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PERIODIC SOLUTIONS OF A NONLINEAR OSCILLATORY SYSTEM WITH TWO DEGREES OF FREEDOM

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Abstract

We study a nonlinear oscillatory system with two degrees of freedom. By using the continuation theorem of coincidence degree theory, some sufficient conditions are obtained to establish the existence of periodic solutions of the system.

1. Introduction

In nonlinear oscillatory systems, periodic motion is very important. But establishing the existence of periodic solutions can be very difficult. Luckily there exist some kinds of periodic solutions in actual physical systems. Consequently, considerable research has been focussed on the development of approximate analytical methods and applications for these specific models. These methods however, have the disadvantage of being computationally intensive. For example, we refer to [1] in which the Lindsted-Poincare method is presented, as well as methods of averaging, the method of multiples scales *etc.* Generally these methods are effective for weakly nonlinear systems with a single degree of freedom. For systems with multiple degrees of freedom, researchers usually resort to numerical methods. These however have large errors for systems with nonlinearities. Hence it is very important to study the existence and uniqueness of periodic solutions for nonlinear systems.

Jun Liu [11] investigated periodic motion and stability for a class of nonlinear oscillating systems with two degrees of freedom. This model can be expressed by two mutual coupling second-order nonlinear differential equations, and these systems are

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usually governed by the differential equations

$$\begin{cases} \ddot{x} + P(t, x, \dot{x}, y, \dot{y})\dot{x} + Q(t, x, y)x = f(x, y)\cos\omega t, \\ \ddot{y} + \Phi(t, x, \dot{x}, y, \dot{y})\dot{y} + H(t, x, y)y = g(x, y)\cos\omega t, \end{cases} \tag{1.1}$$

where $P(t, x, \dot{x}, y, \dot{y})\dot{x}$ and $\Phi(t, x, \dot{x}, y, \dot{y})\dot{y}$ represent resistance, $Q(t, x, y)x$ and $H(t, x, y)y$ represent potential, $f(x, y)\cos\omega t$ and $g(x, y)\cos\omega t$ are periodical stiff external forces, and ω is horn-frequency.

Equations (1.1) can describe many physical phenomena and are very important in both theory and applications. This nonlinear system has been extensively applied to engineering systems, such as machine vibrations, dynamically buckled motions of elastic structures, the rolling motion of ships and the motion of rockets and satellites [2]. Investigations of this system are of importance in engineering. In [1, 2, 5, 8, 9, 13], the authors obtained approximate periodic solutions by using a numerical method for a specific system. In [11], Liu, using a Lyapunov function, studied the existence, uniqueness and stability of a periodic solution for system (1.1) and obtained sufficient conditions which guarantee the existence, uniqueness and asymptotic stability of a periodic solution.

To simplify computation, in this paper we pay attention to a special and very important oscillatory system called the damping oscillation. The damping oscillation of two oscillators is usually governed by the following two mutual coupling second-order nonlinear differential equations:

$$\begin{cases} m_1 x_1'' + (c_1^* + c_2^*)x_1' - c_2^* x_2' + f_1^*(x_1, x_2) = p_1^*(t), \\ m_2 x_2'' - c_2^* x_1' + (c_2^* + c_3^*)x_2' + f_2^*(x_1, x_2) = p_2^*(t), \end{cases} \tag{1.2}$$

where m_1 and m_2 are the masses of two oscillators, x_1 and x_2 are the displacements leaving the equipoise of two masses m_1 and m_2, f_1^* and f_2^* are the external exciting forces based on the two masses m_1 and m_2, c_1^*, c_2^* and c_3^* are linear coefficients of damping, and p_1^* and p_2^* are the nonlinear elastic forces caused by springs.

A time delay effect is very important in damping oscillations and can often impact on or change the status of a damping oscillatory system. For this physical reason, we take into account the time delay effect in system (1.2). Hence (1.2) can be rewritten as follows:

$$\begin{cases} m_1 x_1''(t) + (c_1^* + c_2^*)x_1'(t) - c_2^* x_2'(t) \\ \qquad + f_1^*(x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2)) = p_1^*(t), \\ m_2 x_2''(t) + (c_2^* + c_3^*)x_2'(t) - c_2^* x_1'(t) \\ \qquad + f_2^*(x_1(t), x_2(t), x_2(t - \tau_3), x_2(t - \tau_4)) = p_2^*(t). \end{cases} \tag{1.3}$$

