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SINGULARITIES OF QUADRATIC DIFFERENTIALS AND EXTREMAL TEICHMÜLLER MAPPINGS DEFINED BY DEHN TWISTS

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Abstract

Let S be a Riemann surface of finite type. Let ω be a pseudo-Anosov map of S that is obtained from Dehn twists along two families $\{A, B\}$ of simple closed geodesics that fill S. Then ω can be realized as an extremal Teichmüller mapping on a surface of the same type (also denoted by S). Let ϕ be the corresponding holomorphic quadratic differential on S. We show that under certain conditions all possible nonpuncture zeros of ϕ stay away from all closures of once punctured disk components of $S\backslash\{A, B\}$, and the closure of each disk component of $S\backslash\{A, B\}$ contains at most one zero of ϕ. As a consequence, we show that the number of distinct zeros and poles of ϕ is less than or equal to the number of components of $S\backslash\{A, B\}$.

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1. Introduction

According to Thurston [11], some pseudo-Anosov maps on a Riemann surface S of type (p, n), where $3p - 3 + n > 0$, can be constructed through Dehn twists along two simple closed geodesics (with respect to the complete hyperbolic metric with constant curvature -1). Let α and $\beta \subset S$ be two simple closed geodesics. Denote by t_α and t_β the positive Dehn twists along α and β, respectively. We assume that $\{\alpha, \beta\}$ fills S. Thurston [11] (see also [6]) proved that for all positive integers m and n, the composition $t_\alpha^m \circ t_\beta^{-n}$ represents a pseudo-Anosov mapping class on S.

Thurston's method can be extended to prove the following result (see Penner [10]). Let A, B be families of disjoint nontrivial simple closed geodesics on S so that $\{A, B\}$ fills S. Let w be any word consisting of positive Dehn twists along elements of A and negative Dehn twists along elements of B so that the positive Dehn twist along each element of A and the negative Dehn twist along each element of B occur at least once in w. Then w also represents a pseudo-Anosov mapping class, which means that the map w can be evolved into a pseudo-Anosov map ω via an isotopy $H_t(\cdot)$, $0 \leq t \leq 1$. If we choose S properly, the map $\omega : S \to S$ is an absolutely extremal Teichmüller mapping (see Bers [3]).

We call S an ω-minimal surface. Associated with ω there is a holomorphic quadratic differential ϕ on S that may have simple poles at punctures of S. The quadratic differential ϕ defines a flat metric on S. By taking a suitable power if necessary, in this paper we assume without loss of generality that ω fixes all zeros of ϕ. For each nonpuncture zero z_i of ϕ, $\delta_i = H_t(z_i)$ is a Jordan closed curve on S. It is interesting to compare the locations of all possible zeros of ϕ in the ϕ-flat metric to their locations with respect to the complete hyperbolic metric. The aim of this paper is to locate in a rather coarse manner all possible zeros of ϕ in terms of the regions obtained from cutting along the two families $\{A, B\}$ of closed geodesics on S. Write

$$S\backslash\{A, B\} = \{P_1, \ldots, P_u; Q_1, \ldots, Q_v\}, \quad u \geq 1, v \geq 1, \tag{1}$$

where $\{P_1, \ldots, P_u\}$ and $\{Q_1, \ldots, Q_v\}$ are the collections of disk components and once punctured disk components of $S\backslash\{A, B\}$, respectively. The collection $\{Q_1, \ldots, Q_v\}$ is empty if and only if S is compact. With the notation above, the main result of this paper is as follows.

THEOREM 1.1. *Let S be an ω-minimal surface, and let ϕ be the corresponding quadratic differential on S. Assume that ω leaves each zero of ϕ fixed. Then:*

(1) *each nonpuncture zero z_i of ϕ, if δ_i is a null curve on S, lies in the complement of the closure of $Q_1 \cup \cdots \cup Q_v$ in S;*
(2) *the closure of each disk component P_i contains at most one such zero z_i, with δ_i being a null curve.*

In particular, if $S\backslash\{A, B\}$ consists of once punctured disk components only, then either each zero z_i is a puncture, or δ_i is a nontrivial curve.

REMARK. By the Riemann–Roch theorem (see, for example, [5]), if $p \geq 2$, then ϕ has at least one zero on the compactification $\bar{S}$ of S.

As a consequence of Theorem 1.1, we obtain the following result.

COROLLARY 1.2. *The total number of poles and distinct zeros z_i with δ_i being null curves is no more than the number $u + v$ of the components of $S\backslash\{A, B\}$.*

The idea of the proof of Theorem 1.1 is as follows. A nonpuncture zero z_0 of ϕ on S gives rise to a holomorphic embedding of a Teichmüller geodesic $\mathcal{L} \subset T(S)$ into the Bers fiber space $F(S)$ over $T(S)$. Let $\hat{\mathcal{L}} \subset F(S)$ denote the image of $\mathcal{L}$ under the embedding. With the help of the Bers isomorphism φ of $F(S)$ onto another Teichmüller space $T(\dot{S})$ for $\dot{S} = S\backslash\{\text{a point}\}$, $\mathcal{L}$ can be further embedded into $T(\dot{S})$. By invariance of metrics, one shows that $\varphi(\hat{\mathcal{L}})$ is a Teichmüller geodesic (Lemma 3.1).

On the other hand, [3, Theorem 5] states that a modular transformation θ on $T(\dot{S})$ keeps a Teichmüller geodesic invariant if and only if θ is hyperbolic. Now suppose that $z_0 \in S$ lies in Q_1, say; then one constructs a nonhyperbolic modular transformation θ on $T(\dot{S})$, keeping $\varphi(\hat{\mathcal{L}})$ invariant (Theorem 4.2). It follows from Bers' theorem that $\varphi(\hat{\mathcal{L}})$ is not a Teichmüller geodesic, which leads to a contradiction.

The second statement of Theorem 1.1 follows from Theorem 4.1. Suppose that z_0 and z_1 are two zeros of ϕ in the closure of a disk component P_1. Associated with z_0 and z_1 there are two Teichmüller geodesics $\varphi(\hat{\mathcal{L}}_1)$ and $\varphi(\hat{\mathcal{L}}_2)$ in $T(\dot{S})$ under the Bers isomorphism. Theorem 4.1 asserts the existence of a common hyperbolic modular transformation leaving both $\varphi(\hat{\mathcal{L}}_1)$ and $\varphi(\hat{\mathcal{L}}_2)$ invariant. This contradicts the fact that there is only one invariant geodesic under a hyperbolic transformation.

