ON MAXIMUM MATCHINGS IN CUBIC GRAPHS
WITH A BOUNDED NUMBER OF BRIDGE-COVERING PATHS

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It is proved that if $G$ is a connected cubic graph of order $p$
all of whose bridges lie on $r$ edge-disjoint paths of $G$,
then every maximum matching of $G$ contains at least $p/2 - \lfloor 2r/3 \rfloor$
edges. Moreover, this result is shown to be best possible.

1. Introduction and historical background

A matching in a graph $G$ is a set of pairwise nonadjacent (independent)
edges of $G$. A matching with maximum cardinality is a maximum
matching. If $G$ has order $p$, then a matching of cardinality $p/2$ is
called a perfect matching. Graphs with perfect matchings were
categorized by Tutte [5].

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THEOREM A. (Tutte). A graph $G$ has a perfect matching if and only if for every proper subset $S$ of $V(G)$, the number of odd components of $G-S$ does not exceed $|S|$.

Much research has centred around the determination of regular graphs that contain perfect matchings. A well known result on this subject is due to Petersen [4].

THEOREM B. (Petersen). Every cubic graph with at most two bridges contains a perfect matching.

This result cannot be improved, in general, since cubic graphs having three bridges but no perfect matchings exist. The graph of Figure 1 is the unique smallest such graph.

![Figure 1](https://example.com/figure1.png)

Note that the three bridges of the graph of Figure 1 do not lie on a single path. Indeed, since this graph has no perfect matching, this property is necessary, by a result of Errera [3].

THEOREM C. (Errera). If all the bridges of a connected cubic graph $G$ lie on a single path of $G$, then $G$ has a perfect matching.

The goal of this paper is to provide a generalisation of Theorem C by establishing a lower bound on the cardinality of a maximum matching in a connected cubic graph all of whose bridges lie on a specified number of edge-disjoint paths. Towards this end we state the following generalisation (see [1]) of the aforementioned theorem of Tutte.
THEOREM D. Let $G$ be a cubic graph of order $p$ and let $l$ be an integer with $0 \leq l \leq p/2$. Then every maximum matching of $G$ has at least $(p - 2l)/2$ edges if and only if for every proper subset $S$ of $V(G)$, the number of odd components of $G - S$ does not exceed $|S| + 2l$.

2. The main result

We are now prepared to present a bound on the number of edges in a maximum matching in a connected cubic graph $G$ in terms of the number of paths containing the bridges of $G$.

THEOREM 1. If the bridges of a connected cubic graph $G$ lie on $r$ edge-disjoint paths of $G$, then each maximum matching of $G$ contains at least $p/2 - \lfloor 2r/3 \rfloor$ edges.

Proof. Suppose, to the contrary, that $G$ contains a maximum matching $M$ with fewer than $p/2 - \lfloor 2r/3 \rfloor$ edges. By Theorem D there exists a proper subset $S$ of $V(G)$ such that the number $n$ of odd components of $G - S$ exceeds $|S| + 2\lfloor 2r/3 \rfloor$. Let $|S| = k$. Since $p$ is even, $n$ and $k$ are of the same parity, so that

$$n \geq k + 2\lfloor 2r/3 \rfloor + 2.$$  \hfill (4)

Denote the odd components of $G - S$ by $G_1, G_2, \ldots, G_n$. Since $G$ is connected, every component $G_i (1 \leq i \leq n)$ contains at least one vertex that is adjacent to some vertex of $S$. Suppose, without loss of generality, that $G_1, G_2, \ldots, G_t$ denote the odd components of $G - S$ for which there exists exactly one edge $e_i$ joining a vertex in $G_i (1 \leq i \leq t)$ to a vertex of $S$. For $i = t + 1, t + 2, \ldots, n$, then, there are at least three edges joining vertices of $G_i$ to vertices of $S$; otherwise, for some $j (t + 1 \leq j \leq n)$, vertices of $G_j$ are joined to vertices of $S$ by exactly two edges, implying that $G_j$ has an odd number of odd vertices, which is not possible.