When $x_1(t) = x_2(t)$, system (1.3) reduces to the following second-order delayed differential equation:

$$ax''(t) + bx'(t) + f(x(t), x(t-\tau)) = p(t). \tag{1.4}$$

When $b = 0$, Huang [7] discussed the Duffing equation (1.4) and proved the existence of a 2π-periodic solution of (1.4) under the condition that $|f(0, x)| \le M$ ($M > 0$ is a constant) and other conditions on $f(0, x)$. In the case when $b = 0$,

$$f(x(t), x(t-\tau)) = m^2 x(t) + g(x(t-\tau))$$

(m is a nonzero integer), Ma [12] also investigated (1.4) and established the existence of a 2π-periodic solution of this equation under the condition $|g(x)| \le M$ as well as other conditions on $g(x)$. When $f(x(t), x(t-\tau)) = abx(t) - f(x(t-1))$, $p(t) = 0$, an der Heiden [6] obtained an existence theorem for a nonconstant periodic solution of (1.4). In the case when $f(x(t), x(t-\tau)) = cx(t) + g(x(t-\tau))$, Zhang [16] investigated (1.4) and established the existence of a 2π-periodic solution of (1.4) under the condition that $|g(x)| \le M^* + c^*|x(t)|$ ($M^* > 0$ and $c^* > 0$ are two constants) as well as other conditions on a, b, c and c^*.

Liu [10] established the existence of a 2π-periodic solution for the equation

$$x'(t) = f(x'(t-\tau), x(t-\tau))$$

under the condition that $|f(x_1, x_2)| \le p_1 + p_2|x_1| + p_3|x_2| \pm f(x_1, x_2)$, (here $p_1 > 0$, $p_2 \ge 0$ and $p_3 \ge 0$ are three constants). Hence it is natural to consider the existence of a 2π-periodic solution of (1.4) under the condition that

$$|f(x_1, x_2)| \le p_1 + p_2|x_1| + p_3|x_2| \pm f(x_1, x_2).$$

Thus we will also consider the existence of a 2π-periodic solution of system (1.3) under this condition since (1.3) is a very important damping oscillatory system. Denote

$$\frac{c_1^* + c_2^*}{m_1} = c_1, \quad \frac{c_2^*}{m_1} = c_2, \quad \frac{f_1^*}{m_1} = f_1, \quad \frac{p_1^*}{m_1} = p_1,$$
$$\frac{c_2^*}{m_2} = c_2, \quad \frac{c_2^* + c_3^*}{m_2} = c_3, \quad \frac{f_2^*}{m_2} = f_2, \quad \frac{p_2^*}{m_2} = p_2.$$

Then system (1.3) can be rewritten as follows:

$$\begin{cases} x_1''(t) + c_1 x_1'(t) - c_2 x_2'(t) + f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) = p_1(t), \\ x_2''(t) - c_2 x_1'(t) + c_3 x_2'(t) + f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4)) = p_2(t), \end{cases} \tag{1.5}$$

where τ_i ($i = 1, 2, 3, 4$) are nonnegative constants, c_i ($i = 1, 2, 3$) are positive constants, $f_i : R^4 \to R$ ($i = 1, 2$) are continuous functions, and $p_i : R \to R$ ($i = 1, 2$) are continuous 2π-periodic functions.

On the existence of a periodic solution of system (1.5), fewer results are available in the present literature. In this paper, our purpose is to establish existence results for periodic solutions for system (1.5) using mathematical methods and techniques. For the above delayed system (1.5), approaches related to monotonicity fail, and the bifurcation technique cannot be applied to this system because $p_i(t)$ ($i = 1, 2$) are time-dependent functions. Thus it seems quite natural and reasonable to pursue an alternative approach. We will apply a continuation theorem [3] in coincidence degree theory to establish the existence of periodic solutions for system (1.5). For work concerning the existence of periodic solutions of delay differential equations using coincidence degree theory, we refer to [4, 7, 12, 14–16] and the references cited therein.

To make use of the continuation theorem of coincidence degree theory, we need to introduce some notation.

Let X, Y be real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm mapping of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by L_p the restriction of L to $\operatorname{Dom} L \cap \operatorname{Ker} P$, denote by $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ the inverse of L_p, and denote by $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

For convenience, we also cite the continuation theorem [3, page 40] below.

LEMMA 1.1. *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$ (that is, $QN : \overline{\Omega} \to Y$ and $K_p(I - Q)N : \overline{\Omega} \to Y$ are compact). Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$ (*here* deg *is Brouwer degree*).

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

2. Existence of periodic solutions

Before stating our results, we denote the Euclidean norm by $|\cdot|$ in R. Now we are in a position to state and prove our main results.

THEOREM 2.1. *Assume that the following conditions hold:*

(i) *There exist eight nonnegative constants* $\alpha_i, \beta_i, r_i, p_i$ *and two positive constants* q_i $(i = 1, 2)$ *such that, for all* $(x_1, x_2, x_3, x_4) \in R^4$ *and* $i = 1, 2$,

$$|f_i(x_1, x_2, x_3, x_4)| \le \alpha_i|x_1| + \beta_i|x_2| + r_i|x_3| + p_i|x_4| + q_i + f_i(x_1, x_2, x_3, x_4),$$

or

$$|f_i(x_1, x_2, x_3, x_4)| \le \alpha_i|x_1| + \beta_i|x_2| + r_i|x_3| + p_i|x_4| + q_i - f_i(x_1, x_2, x_3, x_4);$$