2. Preliminaries

We begin by reviewing some basic properties in Teichmüller theory. Let $\mathbf{H}$ denote the hyperbolic plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ endowed with the hyperbolic metric

$$ds = \frac{|dz|}{\operatorname{Im} z}.$$

Write $\overline{\mathbf{H}} = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ and let $\varrho : \mathbf{H} \to S$ be the universal covering with covering group G. Then G is a torsion-free finitely generated Fuchsian group of the first kind with $\mathbf{H}/G = S$.

Let $M(G)$ be the set of Beltrami coefficients for G. That is, $M(G)$ consists of measurable functions μ defined on $\mathbf{H}$ and satisfying the following two properties:

(i) $\|\mu\|_\infty = \operatorname{ess.sup}\{|\mu(z)| : z \in \mathbf{H}\} < 1$; and
(ii) $\mu(g(z))\overline{g'(z)}/g'(z) = \mu(z)$ for all $g \in G$.

According to Ahlfors and Bers [1], for every $\mu \in M(G)$, there are normalized quasiconformal maps w_μ and w^μ of $\mathbb{C}$ onto itself such that for $z \in \mathbf{H}$, $\partial_{\bar{z}} w_\mu(z)/\partial_z w_\mu(z) = \mu(z)$ and $\partial_{\bar{z}} w^\mu(z)/\partial_z w^\mu(z) = \mu(z)$; and for $z \in \overline{\mathbf{H}}$, $\partial_{\bar{z}} w_\mu(z)/\partial_z w_\mu(z) = \overline{\mu(\bar{z})}$ and $\partial_{\bar{z}} w^\mu(z)/\partial_z w^\mu(z) = 0$.

Note that w_μ maps $\mathbf{H}$ onto $\mathbf{H}$ while w^μ maps $\mathbf{H}$ onto an arbitrary quasidisk. Two elements μ and ν in $M(G)$ are said to be equivalent if $w_\mu|_{\partial\mathbf{H}} = w_\nu|_{\partial\mathbf{H}}$, or equivalently, $w^\mu|_{\partial\mathbf{H}} = w^\nu|_{\partial\mathbf{H}}$. The equivalence class of μ is denoted by $[\mu]$. The Teichmüller space $T(S)$, where $S = \mathbf{H}/G$, is defined to be the space of equivalence classes $[\mu]$ of Beltrami coefficients $\mu \in M(G)$. It is well known that $T(S)$ is a complex manifold of dimension $3p - 3 + n$. The Teichmüller distance $\langle[\mu], [\nu]\rangle$ between two points $[\mu]$ and $[\nu] \in T(S)$ is defined by

$$\langle[\mu], [\nu]\rangle = \tfrac{1}{2} \inf\{\log K(w_\mu \circ w_\nu^{-1})\},$$

where K is the maximal dilatation of $w_\mu \circ w_\nu^{-1}$ on $\mathbf{H}$ and the infimum is taken through the homotopy class of $w_\mu \circ w_\nu^{-1}$ that fixes each point in $\partial\mathbf{H}$. The set $Q(G)$ of integrable quadratic differentials consists of holomorphic functions $\phi(z)$ on $\mathbf{H}$ such that

$$(\phi \circ g)(z)g'(z)^2 = \phi(z) \quad \forall z \in \mathbf{H} \text{ and all } g \in G$$

and

$$\|\phi\| = \iint_{\Delta} |\phi(z)|\, dx\, dy = 1,$$

where $\Delta \subset \mathbf{H}$ is a fundamental region of G. Every $\phi \in Q(G)$ can be projected to a meromorphic quadratic differential on $\overline{S}$ that may have simple poles at punctures of S, which is also denoted by ϕ. The differential ϕ assigns to each uniformizing parameter z a holomorphic function $\phi(z)$ such that $\phi(z)\, dz^2$ is invariant under a change of local coordinates. Away from zeros of ϕ there are naturally defined coordinates so that ϕ defines a flat metric that is Euclidean near every nonzero point z. Associated with each ϕ there are horizontal and vertical trajectories defined by $\phi(z)\, dz^2 > 0$ and $\phi(z)\, dz^2 < 0$, respectively. For any $t \in (-1, 1)$ and any $\phi \in Q(G)$, we have that $t(\bar{\phi}/|\phi|) \in M(G)$. The set

$$\left[t\frac{\bar{\phi}}{|\phi|}\right] \in T(S), \quad t \in (-1, 1), \tag{2}$$

is called a Teichmüller geodesic. If t in (2) is replaced by a complex variable $z \in \mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$, we obtain a complex version of the geodesic that is also called a Teichmüller disk.

Notice that every self-map ω of S induces a mapping class and thus a modular transformation χ that acts on $T(S)$. The collection of all such modular transformations form a group Mod_S that is discrete and isomorphic to the group of biholomorphic automorphisms of $T(S)$ when S is not of type (0, 3), (0, 4), (1, 1), (1, 2), and (2, 0); see Royden [9] and Earle–Kra [4] for more details.

For each $\chi \in \mathrm{Mod}_S$, Bers [3] introduced an index

$$a(\chi) = \inf_{[\mu] \in T(S)} \langle [\mu], \chi([\mu]) \rangle.$$

Throughout this paper, we consider those modular transformations χ for which $a(\chi) > 0$. There are two cases: $a(\chi)$ is achieved and $a(\chi)$ is not achieved. In the former case, χ is called hyperbolic. In the latter case, χ is called pseudo-hyperbolic. If χ is hyperbolic, then by [3, Theorem 5], $a(\chi)$ assumes its value on any point in a geodesic $\mathcal{L}$. The transformation χ keeps $\mathcal{L}$ invariant. Conversely, if an element $\chi \in \mathrm{Mod}_S$ keeps a Teichmüller geodesic $\mathcal{L}$ invariant, then χ must be hyperbolic. In this case, χ is induced by a self-map of S, and for each Riemann surface S on $\mathcal{L}$, χ is realized as an absolutely extremal self-mapping ω of S. Associated with the map ω there is an integrable meromorphic quadratic differential ϕ on the compactification of S which is holomorphic on S and may have simple poles at punctures of S (see Bers [3]). Furthermore, ω leaves invariant both horizontal and vertical trajectories defined by ϕ.

Topologically, the map ω that associates with a pair of transverse measured foliations determined by the quadratic differential ϕ is also called pseudo-Anosov. By Thurston [11], the set of pseudo-Anosov mapping classes on S consists of all possible nonperiodic mapping classes that do not keep any finite set of disjoint simple nontrivial closed geodesics invariant.