Let $P_1, P_2, \ldots, P_r$ denote $r$ edge-disjoint paths of $G$ which contain all the bridges of $G$. Then for every $i (1 \leq i \leq r)$, at most two bridges of $G$ that lie on $P_i$ are in the set $\{e_1, e_2, \ldots, e_t\}$. Hence $t \leq 2r$. Since at least $t + 3(n - t) = 3n - 2t$ edges join vertices of
Another bound (see [2]) for the number of edges in a maximum matching in a connected cubic graph \( G \) depends only on the number of bridges in \( G \).

**THEOREM E.** Every maximum matching in a connected cubic graph of order \( p \) with fewer than \( 3(\lambda + 1) \) bridges (\( \lambda \geq 0 \)) has at least \( (p - 2\lambda)/2 \) edges.

If the bridges of a connected cubic graph lie on sufficiently few paths, then the bound provided in Theorem 1 on the number of edges in a maximum matching is an improvement on the bound provided in Theorem E. A specific statement of this improved result is given next.

**COROLLARY 1.** Let \( G \) be a connected cubic graph of order \( p \) having \( m \) bridges, and let \( \lambda \geq 0 \) be an integer such that \( 3\lambda \leq m < 3(\lambda + 1) \).

If these bridges lie on \( r \) edge-disjoint paths, where \( \lfloor 2r/3 \rfloor < \lambda \), then the number of edges in a maximum matching of \( G \) is at least \( p/2 - \lfloor 2r/3 \rfloor \).

The result in Corollary 1 can be shown to be sharp, which we do next. Since the case \( \lambda = 0 \) corresponds to the existence of at most 2 bridges in a connected cubic graph, and sharpness is already known, we consider \( \lambda \geq 1 \) to be given, and choose the maximum \( r \) with \( r \equiv 0 \pmod{3} \), say \( r = 3s \), such that \( \lfloor 2r/3 \rfloor < \lambda \). Then

\[
  r = \begin{cases} 
    (3\lambda - 6)/2 & \text{if } \lambda \text{ is even}, \\
    (3\lambda - 3)/2 & \text{if } \lambda \text{ is odd}.
  \end{cases}
\]

We show that there exists a connected cubic graph \( G \) of order \( p \) having \( m = 3\lambda + j \) bridges (\( j = 0, 1, 2 \)) all of which lie on \( r \) edge-disjoint paths but no fewer, such that each maximum matching contains \( p/2 - \lfloor 2r/3 \rfloor \) edges.

We begin by constructing a graph \( P_n^* (n \geq 1) \), consisting of graphs \( H_1, H_2, \ldots, H_n \), where \( H_i (1 \leq i \leq n - 1) \) is obtained by deleting an
edge of \( K_4 \) and \( H_n \) is obtained by subdividing an edge of \( K_4 \). Denote the two vertices of degree 2 in \( H_i \) \((1 \leq i \leq n-1)\) by \( u_i \) and \( v_i \) and the vertex of degree 2 in \( H_n \) by \( u_n \). Then \( P^*_n \) is produced by joining \( v_i \) and \( u_{i+1} \) \((1 \leq i \leq n-1)\). Observe that each \( P^*_n(n \geq 1) \) has odd order. Let the graph \( H \) be the \( 12s \)-cycle \( w_1, w_2, \ldots, w_{12s}, w_1 \) to which we add 2s new vertices \( x_1, x_2, \ldots, x_{2s} \), where \( x_i \) is joined to \( w_{6i-5}, w_{6i-3} \) and \( w_{6i-1} \) \((1 \leq i \leq 2s)\). Consider next the graphs \( G_1, G_2, \ldots, G_{6s-1} \), each isomorphic to \( P^*_1 \), and the graph \( G_{6s} \), where

\[
G_{6s} = \begin{cases} 
P^*_1 & \text{if } l \text{ is even}, \\
P^*_2 & \text{if } l \text{ is odd}.
\end{cases}
\]

The desired graph \( G \) is now produced by joining \( w_{2i} \) to the vertex \( u_1 \) in \( G_i \) \((1 \leq i \leq 6s)\) by an edge \( e_i \). Figure 2 illustrates the graph \( G \) for \( k = 3, r = 3, s = 1, m = 9 \) and \( j = 0 \).