(ii) *There exists a constant* $h > 0$ *such that when* $\min\{x_1, x_3\} > h$,

$$f_1(x_1, x_2, x_3, x_4) > \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt, \quad f_1(-x_1, x_2, -x_3, x_4) < \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt$$

and that when $\min\{x_2, x_4\} > h$

$$f_2(x_1, x_2, x_3, x_4) > \frac{1}{2\pi}\int_0^{2\pi} p_2(t)\,dt, \quad f_2(x_1, -x_2, x_3, -x_4) < \frac{1}{2\pi}\int_0^{2\pi} p_2(t)\,dt;$$

(iii) $A_1B_1 > C_1D_1$ *with* $A_1 > 0$; *or*
(iv) $A_2B_2 > C_2D_2$ *with* $A_2 > 0$, *where*

$$\begin{aligned} A_1 &= [1 - 2\pi(c_3 + 2\pi\beta_2 + 2\pi p_2)], & A_2 &= [1 - 4\pi^2(\alpha_2 + r_2) - 2\pi^2(\beta_2 + p_2) - \pi c_2], \\ B_1 &= [1 - 2\pi(c_1 + 2\pi\alpha_1 + 2\pi r_1)], & B_2 &= [1 - 4\pi^2(\alpha_1 + r_1) - 2\pi^2(\beta_1 + p_1) - \pi c_2], \\ C_1 &= 4\pi^2[c_2 + 2\pi(\alpha_2 + r_2)], & C_2 &= [2\pi^2(\beta_1 + p_1) + \pi c_2], \\ D_1 &= [c_2 + 2\pi(\beta_1 + p_1)], & D_2 &= [2\pi^2(\beta_2 + p_2) + \pi c_2]. \end{aligned}$$

Then system (1.5) *has at least one* 2π*-periodic solution.*

In the remainder of this section, we give the proof of Theorem 2.1. In order to apply Lemma 1.1, it is crucial to find the required open and bounded subset in a properly chosen space. This can be achieved by establishing some *a priori* estimates, which are, as will be seen, quite technical. In what follows, we will always let

$$l_i \overset{\text{def}}{=} \max_{t\in[0,2\pi]} |p_i(t)|, \quad i = 1, 2, \qquad \|x\|_2 \overset{\text{def}}{=} \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{1/2}, \quad \forall x \in R.$$

PROOF OF THEOREM 2.1. First we prove Theorem 2.1 in the case when conditions (i), (ii) and (iii) hold. Without loss of generality, we assume that the first inequality in condition (i) holds. In order to apply Lemma 1.1 to system (1.5), we consider the spaces

$$X = \{(x_1(t), x_2(t))^T \in C^1(R, R^2) : x_i(t + 2\pi) = x_i(t), i = 1, 2\} \quad \text{and}$$
$$Y = \{(x_1(t), x_2(t))^T \in C(R, R^2) : x_i(t + 2\pi) = x_i(t), i = 1, 2\}$$

respectively equipped with the norms

$$\|(x_1, x_2)^T\|_X = \max_{t\in[0,2\pi]} |x_1(t)| + \max_{t\in[0,2\pi]} |x_2(t)| + \max_{t\in[0,2\pi]} |x_1'(t)| + \max_{t\in[0,2\pi]} |x_2'(t)|$$

and

$$\|(x_1, x_2)^T\|_Y = \max_{t\in[0,2\pi]} |x_1(t)| + \max_{t\in[0,2\pi]} |x_2(t)|.$$

With the above norms, X and Y are Banach spaces. Define the operators L and N by

$$N\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -c_1x_1'(t) + c_2x_2'(t) - f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) + p_1(t) \\ c_2x_1'(t) - c_3x_2'(t) - f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4)) + p_2(t) \end{bmatrix}$$

and

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix}.$$

Define two project operators as follows:

$$P : X \to X, \quad P\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} x_1(t)\,dt \\ \frac{1}{2\pi}\int_0^{2\pi} x_2(t)\,dt \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in X,$$

$$Q : Y \to Y, \quad Q\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} x_1(t)\,dt \\ \frac{1}{2\pi}\int_0^{2\pi} x_2(t)\,dt \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in Y.$$

Since $\operatorname{Ker} L = R^2$ and

$$\operatorname{Im} L = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in X : \int_0^{2\pi} x_i(t)\,dt = 0,\ i = 1, 2\right\},$$

$\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \dim Y/\operatorname{Im} L = 2$. Therefore L is a Fredholm mapping of index zero.