The Bers fiber space $F(S)$ over $T(S)$ is the collection of pairs

$$\{([\mu], z) \mid [\mu] \in T(S), z \in w^{\mu}(\mathbf{H})\}.$$

The natural projection $\pi : F(S) \to T(S)$ is holomorphic. We fix a point $a \in S$ and let $\dot{S} = S \backslash \{a\}$. Theorem 9 of [2] states that there is an isomorphism $\varphi : F(S) \to T(\dot{S})$ that is unique up to a modular transformation of $T(\dot{S})$.

Let $\chi \in \mathrm{Mod}_S$ be induced by a map $\omega : S \to S$. We lift the map ω to a map $\hat{\omega} : \mathbf{H} \to \mathbf{H}$. The map $\hat{\omega}$ has the property that $\hat{\omega} G \hat{\omega}^{-1} = G$. Suppose that $\omega' : S \to S$ is another map isotopic to ω. As usual, the map ω' can also be lifted to a map $\hat{\omega}' : \mathbf{H} \to \mathbf{H}$ that is isotopic to $\hat{\omega}$ by an isotopy fixing each point in $\partial \mathbf{H}$. That is, $\hat{\omega}$ and $\hat{\omega}'$ induce the same automorphism of G. In this case, $\hat{\omega}$ and $\hat{\omega}'$ are said to be equivalent and we denote the equivalence class of $\hat{\omega}$ by $[\hat{\omega}]$.

Using the map $\hat{\omega}$, one constructs a biholomorphic map θ of $F(S)$ onto itself by the formula

$$\theta([\mu], z) = ([\nu], w^{\nu} \circ \hat{\omega} \circ (w^{\mu})^{-1}(z)) \quad \text{for every pair } ([\mu], z) \in F(S), \tag{3}$$

where ν is the Beltrami coefficient of $w^{\mu} \circ \hat{\omega}^{-1}$.

LEMMA 2.1 (Bers [2]). *Let $\hat{\omega}$ and $\hat{\omega}'$ be in the same equivalence class. If $\hat{\omega}$ is replaced by $\hat{\omega}'$, the resulting map θ defined as (3) is unchanged. In other words, θ depends only on the equivalence class* $[\hat{\omega}]$.

Therefore, the map θ is uniquely determined by $[\omega]$. Lemmas 3.1 to 3.5 of Bers [2] demonstrate that θ is a holomorphic automorphism of $F(S)$ that preserves the fiber structure and that all such θs form a group $\mathrm{mod}(S)$ acting on $F(S)$ faithfully.

Note that each element $g \in G$ acts on $F(S)$ by the formula

$$g([\mu], z) = ([\mu], w^{\mu} \circ g \circ (w^{\mu})^{-1}(z)).$$

In this way, the group G is regarded as a normal subgroup of $\mathrm{mod}(S)$, and the quotient $\mathrm{mod}(S)/G$ is isomorphic to the modular group Mod_S. Let $i : \mathrm{mod}(S) \to \mathrm{Mod}_S$ denote the natural projection that is induced by the holomorphic projection $\pi : F(S) \to T(S)$.

By [2, Theorem 10], the Bers isomorphism $\varphi : F(S) \to T(\dot{S})$ induces an isomorphism φ^* of $\mathrm{mod}(S)$ onto the subgroup Mod_S^a of $\mathrm{Mod}_{\dot{S}}$ that fixes the distinguished puncture a via the formula

$$\mathrm{mod}(S) \ni [\hat{\omega}] \xrightarrow{\varphi^*} \varphi \circ [\hat{\omega}] \circ \varphi^{-1} \in \mathrm{Mod}_S^a.$$

The image of $[\hat{\omega}]$ in Mod_S^a under φ^* is denoted by $[\hat{\omega}]^*$.

3. Invariant geodesics embedded into another Teichmüller space via a Bers isomorphism

In this section, we assume that $\chi \in \mathrm{Mod}_S$ is a hyperbolic transformation that keeps a Teichmüller geodesic $\mathcal{L} \subset T(S)$ invariant. We further assume that $[0] \in \mathcal{L}$ is represented by S. Choose $\phi \in Q(G)$ so that

$$\mathcal{L} = \left\{ \left[t \frac{\bar{\phi}}{|\phi|} \right], t \in (-1, 1) \right\}.$$

Write $\mu = \bar{\phi}/\phi$. Choose $\hat{\omega} : \mathbf{H} \to \mathbf{H}$ that projects to $\omega : S \to S$ that induces χ. We assume that ω is an absolutely extremal Teichmüller mapping on S. By an argument of Kra [7], there is a hyperbolic Möbius transformation M that leaves invariant $(-1, 1)$ as well as $\mathbf{D}$ and satisfies the equation

$$\chi([t\mu]) = [\text{Beltrami coefficient of } w^{t\mu} \circ \hat{\omega}^{-1}] = [M(t)\mu] \quad \forall t \in \mathbf{D}.$$

Suppose that $z_0 \in S$ is a zero of ϕ. Let $\hat{z}_0 \in \mathbf{H}$ be such that $\varrho(\hat{z}_0) = z_0$. Let

$$\hat{\mathcal{L}} = \{([t\mu], w^{t\mu}(\hat{z}_0)), t \in (-1, 1)\} \subset F(S). \tag{4}$$

It is easy to see that the projection $\pi : F(S) \to T(S)$ defines an embedding of $\hat{\mathcal{L}}$ into $T(S)$ with $\mathcal{L} = \pi(\hat{\mathcal{L}})$.

The following result is well known and the argument is implicitly given in [7].

LEMMA 3.1. *The image $L = \varphi(\hat{\mathcal{L}})$ under the Bers isomorphism $\varphi : F(S) \to T(\dot{S})$ is a Teichmüller geodesic.*

PROOF. By definition, $\mathcal{L} = \mathcal{L}(t) \subset T(S)$, $t \in (-1, 1)$, is an isometric embedding. For any two points $x, y \in \mathcal{L}$, let $\hat{x}, \hat{y} \in \hat{\mathcal{L}}$ be such that $\pi(\hat{x}) = x$ and $\pi(\hat{y}) = y$. Let $x^* = \varphi(\hat{x})$ and $y^* = \varphi(\hat{y})$. By complexifying there is a Teichmüller disk $D \subset T(S)$ with $\mathcal{L} \subset D$ and a holomorphic map $s : D \to F(S)$ defined by sending the point $[z\mu]$, $z \in \mathbf{D}$ and $\mu = \bar{\phi}/|\phi|$, to the point $([z\mu], w^{z\mu}(z_0))$. It is easy to check that $\hat{\mathcal{L}} \subset s(D)$ with $s(x) = \hat{x}$ and $s(y) = \hat{y}$. Now $\varphi \circ s : D \to T(\dot{S})$ is holomorphic and is distance nonincreasing. Therefore, we obtain

$$\langle x^*, y^* \rangle \leq \langle x, y \rangle. \tag{5}$$

Notice that the natural projection $\pi : F(S) \to T(S)$ is holomorphic, so $\pi \circ \varphi^{-1} : T(\dot{S}) \to T(S)$ is holomorphic and hence distance nonincreasing. It follows that $\langle x, y \rangle \leq \langle x^*, y^* \rangle$. Combining with (5), we conclude that

$$\langle x, y \rangle = \langle x^*, y^* \rangle \quad \text{for any two points } x, y \in \mathcal{L}.$$