![Figure 2](https://doi.org/10.1017/S0004972700003737) Published online by Cambridge University Press
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Clearly \( G \) is connected, cubic, and each edge \( e \) (\( 1 \leq i \leq 6 \theta \)) is a bridge of \( G \). Further, since \( G_{6 \theta} \) contains \( 6 + j \) or \( 3 + j \) bridges, depending on whether \( \ell \) is even or odd, respectively, it follows that \( G \) contains exactly \( 6 \theta + 6 + j \) or \( 6 \theta + 3 + j \) bridges, according to whether \( \ell \) is even or odd. Since the bridges of \( G \) lie on \( r = 3 \theta \) edge-disjoint paths, Corollary 1 implies that every maximum matching of \( G \) contains at least \( p/2 - \lfloor 2r/3 \rfloor \) edges.

It remains to be shown that every maximum matching of \( G \) contains at most \( p/2 - \lfloor 2r/3 \rfloor \) edges. We use Theorem D to prove this statement. Let \( S = \{ \omega_{2i} \mid 1 \leq i \leq 6 \theta \} \cup \{ x_1, x_2, \ldots, x_{2 \theta} \} \). Then \( |S| = 8 \theta \), and

\[
G - S = \begin{cases} 
6 \theta K_2 \cup (6 \theta - 1)P_1 \cup P_4^* & \text{if } \ell \text{ is even,} \\
6 \theta K_2 \cup (6 \theta - 1)P_1^* \cup P_4 & \text{if } \ell \text{ is odd.}
\end{cases}
\]

Therefore, \( G - S \) contains \( 12 \theta = |S| + 4 \theta \) odd components. Theorem D now implies that every maximum matching of \( G \) contains at most \( p/2 - 2 \theta = p/2 - \lfloor 2r/3 \rfloor \) edges. Hence every maximum matching of \( G \) contains exactly \( p/2 - \lfloor 2r/3 \rfloor \) edges.

The cases where \( r \equiv 1 \pmod{3} \) or \( r \equiv 2 \pmod{3} \) can be handled in a similar manner. If \( r \equiv 1 \pmod{3} \), the maximum \( r \) with \( \lfloor 2r/3 \rfloor < \ell \) is given by

\[
r = \begin{cases} 
(3 \ell - 4)/2 & \text{if } \ell \text{ is even,} \\
(3 \ell - 1)/2 & \text{if } \ell \text{ is odd.}
\end{cases}
\]

Further, the maximum \( r \) for \( r \equiv 2 \pmod{3} \) and \( \lfloor 2r/3 \rfloor < \ell \) satisfies

\[
r = \begin{cases} 
(3 \ell - 2)/2 & \text{if } \ell \text{ is even,} \\
(3 \ell + 1)/2 & \text{if } \ell \text{ is odd.}
\end{cases}
\]

Then using a construction similar to the one described for \( r \equiv 0 \pmod{3} \) we can show, for the above choices of \( r \), that there is a graph \( G \) having \( m + j \) bridges (\( j = 0, 1, 2 \) and \( 3 \ell \leq m < 3(\ell + 1) \)) all of which lie on \( r \) edge-disjoint paths and where every maximum matching of \( G \) has \( p/2 - \lfloor 2r/3 \rfloor \) edges. Consequently, the result stated in Corollary 1 is the best possible.
Maximum matchings in cubic graphs

References


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