For the above L and N, $Lx = \lambda Nx$ reads

$$\begin{cases} x_1''(t) + \lambda c_1x_1'(t) - \lambda c_2x_2'(t) + \lambda f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) = \lambda p_1(t), \\ x_2''(t) - \lambda c_2x_1'(t) + \lambda c_3x_2'(t) + \lambda f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4)) = \lambda p_2(t). \end{cases} \tag{2.1}$$

Suppose that $(x_1(t), x_2(t))^T \in X$ is a solution of system (2.1) for a certain $\lambda \in (0, 1)$. Integrating (2.1) over $[0, 2\pi]$ gives

$$\int_0^{2\pi} f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2))\,dt = \int_0^{2\pi} p_1(t)\,dt \tag{2.2}$$

and

$$\int_0^{2\pi} f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4))\,dt = \int_0^{2\pi} p_2(t)\,dt. \tag{2.3}$$

From condition (ii) in Theorem 2.1 and (2.2), it follows that there exist $\tau \in \{0, \tau_1\}$, a point $t^* \in [0, 2\pi]$ and a constant $\overline{M} > 0$ such that

$$x_1(t^* - \tau) \leq \overline{M}. \tag{2.4}$$

If this is not true, then for all $M > 0$ and $t \in [0, 2\pi]$, we have $x_1(t) > M$ and $x_1(t - \tau_1) > M$, which, in view of condition (ii) in Theorem 2.1, contradicts (2.2). Thus (2.4) holds. Denoting $t^* - \tau = \xi_1 + 2\pi k$, where $\xi_1 \in [0, 2\pi]$ and k is an integer, then

$$x_1(\xi_1) \leq \overline{M}. \tag{2.5}$$

Similarly, we can find a point $\xi_2 \in [0, 2\pi]$ and a constant $N > 0$ such that

$$x_1(\xi_2) \geq -N. \tag{2.6}$$

Then from (2.5) and (2.6) we can obtain for all $t \in [0, 2\pi]$,

$$x_1(t) = x_1(\xi_1) - \int_{\xi_1}^{t} x_1'(s)\,ds \leq x_1(\xi_1) + \int_0^{2\pi} |x_1'(t)|\,dt \leq \overline{M} + \int_0^{2\pi} |x_1'(t)|\,dt$$

and

$$x_1(t) = x_1(\xi_2) - \int_{\xi_2}^{t} x_1'(s)\,ds \geq -N - \int_0^{2\pi} |x_1'(t)|\,dt.$$

Consequently

$$\begin{aligned} |x_1(t)| &\leq \max\left\{\overline{M} + \int_0^{2\pi} |x_1'(t)|\,dt,\ N + \int_0^{2\pi} |x_1'(t)|\,dt\right\} \\ &< \overline{M} + N + \int_0^{2\pi} |x_1'(t)|\,dt \stackrel{\text{def}}{=} d_1 + \int_0^{2\pi} |x_1'(t)|\,dt. \end{aligned} \tag{2.7}$$

By condition (ii) in Theorem 2.1 and (2.3), using the same argument as for obtaining (2.7) gives

$$\begin{aligned} |x_2(t)| &\leq \max\left\{M^* + \int_0^{2\pi} |x_2'(t)|\,dt,\ N^* + \int_0^{2\pi} |x_2'(t)|\,dt\right\} \\ &< M^* + N^* + \int_0^{2\pi} |x_2'(t)|\,dt \stackrel{\text{def}}{=} d_2 + \int_0^{2\pi} |x_2'(t)|\,dt, \end{aligned} \tag{2.8}$$

where M^* and N^* are two positive constants.

From the first equation of system (2.1), we obtain

$$\begin{aligned} \int_0^{2\pi} |x_1''(t)|\,dt \leq{}& c_1 \int_0^{2\pi} |x_1'(t)|\,dt + \int_0^{2\pi} |p_1(t)|\,dt + c_2 \int_0^{2\pi} |x_2'(t)|\,dt \\ &+ \int_0^{2\pi} \big[\alpha_1|x_1(t)| + \beta_1|x_2(t)| + r_1|x_1(t-\tau_1)| + p_1|x_2(t-\tau_2)| \\ &+ q_1 + f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2))\big]\,dt, \end{aligned}$$

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which, together with (2.2), implies that

$$\int_0^{2\pi} |x_1''(t)|\,dt \le c_1 \int_0^{2\pi} |x_1'(t)|\,dt + c_2 \int_0^{2\pi} |x_2'(t)|\,dt + 2\pi(2l_1 + q_1)$$
$$+ 2\pi(\alpha_1 + r_1) \max_{t\in[0,2\pi]} |x_1(t)| + 2\pi(\beta_1 + p_1) \max_{t\in[0,2\pi]} |x_2(t)|. \quad (2.9)$$