Hence $L = L(t)$ must also be an isometric embedding, which says that L is also a Teichmüller geodesic, as claimed. □

By [7], the element $\theta = [\hat{\omega}] \in \text{mod}(S)$ acts on $\hat{\mathcal{L}}$ via the formula

$$\theta([t\mu], w^{t\mu}(\hat{z}_0)) = ([M(t)\mu], w^{M(t)\mu} \circ \hat{\omega}(\hat{z}_0)) \quad \forall([t\mu], w^{t\mu}(\hat{z}_0)) \in \hat{\mathcal{L}}.$$

From Lemma 2.1, one shows that the image $\theta(\hat{\mathcal{L}})$ in $F(S)$ only depends on $[\hat{\omega}]$. In summary, we have the following result.

LEMMA 3.2 (Kra [7]). *Suppose that $z_0 \in S$ is a zero of ϕ. An element $\theta \in$ mod(S) keeps the line $\hat{\mathcal{L}}$ invariant if the representative $\hat{\omega}$ of θ satisfies the condition that $\hat{\omega}(\hat{z}_0) = \hat{z}_0$.*

4. Proof of Theorem 1.1

Let $\mathcal{L} \subset T(S)$ be a geodesic invariant under a hyperbolic transformation χ. Assume that ϕ has a nonpuncture zero z_0. Let $\hat{z}_0 \in \mathbf{H}$ be such that $\varrho(\hat{z}_0) = z_0$. Theorem 1.1 follows from Lemma 3.1 and the following two results.

THEOREM 4.1. *If $z_0 \in P_i$ for some i for which $1 \le i \le u$, then there is an element θ in* mod(S) *such that $\theta^* = \varphi^*(\theta)$ is hyperbolic with the following properties:*

(1) *θ projects to χ under the projection induced by $\pi : F(S) \to T(S)$; and*
(2) *the lift $\hat{\mathcal{L}}$ of $\mathcal{L}$ that passes through $\hat{z}_0$ and is defined by (4) is an invariant line under θ.*

THEOREM 4.2. *If $z_0 \in Q_j$ for some j with $1 \le j \le v$, and δ_0 is a null curve, then there is an an element θ in* mod(S) *such that $\theta^* = \varphi^*(\theta)$ is nonhyperbolic with properties* (1) *and* (2) *in Theorem 4.1.*

PROOF OF THEOREM 1.1. (1) If S is compact, there is nothing to prove. So we assume that $n \ge 1$ and $z_0 \in S$ is a zero of ϕ that is not a puncture of S, and that δ_0 is a null curve. We further assume that Q_1 is a component of $S \backslash \{A, B\}$ that contains z_0 and a puncture z_1. Clearly, $z_0 \ne z_1$. Let $\hat{\mathcal{L}}$ be defined in (4). By Lemma 3.1, $\varphi(\hat{\mathcal{L}}) \subset T(\dot{S})$ is a Teichmüller geodesic. By Theorem 4.2, $\varphi(\hat{\mathcal{L}}) \subset T(\dot{S})$ is invariant under an element θ^* that is not a hyperbolic modular transformation on $T(\dot{S})$. Hence by [3, Theorem 5], $\varphi(\hat{\mathcal{L}})$ is not a Teichmüller geodesic in $T(\dot{S})$. This contradiction proves (1) of Theorem 1.1.

To prove (2), we assume that there are two zeros z_0 and z_1 lying in the closure of a disk component P_1, say. Let $\hat{P}_1 \subset \mathbf{H}$ be such that $\varrho|_{\hat{P}_1} : \hat{P}_1 \to P_1$ is a homeomorphism. Let $\hat{z}_0, \hat{z}_1 \in \hat{P}_1$ be such that $\varrho(\hat{z}_0) = z_0$ and $\varrho(\hat{z}_1) = z_1$. Let $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$ be the lines passing through $\hat{z}_0$ and $\hat{z}_1$, respectively. From Lemma 3.1, $\varphi(\hat{\mathcal{L}}_0)$ and $\varphi(\hat{\mathcal{L}}_1)$ are distinct Teichmüller geodesics in $T(\dot{S})$.

Let θ_0^* and θ_1^* denote the hyperbolic transformations obtained from $\hat{z}_0$ and $\hat{z}_1$ respectively (Theorem 4.1). Since $\hat{z}_0$ and $\hat{z}_1 \in \hat{P}_1$, by construction of Theorem 4.1, if δ_0 is trivial, then $\theta_0^* = \theta_1^*$. Set $\theta^* = \theta_0^* = \theta_1^*$. By Lemma 3.2, θ^* keeps both $\varphi(\hat{\mathcal{L}}_0)$ and $\varphi(\hat{\mathcal{L}}_1)$ invariant. It follows from [3, Theorem 5] that θ^* is hyperbolic. This contradicts the uniqueness of the invariant geodesic of a hyperbolic transformation. This proves (2). □

PROOF OF COROLLARY 1.2. Note that the closure $\bar{P}_i$ or $\bar{Q}_j$ of each component P_i or Q_j in expression (1) is a polygon with geodesic boundary segments (with respect to the hyperbolic metric on S). By the argument of Theorem 1.1, each $\bar{P}_i$ or $\bar{Q}_j$ cannot contain more than one zero z_j with δ_j a null curve. Each pole of ϕ must be a puncture of some Q_j. By Theorem 1.1(1), each $\bar{Q}_j$ contains at most one zero z_j that is the puncture of Q_j. In this case, z_j cannot be a pole of ϕ. Moreover, if there are components P_i and P_j in expression (1) with the null curve property so that a zero of ϕ lies in $\bar{P}_i \cap \bar{P}_j$, then there do not exist any other zeros in either P_i or P_j. If a zero z_0 with δ_0 a null curve lies in the intersections of α_i and β_j for some $\alpha_i \in A$ and $\beta_j \in B$, then the closure of a polygon in expression (1) one of whose vertices is z_0 does not include any other zeros with the null curve property. Overall, we conclude that the total number of poles and distinct zeros z_i with δ_i being null curves is no more than the number of components of $S \backslash \{A, B\}$. □

The rest of this paper is devoted to the proof of Theorems 4.1 and 4.2.

