X$ is regular.

Finally, we prove $X$ is a $wM$-space. Let $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$. Now we want to show that $(x_n)$ has a cluster point. Let $p_n \in g(n, z_n) \cap g(n, y_n)$. Since $p_n \in g(n, z_n)$ and $x \in g(n, z_n)$, $\{x, p_n\} \subseteq g(n, z_n)$. Therefore, $x$ is a cluster point of the sequence $(p_n)$. There is a subsequence $(m(n))$ of the sequence $(n)$ such that $p_{m(n)} \in g(n, x)$, which implies that $x \in g(n, p_{m(n)})$. We
have \( p_{m(n)} \in g(n, y_{m(n)}) \), so \( y_{m(n)} \in g(m(n), p_{m(n)}) \subset g(n, p_{m(n)}) \), so \( x \) is a cluster point of the sequence \( \langle y_{m(n)} \rangle \). Therefore, there is a subsequence \( \langle m(n)(k) \rangle \) of the sequence \( \langle m(n) \rangle \) such that \( y_{m(n)(k)} \in g(k, x) \) for all \( k \) and hence \( x \in g(k, y_{m(n)(k)}) \) for all \( k \). Since

\[
\{x, y_{m(n)(k)} \} \subset g(n, p_{m(n)})\]

for all \( k \) and hence \( x \) is the cluster point of the sequence \( \langle x_{m(n)(k)} \rangle \). Therefore, \( x \) is the cluster point of the sequence \( \langle x_n \rangle \).

We define \( g^1(n, x) = g(n, x) \), and \( g^{k+1}(n, x) = \bigcup \{g(n, y) : y \in g^k(n, x)\} \) for \( k \geq 1 \).

**THEOREM 2.3.** A space \( X \) is metrizable if and only if there is a Heath g-function \( g \) such that

1. if \( x \in g(n, y) \) then \( y \in g(n, x) \);
2. \( \{g^2(n, x_n) : n \in \mathbb{N}\} \) is a local basis at \( x \) for all \( x \in X \).

**PROOF.** Let \( X \) be metrizable space with a sequence \( \{\mathcal{G}_n\}_{n \in \mathbb{N}} \) of open covers of \( X \) satisfying that \( \{st^2(x, \mathcal{G}_n)\} \) is a local base at \( x \) for all \( x \in X \). Put \( g(n, x) = st(x, \mathcal{G}_n) \) for each \( x \in X \) and for each \( n \). Then \( g \) is a COC-map which satisfies (1) and (2), because \( g^2(n, x_n) = st^2(x, \mathcal{G}_n) \).

For the converse, we can prove by induction on \( k \) that if \( \langle x_n \rangle \) is a sequence in \( X \) and \( x \in X \) with \( x_n \in g^k(n, x) \) for all \( n \) then \( x \) is a cluster point of \( \langle x_n \rangle \). From this it follows that if \( U \) open with \( x \in U \) then there is some \( n \) with \( g^4(n, x) \subset U \). Put \( \mathcal{G}_n = \{g(n, x) : x \in X\} \) for \( n \in \mathbb{N} \). Then \( \{st^2(x, \mathcal{G}_n)\} = g^4(n, x) \), so \( \{\mathcal{G}_n\}_{n \in \mathbb{N}} \) is a sequence of open covers such that \( \{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}\} \) is a local base at \( x \) for all \( x \in X \). Hence, by the Moore Metrization theorem [1], \( X \) is metrizable. This completes the proof.

**COROLLARY 2.4.** A space \( X \) is metrizable if and only if there is a Heath g-function \( g \) such that

1. if \( x \in g(n, y) \) then \( y \in g(n, x) \);
2. \( \{g^2(n, x_n) : n \in \mathbb{N}\} \) is a local basis at \( x \) for all \( x \in X \).

**THEOREM 2.5.** A space \( X \) is metrizable if and only if there is a Heath g-function \( g \) such that

1. if \( x \in g(n, y) \) then \( y \in g(n, x) \);
2. \( \bigcap_{n \in \mathbb{N}} g^2(n, x) = \{x\} \);
3. if \( \{x, x_n\} \subset g(n, y_n) \) then the sequence \( \langle x_n \rangle \) has a cluster point.
PROOF. It is easy to prove necessity. To prove sufficiency, we need to prove that $x$ is a cluster point of the sequence $\langle x_n \rangle$. Let $q$ be a cluster point of $\langle x_n \rangle$. Suppose that $q \neq x$. Then there are infinitely many integer $m \geq n$ such that $x_m \in g(n, q)$. Now we have $\{x, x_m\} \subset g(n, y_m)$. By conditions (1) and (2) we get $x \in g(n, y_m)$ and $y_m \in g(n, x_m)$. Therefore, $\{x_m : m \geq n\} \subset g^2(n, x)$, so $q \in \{x_m : m \geq n\} \subset g^2(n, x)$. Thus $q \in \bigcap_{n \in \mathbb{N}} g^2(n, x) = \{x\}$. Hence $q = x$, as required.

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