From the second equation of system (2.1), we obtain

$$\int_0^{2\pi} |x_2''(t)|\,dt \le c_2 \int_0^{2\pi} |x_1'(t)|\,dt + c_3 \int_0^{2\pi} |x_2'(t)|\,dt + \int_0^{2\pi} |p_2(t)|\,dt$$
$$+ \int_0^{2\pi} \big[\alpha_2|x_1(t)| + \beta_2|x_2(t)| + r_2|x_1(t-\tau_3)| + p_2|x_2(t-\tau_4)|$$
$$+ q_2 + f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4))\big]\,dt,$$

which, together with (2.3), implies that

$$\int_0^{2\pi} |x_2''(t)|\,dt \le c_2 \int_0^{2\pi} |x_1'(t)|\,dt + c_3 \int_0^{2\pi} |x_2'(t)|\,dt + 2\pi(2l_2 + q_2)$$
$$+ 2\pi(\alpha_2 + r_2) \max_{t\in[0,2\pi]} |x_1(t)| + 2\pi(\beta_2 + p_2) \max_{t\in[0,2\pi]} |x_2(t)|. \quad (2.10)$$

Substituting (2.7) and (2.8) into (2.9) and (2.10) gives

$$\begin{aligned}\int_0^{2\pi} |x_2''(t)|\,dt &\le c_1 \int_0^{2\pi} |x_1'(t)|\,dt + c_2 \int_0^{2\pi} |x_2'(t)|\,dt \\ &\quad + 2\pi(2l_1 + q_1) + 2\pi(\alpha_1 + r_1)d_1 + 2\pi(\beta_1 + p_1)d_2 \\ &\quad + 2\pi(\alpha_1 + r_1) \int_0^{2\pi} |x_1'(t)|\,dt + 2\pi(\beta_1 + p_1) \int_0^{2\pi} |x_2'(t)|\,dt \\ &= [c_1 + 2\pi(\alpha_1 + r_1)] \int_0^{2\pi} |x_1'(t)|\,dt \\ &\quad + [c_2 + 2\pi(\beta_1 + p_1)] \int_0^{2\pi} |x_2'(t)|\,dt \\ &\quad + 2\pi(2l_1 + q_1 + \alpha_1 d_1 + r_1 d_1 + \beta_1 d_2 + p_1 d_2) \end{aligned} \quad (2.11)$$

and

$$\begin{aligned}\int_0^{2\pi} |x_2''(t)|\,dt &\le [c_2 + 2\pi(\alpha_2 + r_2)] \int_0^{2\pi} |x_1'(t)|\,dt \\ &\quad + [c_3 + 2\pi(\beta_2 + p_2)] \int_0^{2\pi} |x_2'(t)|\,dt \\ &\quad + 2\pi(2l_2 + q_2 + \alpha_2 d_2 + r_2 d_2 + \beta_2 d_2 + p_2 d_2). \end{aligned} \quad (2.12)$$

Since $x_i(0) = x_i(2\pi)$, $i = 1, 2$, then there exist two points $\xi_i \in (0, 2\pi)$ $(i = 1, 2)$ such that $x_i'(\xi_i) = 0$, $i = 1, 2$. Thus for all $t \in [0, 2\pi]$, we have $x_i'(t) = \int_{\xi_i}^t x_i''(s)\,ds$, $i = 1, 2$. Hence

$$|x_i'(t)| \le \int_0^{2\pi} |x_i''(t)\,dt, \quad i = 1, 2. \tag{2.13}$$

Substituting (2.13) into (2.11) and (2.12), we have

$$B_1 \int_0^{2\pi} |x_1''(t)|\,dt \le 2\pi[c_2 + 2\pi(\beta_1 + p_1)] \int_0^{2\pi} |x_2''(t)|\,dt + M_1 \tag{2.14}$$

and

$$A_1 \int_0^{2\pi} |x_2''(t)|\,dt \le 2\pi[c_2 + 2\pi(\alpha_2 + r_2)] \int_0^{2\pi} |x_1''(t)|\,dt + M_2, \tag{2.15}$$

where M_1 and M_2 are two positive constants. From (2.14) and (2.15), we obtain

$$(A_1B_1 - C_1D_1) \int_0^{2\pi} |x_1''(t)|\,dt \le M_1A_1 + 2\pi M_2[c_2 + 2\pi(\beta_1 + p_1)], \tag{2.16}$$

from which, together with (2.15), it follows that there exist two positive constants d_3 and d_4 such that

$$\int_0^{2\pi} |x_1''(t)|\,dt < d_3 \quad \text{and} \quad \int_0^{2\pi} |x_2''(t)|\,dt < d_4. \tag{2.17}$$

From (2.13) and (2.17), it follows that there exist two positive constants R_3 and R_4 such that $|x_1'(t)| < R_3$ and $|x_2'(t)| < R_4$. From (2.7), (2.8) and (2.17), it follows that there exist two positive constants R_1 and R_2 such that $|x_1(t)| < R_1$ and $|x_2(t)| < R_2$.