5. Reducible maps projecting to pseudo-Anosov maps

Let $\dot{S}$ be a Riemann surface as defined in Section 2. We know that $\dot{S}$ has type $(p, n+1)$. Let W be a nonperiodic nonpseudo-Anosov self-map of $\dot{S}$. By Thurston [11], there exists an admissible system

$$\{c_1, c_2, \ldots, c_s\}, \quad s \geq 1, \tag{6}$$

of simple nontrivial geodesics on $\dot{S}$ so that for every i, where $1 \leq i \leq s$, $W(c_i)$ is homotopic to c_j for some j with $1 \leq j \leq s$. Here by 'admissible' we mean that no loop in (6) bounds a once punctured disk and c_i is not homotopic to c_j whenever $i \neq j$. Note that W may permute the components $\{R_1, \ldots, R_q\}$ of $S \backslash \{c_1, c_2, \ldots, c_s\}$, and if W keeps a component R_j invariant, the restriction $W|_{R_j}$ could be either the identity, or periodic, or pseudo-Anosov. Thus there is an integer K such that W^K keeps every c_i and every R_j invariant, and for each j, where $1 \leq j \leq q$, $W^K|_{R_j}$ is either the identity or pseudo-Anosov. If all $W^K|_{R_j}$ are the identity, W^K is a product of powers of positive and negative Dehn twists along certain loops in (6). In general, W^K induces a pseudo-hyperbolic transformation on $T(\dot{S})$. See Bers [3] for details.

We now consider a special case. Let $\theta = [\hat{\omega}]$ be an element of mod(S) that projects to $\chi \in \mathrm{Mod}_S$. We assume that χ is induced by $\omega : S \to S$ that is an absolutely extremal Teichmüller mapping. Let $\phi \in Q(G)$ be the corresponding quadratic differential. By Royden's theorem [9] (see also Earle–Kra [4]), $\theta^* = \varphi^*(\theta)$ is a modular transformation on $T(\dot{S})$. Thus θ^* is induced by a quasiconformal self-map W of $\dot{S}$. The map W is isotopic to ω if W is viewed as a self-map of S. Notice that W is nonperiodic; it may or may not be pseudo-Anosov. Even if W is pseudo-Anosov, $\dot{S}$ may not be the right candidate in $T(\dot{S})$ that makes W an absolutely extremal self-mapping on $\dot{S}$.

LEMMA 5.1. *Assume, with the above notation, that S is not compact and W is not pseudo-Anosov. Then W is reduced by a single closed geodesic c_1 that is a* *boundary of a twice punctured disk $\Omega \subset \dot{S}$ that encloses a. More precisely, if we write $\dot{S} \backslash c_1 = \Omega \cup R$, then $W|_\Omega$ is the identity and $W|_R$ is pseudo-Anosov and essentially the same as ω, and W induces a pseudo-hyperbolic transformation on $T(\dot{S})$.*

PROOF. Let W be reduced by (6), and let γ_i denote the geodesic on S obtained from c_i by adding the puncture a. Since W is isotopic to ω on S, ω keeps the curve system $\{\gamma_1, \ldots, \gamma_{s_0}\}$ invariant, where $s_0 = s$ if neither two elements c_i and c_j bound an a-punctured cylinder, nor does an element c_i project to a trivial loop; $s_0 = s - 1$ otherwise.

Since ω is pseudo-Anosov, the set $\{\gamma_1, \ldots, \gamma_{s_0}\}$ is empty. Hence, the only possibility is that all geodesics in (6) are boundaries of twice punctured disks enclosing a. Since geodesics in (6), if not empty, are disjoint, we must have that $s = 1$ and c_1 in (6) is the boundary of a twice punctured disk.

As a is filled in, c_1 becomes a trivial loop. This means that $W|_R$ is essentially the same as ω. Notice that Ω is a twice punctured disk, and $W|_\Omega$ fixes each boundary component. It follows that $W|_\Omega$ is isotopic to the identity. The lemma is proved. □

The following result, along with Lemma 5.1, establishes the relationship between elements in $\mathrm{mod}(S)$ and nonpseudo-Anosov elements in Mod_S^a via the Bers isomorphism φ^*. Recall that $[\hat{\omega}]^* = \theta^* \in \mathrm{Mod}_S^a$ is induced by $W : \dot{S} \to \dot{S}$.

LEMMA 5.2. *Suppose that S is not compact. Assume that $\omega : S \to S$ is pseudo-Anosov and fixes at least one puncture of S. Then certain nonpseudo-Anosov maps W of $\dot{S}$ exist with the property that W projects to ω. All possible nonpseudo-Anosov maps W projecting to ω are obtained from those $\hat{\omega} : \mathbf{H} \to \mathbf{H}$ that fix a fixed point of a parabolic element T of G. In particular, if $\omega : S \to S$ does not fix any punctures of S, then every W so obtained must also be pseudo-Anosov.*

PROOF. Assume that $\hat{\omega} : \mathbf{H} \to \mathbf{H}$ fixes the fixed point x of a parabolic element $T \in G$. This implies that $\hat{\omega} \circ T = T^k \circ \hat{\omega}$ for some $k \geq 1$. That is,

$$[\hat{\omega}]^* \circ T^* = T^{*k} \circ [\hat{\omega}]^*. \tag{7}$$

From Theorem 2 of [7, 8], T^* is represented by a Dehn twist $t_{\partial\Omega}$ along the boundary $\partial\Omega$ of a twice punctured disk $\Omega \subset \dot{S}$. Let $W : S \to S$ be a map that induces $[\hat{\omega}]^*$. From (7) we obtain

$$t_{W(\partial\Omega)} = t_{\partial\Omega}^k.$$

(In fact, it is easily shown that $k = 1$.) It follows that the map W leaves $\partial\Omega$ invariant. So W is not pseudo-Anosov.

Conversely, assume that W is not pseudo-Anosov. By Lemma 5.1, W is reduced by a single geodesic c that is a boundary of a twice punctured disk. This means that $W \circ t_c = t_c \circ W$. By Theorem 2 of [7, 8] again, there is a parabolic element $T \in G$ such that $T^* = t_c$. Hence, $[\hat{\omega}]^* \circ T^* = T^* \circ [\hat{\omega}]^*$. Thus $\hat{\omega} \circ T^k = T^k \circ \hat{\omega}$ for any integer k. It follows that $\hat{\omega}$ fixes the fixed point of T.

In particular, if $\omega : S \to S$ does not fix any punctures of S, then W must be pseudo-Anosov. □

REMARK. The disk $\Omega \subset \dot{S}$ obtained from Lemma 5.1 contains another puncture $b \neq a$, which is viewed as a puncture of S corresponding to the conjugacy class of T. Conversely, every $[\hat{\omega}] \in \text{mod}(S)$ that fixes a parabolic fixed point of G produces a nonpseudo-Anosov map W on $\dot{S}$ that is characterized in Lemma 5.1.

6. Pseudo-Anosov maps and their lifts defined by geodesics

Let $A = \{\alpha_1, \ldots, \alpha_q\}$ and $B = \{\beta_1, \ldots, \beta_r\}$. Let w be as defined in the Introduction. One writes

$$w = \prod_{i}^{N} (t_{\alpha_1}^{n_{i1}} \circ \cdots \circ t_{\alpha_q}^{n_{iq}} \circ t_{\beta_1}^{-m_{i1}} \circ \cdots \circ t_{\beta_r}^{-m_{ir}}) \tag{8}$$

for a positive integer N and nonnegative integers n_{ij} and m_{ik} with the property that

$$\sum_{i=1}^{N} n_{ij}^2 \neq 0 \quad \text{and} \quad \sum_{i=1}^{N} m_{ik}^2 \neq 0, \tag{9}$$

where $1 \leq i \leq N$, $1 \leq j \leq q$ and $1 \leq k \leq r$. By [10], the map $w : S \to S$ represents a pseudo-Anosov mapping class on S. Let $z \in S \backslash \{A, B\}$. Let Δ be a fundamental region of G and let $\hat{z} = \{\varrho^{-1}(z)\} \cap \Delta$.

Let $\hat{\alpha}_1 \subset \mathbf{H}$ be a geodesic such that $\varrho(\hat{\alpha}_1) = \alpha_1$ and $\Delta \cap \hat{\alpha}_1 \neq \emptyset$. Note that there may be more than one choice for such a geodesic $\hat{\alpha}_1$. The geodesic $\hat{\alpha}_1$ is invariant under a simple hyperbolic element $g_{\hat{\alpha}_1}$ of G. Let $D_{\hat{\alpha}_1}$ and $D'_{\hat{\alpha}_1}$ be the components of $\mathbf{H} \backslash \hat{\alpha}_1$.

To obtain a lift $\tau_{\hat{\alpha}_1}$ of t_{α_1} with the fixed geodesic $\hat{\alpha}_1$, we take an earthquake shifting along $\hat{\alpha}_1$ in such a way that it is the identity on $D'_{\hat{\alpha}_1} \cup \hat{\alpha}_1$, and is $g_{\hat{\alpha}_1}$ on $D_{\hat{\alpha}_1}$ away from a small neighborhood of $\hat{\alpha}_1$. We thus define $\tau_{\hat{\alpha}_1}$ on $\mathbf{H}$ via G-invariance. Note that if $\tau_{\hat{\alpha}_1}$ is a lift obtained in this way, then $g_{\hat{\alpha}_1}^{-1} \circ \tau_{\hat{\alpha}_1}$ or $\tau_{\hat{\alpha}_1} \circ g_{\hat{\alpha}_1}^{-1}$ is also a lift of t_{α_1} defined by the other component $D'_{\hat{\alpha}_1}$. Thus one may assume without loss of generality that $\tau_{\hat{\alpha}_1}(\hat{z}) = \hat{z}$ and $\hat{z} \in D'_{\hat{\alpha}_1}$.

The construction of $\tau_{\hat{\alpha}_1}$ gives rise to a collection $E_{\hat{\alpha}_1}$ of half-planes, among which a partial order can be naturally defined. There are infinitely many disjoint maximal elements of $E_{\hat{\alpha}_1}$ and for each maximal element $D_{\hat{\alpha}_1} = D^1_{\hat{\alpha}_1}$ of $E_{\hat{\alpha}_1}$, there are infinitely many second-level elements $D^2_{\hat{\alpha}_1} \subset D^1_{\hat{\alpha}_1}$; and for each such $D^2_{\hat{\alpha}_1}$, there are infinitely many third-level elements $D^3_{\hat{\alpha}_1}$ in $D^2_{\hat{\alpha}_1}$, and so on.

The quasiconformal homeomorphism $\tau_{\hat{\alpha}_1}$ restricts to the identity on the complement of disjoint union of all maximal elements of $E_{\hat{\alpha}_1}$ in $\mathbf{H}$; it is quasiconformal with Beltrami coefficient supported on (disjoint) neighborhoods of $\hat{\alpha}_1$ and its G-translations. Moreover, from the construction, $\tau_{\hat{\alpha}_1}(\hat{y}) = \hat{y}$ for points $\hat{y}$ on the boundaries of all maximal elements of $E_{\hat{\alpha}_1}$. $\tau_{\hat{\alpha}_1}$ naturally extends to a quasisymmetric mapping of $\partial\mathbf{H}$ onto $\partial\mathbf{H}$ that fixes infinitely many hyperbolic fixed points of G and infinitely many parabolic fixed points if S is not compact.

Let $\hat{\alpha}_1, \ldots, \hat{\alpha}_q \subset \mathbf{H}$ be the geodesics such that $\Delta \cap \hat{\alpha}_j \neq \emptyset$ for $j = 1, \ldots, q$. Since $\alpha_1, \ldots, \alpha_q$ are pairwise disjoint, $\hat{\alpha}_1, \ldots, \hat{\alpha}_q$ are pairwise disjoint as well. Since $z \in S\backslash\{A, B\}$, the maximal elements $D_{\hat{\alpha}_1}, \ldots, D_{\hat{\alpha}_q}$ can be properly chosen so that

$$\hat{z} \in \Delta\backslash\{\text{all maximal elements of } E_{\hat{\alpha}_1}, \ldots, E_{\hat{\alpha}_q}\}. \tag{10}$$

Notice that the simple closed geodesics $\alpha_1, \ldots, \alpha_q$ are pairwise disjoint and that the region $\Delta\backslash\{\text{all maximal elements of } E_{\hat{\alpha}_1}, \ldots, E_{\hat{\alpha}_q}\}$ is not empty, by [12, Lemma 4], $\tau_{\hat{\alpha}_{j_1}}$ commutes with $\tau_{\hat{\alpha}_{j_2}}$ for $j_1, j_2 = 1, \ldots, q$. Now for a nonnegative integer tuple $\sigma_i = (n_{i1}, \ldots, n_{iq})$ that satisfies (9), we define

$$\hat{T}_A^{\sigma_i} = \tau_{\hat{\alpha}_1}^{n_{i1}} \circ \tau_{\hat{\alpha}_2}^{n_{i2}} \circ \cdots \circ \tau_{\hat{\alpha}_q}^{n_{iq}}, \quad 1 \leq i \leq N. \tag{11}$$

We see that $\hat{T}_A^{\sigma_i}$ does not depend on the order of those $\tau_{\hat{\alpha}_1}^{n_{i1}}, \ldots, \tau_{\hat{\alpha}_q}^{n_{iq}}$.