Clearly, R_i $(i = 1, 2, 3, 4)$ are independent of λ. Denote

$$M = R_1 + R_2 + R_3 + R_4 + C,$$

where $C > 0$ is taken sufficiently large such that $M > 2h$. Now we take $\Omega = \{(x_1, x_2)^T \in X : \|(x_1, x_2)^T\|_X < M\}$. By an easy computation, we find that the inverse K_p of L_p has the form $K_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$,

$$K_p \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \int_0^t ds \int_0^s x_1(u)\,du - \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t ds \int_0^s x_1(u)\,ds \\ \int_0^t ds \int_0^s x_2(u)\,du - \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t ds \int_0^s x_2(u)\,ds \end{bmatrix}, \quad \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \overline{\Omega}.$$

It is easy to show that QN and $K_p(I - Q)N$ are continuous by the Lebesgue theorem. Moreover, $QN(\overline{\Omega})$, $K_p(I - Q)N(\overline{\Omega})$ are relatively compact for a bounded set $\Omega \subset X$ by the Arzela-Ascoli theorem. Therefore N is L-compact on $\overline{\Omega}$. Condition (a) in

Lemma 1.1 is now satisfied. When $(x_1, x_2)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(x_1, x_2)^T$ is a constant vector in R^2 with $|x_1| + |x_2| = M$. Then

$$QN\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -f_1(x_1, x_2, x_1, x_2) + \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt \\ -f_2(x_1, x_2, x_1, x_2) + \frac{1}{2\pi}\int_0^{2\pi} p_2(t)\,dt \end{bmatrix}.$$

Therefore, when $(x_1, x_2) \in \partial\Omega \cap \operatorname{Ker} L$, by condition (ii) in Theorem 2.1,

$$QN\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This satisfies condition (b) in Lemma 1.1. In order to show that condition (c) in Lemma 1.1 is satisfied, we define the homotopy $\phi : \operatorname{Dom} L \times [0, 1] \to X$ by

$$\phi(x_1, x_2, \mu^*) = \mu^*(x_1, x_2)^T + (1 - \mu^*)\begin{bmatrix} f_1(x_1, x_2, x_1, x_2) - \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt \\ f_2(x_1, x_2, x_1, x_2) - \frac{1}{2\pi}\int_0^{2\pi} p_2(t)\,dt \end{bmatrix},$$

where $x_1, x_2 \in R$, $\mu^* \in [0, 1]$.

When $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \partial\Omega \cap \operatorname{Ker} L$, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is a constant vector in R^2 with $|x_1| + |x_2| = M$. We will show that when $(x_1, x_2)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\phi(x_1, x_2, \mu^*) \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. In view of the constant vector $(x_1, x_2)^T$ satisfying $|x_1| + |x_2| = M > 2h$, we have $|x_1| > h$ or $|x_2| > h$. We assume that $|x_1| > h$ without loss of generality. When $x_1 > h$, condition (ii) gives

$$f_1(x_1, x_2, x_1, x_2) > \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt,$$

thus

$$\mu^* x_1 + (1 - \mu^*)\left[f_1(x_1, x_2, x_1, x_2) - \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt\right] > 0.$$

Similarly, when $-x_1 > h$, we have

$$u^* x_1 + (1 - \mu^*)\left[f_1(x_1, x_2, x_1, x_2) - \frac{1}{2\pi}\int_0^{2\pi} p_1(t)\,dt\right] < 0.$$

Therefore when $(x_1, x_2)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\phi(x_1, x_2, \mu^*) \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. By topological degree theory and by taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(x_1, x_2) \to (x_1, x_2)$, we have

$$\begin{aligned} &\deg(QN(x_1, x_2)^T, \Omega \cap \operatorname{Ker} L, (0, 0)^T) \\ &\quad = \deg((-x_1, -x_2)^T, \Omega \cap \operatorname{Ker} L, (0, 0)^T) \neq 0. \end{aligned}$$

Hence condition (c) in Lemma 1.1 is satisfied. Therefore system (1.5) has at least one 2π-periodic solution.

Next we prove Theorem 2.1 in the case when conditions (i), (ii) and (iv) hold. We assume that the first inequality in condition (i) of Theorem 2.1 holds. In order to apply Lemma 1.1 to system (1.5), we define the same Banach spaces X and Y, operators L and N, project operators P and Q as those defined in the proof of Theorem 2.1. For the above L and N, $Lx = \lambda Nx$ reads

$$\begin{cases} x_1''(t) + \lambda c_1 x_1'(t) - \lambda c_2 x_2'(t) \\ \qquad + \lambda f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) = \lambda p_1(t), \\ x_2''(t) - \lambda c_2 x_1'(t) + \lambda c_3 x_2'(t) \\ \qquad + \lambda f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4)) = \lambda p_2(t). \end{cases} \tag{2.18}$$

Suppose that $(x_1(t), x_2(t))^T \in X$ is a solution of system (2.18) for a certain $\lambda \in (0, 1)$. From the proofs of (2.2), (2.3), (2.7) and (2.8) in the proof of Theorem 2.1, we have

$$|x_i(t)| < d_i + \int_0^{2\pi} |x_i'(t)|\, dt, \quad i = 1, 2, \tag{2.19}$$

$$\int_0^{2\pi} f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2))\, dt = \int_0^{2\pi} p_1(t)\, dt \tag{2.20}$$

and

$$\int_0^{2\pi} f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4))\, dt = \int_0^{2\pi} p_2(t)\, dt. \tag{2.21}$$