Similarly, let $\hat{\beta}_1, \ldots, \hat{\beta}_r \subset \mathbf{H}$ be the geodesics such that $\Delta \cap \hat{\beta}_k \neq \emptyset$ for $k = 1, \ldots, r$. The maximal elements $D_{\hat{\beta}_1}, \ldots, D_{\hat{\beta}_r}$ can also be properly chosen so that

$$\hat{z} \in \Delta\backslash\{\text{all maximal elements of } E_{\hat{\beta}_1}, \ldots, E_{\hat{\beta}_r}\}. \tag{12}$$

For a nonnegative integer tuple $\lambda_i = (m_{i1}, \ldots, m_{ir})$ that satisfies (9), we define

$$\hat{T}_B^{-\lambda_i} = \tau_{\hat{\beta}_1}^{-m_{i1}} \circ \tau_{\hat{\beta}_2}^{-m_{i2}} \circ \cdots \circ \tau_{\hat{\beta}_r}^{-m_{ir}}, \quad 1 \leq i \leq N. \tag{13}$$

Again, $\hat{T}_B^{-\lambda_i}$ does not depend on the order of those $\tau_{\hat{\beta}_1}^{-m_{i1}}, \ldots, \tau_{\hat{\beta}_r}^{-m_{ir}}$.

More precisely, we assume that z lies in one component R of expression (1). The component R is either P_i for some i with $1 \leq i \leq u$, or Q_j for some j with $1 \leq j \leq u$. Since $\varrho : \mathbf{H} \to S$ is a local homeomorphism, there is a nonempty subset Σ_R of Δ such that $\hat{z} \in \Sigma_R$ and $\varrho|_{\Sigma_R} : \Sigma_R \to R$ is a homeomorphism. As we remarked earlier, there is more than one choice of each geodesic $\hat{\alpha}_j$ that meets Δ so that $\varrho(\hat{\alpha}_j) = \alpha_j$. In any case, there are only finitely many maximal elements of $E_{\hat{\alpha}_j}$ and $E_{\hat{\beta}_k}$ that intersect Δ. The region Σ_R can be obtained from the fundamental region Δ with the removal of all such (finitely many) maximal elements of $E_{\hat{\alpha}_j}$ and $E_{\hat{\beta}_k}$ for $1 \leq j \leq q$ and $1 \leq k \leq r$. We now consider the map

$$\hat{T}_{\Delta,R} = \prod_i^N (\hat{T}_A^{\sigma_i} \circ \hat{T}_B^{-\lambda_i}). \tag{14}$$

LEMMA 6.1. *With the above construction, the map $\hat{T}_{\Delta,R}$ defined as (14) is a lift of w and fixes any point $\hat{z} \in \Sigma_R$. Furthermore, if Δ' is another fundamental region of G, then there is an element $h \in G$ sending Δ onto Δ' so that*

$$h \circ (\hat{T}_{\Delta,R}) \circ h^{-1} = \hat{T}_{\Delta',R}.$$

PROOF. By construction, $\tau_{\hat{\alpha}_j}$ and $\tau_{\hat{\beta}_k}$ are lifts of t_{α_j} and t_{β_k}, respectively. One obtains

$$\varrho \circ \tau_{\hat{\alpha}_j} = t_{\alpha_j} \circ \varrho \quad \text{and} \quad \varrho \circ \tau_{\hat{\beta}_k} = t_{\beta_k} \circ \varrho.$$

From (11), (13) and (14), one calculates that $\varrho \circ \hat{T}_{\Delta,R} = w \circ \varrho$. This says that $\hat{T}_{\Delta,R}$ is a lift of w.

Clearly, w has the property that $w(z) = z$ for $z \in S \backslash \{A, B\}$. It is immediate that $\hat{T}_{\Delta,R}$ fixes any point $\hat{z} \in \Sigma_R$. The last statement is also trivial. □

In what follows we fix a fundamental region Δ of G. From Lemma 6.1, each component R of $S \backslash \{A, B\}$ corresponds to an element $[\hat{T}_{\Delta,R}]$ in $\mathrm{mod}(S)$ such that $\hat{T}_{\Delta,R}|_{\Sigma_R}$ is the identity. We thus obtain an injection:

$$\{P_1, \ldots, P_u; Q_1, \ldots, Q_v\} \ni R \longmapsto [\hat{T}_{\Delta,R}] \in \mathrm{mod}(S).$$

If $R = P_i$ for some i with $1 \le i \le u$, the region Σ_R stays away from $\partial \mathbf{H}$.

LEMMA 6.2. *Suppose that R contains a zero z_i of ϕ with the property that the curve δ_i is trivial. Then $[\hat{T}_{\Delta,R}]^* \in \mathrm{Mod}_S^a$ is hyperbolic.*

PROOF. Let $\mathcal{L} \subset T(S)$ be the invariant geodesic under the hyperbolic mapping class χ. Let $\hat{z}_i \in \Sigma_R$ be such that $\varrho(\hat{z}_i) = z_i$. Let $\hat{\mathcal{L}} \subset F(S)$ be defined by (4), and let $\hat{\omega}$ be the lift of ω that fixes $\hat{z}_i$. By assumption, $[\hat{\omega}] = [\hat{T}_{\Delta,R}]$. Thus from Lemma 3.2, we see that $[\hat{T}_{\Delta,R}]^*$ keeps $\varphi(\hat{\mathcal{L}}) \subset T(\dot{S})$ invariant. By Lemma 3.1, $\varphi(\hat{\mathcal{L}})$ is a Teichmüller geodesic. By [3, Theorem 5], $[\hat{T}_{\Delta,R}]^*$ is hyperbolic, as asserted. □

REMARK. From Lemmas 6.2 and 5.2, for a disk component R containing a zero of ϕ and for arbitrary fundamental region Δ, we conclude that $\hat{T}_{\Delta,R}$ does not fix any parabolic fixed point of G. A direct proof of this fact is difficult.

In the case of $R = Q_j$ for some j, where $1 \le j \le v$, the set Σ_R touches $\partial \mathbf{H}$ at the fixed point of a parabolic element of G corresponding to the puncture z_j of Q_j. The following lemma handles this case. Let $z \in R$, and $\hat{z} \in \Sigma_R$ be such that $\rho(\hat{z}) = z$.

LEMMA 6.3. *Under the above condition, the map $\hat{T}_{\Delta,R}$ fixes both $\hat{z}$ and the fixed point of a parabolic element of G.*

PROOF. From Lemma 6.1, the map $\hat{T}_{\Delta,R}$ fixes $\hat{z}$ for $\hat{z} \in \Sigma_R$. Note that the boundary of Σ_R consists of portions of some translations of $\hat{\alpha}_j$ and $\hat{\beta}_k$. Let z_j be the puncture of Q_j. We can draw a path γ in Q_j that connects from z to z_j without intersecting any boundary components of Q_j. In particular, γ is disjoint from any element in A or B.

Now we can lift the path γ to a path $\hat{\gamma}$ in $\mathbf{H}$ that connects from $\hat{z}$ to a parabolic vertex v_j of Δ (corresponding the puncture z_j). Since γ does not intersect $\{A, B\}$, $\hat{\gamma}$ avoids all maximal elements of $E_{\hat{\alpha}_j}$ and $E_{\hat{\beta}_k}$ for $1 \le j \le q$ and $1 \le k \le r$. But since $\hat{T}_{\Delta,R}$ fixes $\hat{z}$ as well as any other points in $\hat{\gamma}$, by continuity, we conclude that $\hat{T}_{\Delta,R}$ fixes v_j, as asserted. □

As an immediate consequence of Lemma 6.3, we obtain the following result.

LEMMA 6.4. *Under the same condition of Lemma 6.3, $[\hat{T}_{\Delta,R}]^*$ is a pseudo-hyperbolic modular transformation on $T(\dot{S})$.*

PROOF. The lemma follows from Lemmas 6.3 and 5.2. □

7. Proof of Theorems 4.1 and 4.2

We assume that S is noncompact. Recall that $Q_1, \ldots, Q_v$ obtained from (1) are all possible once punctured disk components of $S \backslash \{A, B\}$.

Since a word w defined by (8) represents a pseudo-Anosov mapping class (see Penner [10]), we see that w is isotopic to a pseudo-Anosov map ω. By assumption, the map ω fixes nonpuncture zeros. Let $\chi \in \mathrm{Mod}_S$ be induced by ω. Then χ is hyperbolic in the sense of Bers [3]. It follows from [3, Theorem 5] that there is a Teichmüller geodesic $\mathcal{L}$ in $T(S)$ such that $\chi(\mathcal{L}) = \mathcal{L}$.

Let $x \in \mathcal{L}$ be represented by S. Then $\omega : S \to S$ is an absolutely extremal Teichmüller mapping. Let $z_0 \in \bar{S}$ be a zero of ϕ so that δ_0 is trivial. Note that some zeros could be punctures of S. Suppose that $z_0 \in Q_1$ is a nonpuncture zero of ϕ. Let z_1 denote the puncture of Q_1. Then $z_0 \neq z_1$. For any point $\hat{z}_0 \in \varrho^{-1}(z_0)$, we can choose a fundamental region Δ of G so that $\hat{z}_0 \in \Delta$. Since Q_1 contains a puncture, there is a parabolic vertex v_1 of Δ in $\partial \mathbf{H}$ that corresponds to the puncture z_1.

By Lemma 6.1, the map w can be lifted to $\hat{T}_{\Delta,R}$ that fixes $\hat{z}_0$. From Lemma 6.3, $\hat{T}_{\Delta,R}$ fixes v_1. Note that ω is isotopic to w. By assumption, an isotopy $H_t(\cdot)$ on S connecting ω and w can be constructed to leave z_0 fixed. Now ω can be lifted to $\hat{\omega}$ so that $\hat{\omega}(\hat{z}_0) = \hat{z}_0$. Also, the isotopy $H_t(\cdot)$ can be lifted to an isotopy $\hat{H}_t(\cdot)$ that satisfies the following properties: (i) for all $0 \leq t \leq 1$, $\hat{H}_t(\cdot) \circ G \circ \hat{H}_t(\cdot)^{-1} = G$; (ii) for all $0 \leq t \leq 1$, $\hat{H}_t(\hat{z}_0) = \hat{z}_0$; and (iii) $\hat{H}_0(\cdot) = \hat{\omega}$.

Since $\hat{T}_{\Delta,R}$ is a lift of w, there is an element $h \in G$ such that $\hat{H}_1(\cdot) = h \circ \hat{T}_{\Delta,R}$. Obviously, $\hat{\omega}|_{\partial \mathbf{H}} = h \circ \hat{T}_{\Delta,R}|_{\partial \mathbf{H}}$. Since both $\hat{\omega}$ and $\hat{T}_{\Delta,R}$ fix $\hat{z}_0$, $h(\hat{z}_0) = \hat{z}_0$. Hence $h = id$. It follows that $\hat{\omega}|_{\partial \mathbf{H}} = \hat{T}_{\Delta,R}|_{\partial \mathbf{H}}$ and thus $[\hat{\omega}] = [\hat{T}_{\Delta,R}]$. We conclude that $\hat{\omega}$ also fixes v_1. By Lemma 5.2, $[\hat{\omega}]^* \in \mathrm{Mod}_S^a$ is not pseudo-Anosov. Set $\theta = [\hat{\omega}] = [\hat{T}_{\Delta,R}]$. We claim that θ satisfies conditions (1) and (2) of Theorem 4.1. Indeed, condition (1) is clear. Since $\hat{\omega}$ fixes $\hat{z}_0$, by Lemma 3.2, θ keeps $\hat{\mathcal{L}} \subset F(S)$ (defined in (4)) invariant. So condition (2) holds. This completes the proof of Theorem 4.2.

The proof of Theorem 4.1 is similar. Suppose that $z_0 \in P_1$, where z_0 is a zero of ϕ so that δ_0 is trivial. From Lemma 6.1 again, the map w can be lifted to $\hat{T}_{\Delta,R}$ that fixes $\hat{z}_0$. From Lemma 6.2 and the same argument as above, $[\hat{T}_{\Delta,R}]^*$ is hyperbolic and satisfies conditions (1) and (2) of Theorem 4.1. If the condition that δ_0 is trivial is not assumed, then we can only get that $[\hat{\omega}]$ is hyperbolic and satisfies those conditions of Theorem 4.1.

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