Multiplying the first equation of (2.18) by $x_1(t)$ and integrating over $[0, 2\pi]$ gives

$$-\|x_1'\|_2^2 = \lambda \int_0^{2\pi} x_1(t) \left[-c_2 x_2'(t) + f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) - p_1(t)\right] dt,$$

which, together with (2.20), implies that

$$\begin{aligned} \|x_1'\|_2^2 \le & \int_0^{2\pi} |x_1(t)| \big[c_2 |x_2'(t)| + \alpha_1 |x_1(t)| + \beta_1 |x_2(t)| + r_1 |x_1(t-\tau_1)| \\ & + p_1 |x_2(t-\tau_2)| + f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) + q_1 + l_1\big] dt \\ \le & \max_{t\in[0,2\pi]} |x_1(t)| \left[c_2 \int_0^{2\pi} |x_2'(t)|\, dt + (\alpha_1 + r_1) \int_0^{2\pi} |x_1(t)|\, dt \right. \\ & \left. + (\beta_1 + p_1) \int_0^{2\pi} |x_2(t)|\, dt + 2\pi(q_1 + l_1)\right]. \end{aligned} \tag{2.22}$$

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Substituting (2.19) into (2.22) gives

$$\begin{aligned}\|x_1'\|_2^2 &\le \left(d_1 + \int_0^{2\pi} |x_1(t)|\,dt\right)\Big[2\pi(\alpha_1 + r_1)d_1 + 2\pi(\beta_1 + p_1)d_2 + 2\pi(q_1 + l_1) \\ &\quad + 2\pi(\beta_1 + p_1)\int_0^{2\pi} |x_2'(t)|\,dt + c_2\int_0^{2\pi} |x_2'(t)|\,dt \\ &\quad + 2\pi(\alpha_1 + r_1)\int_0^{2\pi} |x_1'(t)|\,dt\Big] \\ &\le \left(d_1 + \sqrt{2\pi}\|x_1'\|_2\right)\Big[A + 2\pi\sqrt{2\pi}(\beta_1 + p_1)\|x_2'\|_2 + \sqrt{2\pi}c_2\|x_2'\|_2 \\ &\quad + 2\pi\sqrt{2\pi}(\alpha_1 + r_1)\|x_1'\|_2\Big],\end{aligned}$$

where $A > 0$ is a constant. Thus by using the inequality

$$\frac{2}{\sqrt{\varepsilon}}\sqrt{\varepsilon}ab \le \frac{a^2}{\varepsilon} + \varepsilon b^2, \quad 2ab \le a^2 + b^2,\ \varepsilon > 0,$$

we obtain

$$\begin{aligned}\left[1 - 4\pi^2(\alpha_1 + r_1)\right]\|x_1'\|_2^2 &\le d_1 A + \left[(2\pi)^{3/2}d_1(\alpha_1 + r_1) + \sqrt{2\pi}A\right]\|x_1'\|_2 \\ &\quad + \left[(2\pi)^{3/2}d_1(\beta_1 + p_1) + \sqrt{2\pi}c_2 d_1\right]\|x_2'\|_2 \\ &\quad + \left[4\pi^2(\beta_1 + p_1) + 2\pi c_2\right]\|x_1'\|_2\|x_2'\|_2 \\ &\le d_1 A + \frac{1}{2\varepsilon}\left[(2\pi)^{3/2}d_1(\alpha_1 + r_1) + \sqrt{2\pi}A\right] \\ &\quad + \frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_1(\alpha_1 + r_1) + \sqrt{2\pi}A\right]\|x_1'\|_2^2 \\ &\quad + \frac{1}{2\varepsilon}\left[(2\pi)^{3/2}d_1(\beta_1 + p_1)d_1 + \sqrt{2\pi}c_2 d_1\right] \\ &\quad + \frac{\varepsilon}{2}\left[(2\pi)^{3/2}(\beta_1 + p_1)d_1 + \sqrt{2\pi}c_2 d_1\right]\|x_2'\|_2^2 \\ &\quad + \left[2\pi^2(\beta_1 + p_1) + \pi c_2\right]\left(\|x_1'\|_2^2 + \|x_2'\|_2^2\right),\end{aligned}$$

here ε is a chosen positive constant, which implies that

$$\left\{B_2 - \frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_1(\alpha_1 + r_1) + \sqrt{2\pi}A\right]\right\}\|x_1'\|_2^2 \le B + C\varepsilon\|x_2'\|_2^2 + C_2\|x_2'\|_2^2, \qquad (2.23)$$

where B and C are two positive constants.

Multiplying the second equation of (2.18) by $x_2(t)$ and integrating over $[0, 2\pi]$ gives

$$-\|x_2'\|_2^2 = \lambda\int_0^{2\pi} x_2(t)[-c_2 x_1'(t) + f_2(x_1(t), x_2(t), x_1(t - \tau_3), x_2(t - \tau_4)) - p_2(t)]dt,$$

from which a parallel argument to (2.23) shows that

$$\left\{A_2-\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_2(\alpha_2+r_2)+\sqrt{2\pi}A^*\right]\right\}\|x_2'\|_2^2\le B^*+C^*\varepsilon\|x_1'\|_2^2+D_2\|x_1'\|_2^2, \quad (2.24)$$

where B^*, C^* and A^* are three positive constants. In view of condition (iv) in Theorem 2.1, we can choose ε small enough such that

$$B_2>\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_1(\alpha_1+r_1)+\sqrt{2\pi}A\right]$$

and

$$\left\{A_2-\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_2(\alpha_2+r_2)+\sqrt{2\pi}A^*\right]\right\}\left\{B_2-\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_1(\alpha_1+r_1)+\sqrt{2\pi}A\right]\right\}$$
$$>C\varepsilon[C^*\varepsilon+2\pi(\beta_2+p_2)+c_2\pi]+C^*\varepsilon C_2+C_2D_2. \quad (2.25)$$

From (2.23)–(2.25), it follows that

$$\left\{A_2-\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_2(\alpha_2+r_2)+\sqrt{2\pi}A^*\right]\right\}$$
$$\times\left\{B_2-\frac{\varepsilon}{2}\left[(2\pi)^{3/2}d_1(\alpha_1+r_1)+\sqrt{2\pi}A\right]\right\}\|x_1'\|_2^2$$
$$\le D+\left\{C\varepsilon[C^*\varepsilon+2\pi(\beta_2+p_2)+c_2\pi]+C^*\varepsilon C_2+C_2D_2\right\}\|x_1'\|_2^2, \quad (2.26)$$

where $D>0$ is a constant.

The rest of the proof is similar to that of Theorem 2.1 and we omit it. □

3. Example

Now an example is given to illustrate our results.

EXAMPLE 1. Consider the following nonlinear oscillatory system of two degrees of freedom:

$$\begin{cases}x_1''(t)+c_1x_1'(t)-c_2x_2'(t)+e^{x_1(t)+x_1(t-\tau_1)+\frac{1}{2}\sin^2x_2(t-\tau_2)+\frac{1}{2}\sin^2x_2(t)}=1+\sin t,\\ x_2''(t)-c_2x_1'(t)+c_3x_2'(t)+e^{x_2(t)+x_2(t-\tau_4)+\frac{1}{2}\sin^2x_1(t-\tau_3)+\frac{1}{2}\sin^2x_1(t)}=1+\sin t,\end{cases} \quad (3.1)$$

where τ_i $(i=1,2,3,4)$ and c_i $(i=1,2,3)$ are seven positive constants.

Since $p_1(t)=1+\sin t$, $p_2(t)=1+\sin t$,

$$f_1(x_1(t),x_2(t),x_1(t-\tau_1),x_2(t-\tau_2))=e^{x_1(t)+x_1(t-\tau_1)+\frac{1}{2}\sin^2x_2(t-\tau_2)+\frac{1}{2}\sin^2x_2(t)}>0$$

and

$$f_2(x_1(t),x_2(t),x_1(t-\tau_3),x_2(t-\tau_4))=e^{x_2(t)+x_2(t-\tau_4)+\frac{1}{2}\sin^2x_1(t-\tau_3)+\frac{1}{2}\sin^2x_1(t)}>0,$$

then there exist $\alpha_i = \beta_i = r_i = p_i = 0$, $q_i = 1$, $i = 1, 2$, such that condition (i) in Theorem 2.1 is satisfied, and if we take $(1 - 2\pi c_3)(1 - 2\pi c_1) > 4\pi^2 c_2^2$, with $2\pi c_1 < 1$, then condition (iii) is satisfied. When $x_1(t) > 1$ and $x_1(t - \tau_1) > 1$,

$$f_1(x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)) > e^2 > 1,$$
$$f_1(-x_1(t), x_2(t), -x_1(t-\tau_1), x_2(t-\tau_2)) < e^{-2+1} = e^{-1} < 1.$$

When $x_2(t) > 1$, $x_2(t - \tau_4) > 1$,

$$f_2(x_1(t), x_2(t), x_1(t-\tau_3), x_2(t-\tau_4)) > e^2 > 1,$$
$$f_2(x_1(t), -x_2(t), x_1(t-\tau_3), -x_2(t-\tau_4)) < e^{-2+1} = e^{-1} < 1.$$

However

$$\frac{1}{2\pi}\int_0^{2\pi} p_i(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} (1 + \sin t)\,dt = 1, \quad i = 1, 2.$$

Thus condition (ii) in Theorem 2.1 is satisfied. Therefore, by Theorem 2.1, system (3.1) has at least one 2π-periodic solution.

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