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RESEARCH ARTICLE

Stability conditions on Calabi-Yau double/triple solids

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Abstract

In this paper, we prove a stronger form of the Bogomolov–Gieseker (BG) inequality for stable sheaves on two classes of Calabi–Yau threefolds, namely, weighted hypersurfaces inside the weighted projective spaces $\mathbb{P}(1,1,1,1,2)$ and $\mathbb{P}(1,1,1,1,4)$. Using the stronger BG inequality as a main technical tool, we construct open subsets in the spaces of Bridgeland stability conditions on these Calabi–Yau threefolds.

Contents

1. Introduction

1.1. Motivation and Results

Since Bridgeland [Bri07] defined the notion of stability conditions on derived categories, its construction on a given threefold has been an important open problem. It turned out that, to solve this problem, we need a Bogomolov–Gieseker (BG) type inequality, involving the third Chern character, for certain stable objects in the derived category [BMS16, BMT14, BMSZ17]. There are several classes of threefolds on which we know the existence of Bridgeland stability conditions [BMS16, BMT14, BMSZ17, Kos17,

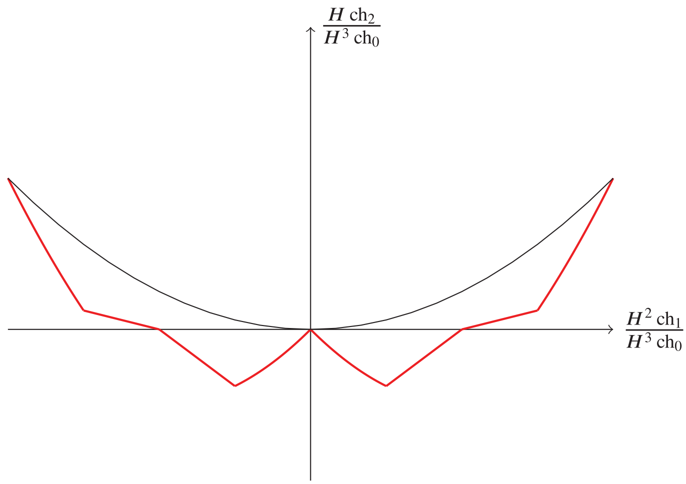

Figure 1. *Strong BG inequality on double/triple cover CY3s.*

Kos20, Li19a, Li19b, Liu21a, Liu21b, MP16a, MP16b, Macr14, Piy17, Sch14]. For K-trivial threefolds, the only known cases are the quintic threefolds [Li19a], abelian threefolds and their étale quotients [BMS16, MP16a, MP16b].

Among them, Li [Li19a] recently treated quintic threefolds, which is one of the most important cases for mirror symmetry. The crucial step in his arguments is to establish the improvement of the classical BG inequality for torsion free slope stable sheaves. Recall that a version of the classical BG inequality is the inequality

$$\frac{H\,\mathrm{ch}_2(E)}{H^3\,\mathrm{ch}_0(E)} \leq \frac{1}{2}\left(\frac{H^2\,\mathrm{ch}_1(E)}{H^3\,\mathrm{ch}_0(E)}\right)^2, \tag{1}$$

where E is a slope stable sheaf with respect to an ample divisor H. For del Pezzo and K3 surfaces, we can easily get the inequality stronger than equation (1), simply by using the Serre duality. In contrast, such an improvement of the BG inequality on Calabi–Yau threefolds is highly nontrivial.

When the first draft of this paper was submitted, the arguments in [Li19a] have been applied only for quintic threefolds. Very recently, Liu [Liu21a] treated Calabi–Yau complete intersections of quadratic and quartic hypersurfaces in $\mathbb{P}^5$ via a similar method. The goal of the present paper is to extend it to two other examples of Calabi–Yau threefolds, namely, general weighted hypersurfaces in the weighted projective spaces $\mathbb{P}(1,1,1,1,2)$ and $\mathbb{P}(1,1,1,1,4)$. We call them as *triple/double cover CY3* since they have finite morphisms to $\mathbb{P}^3$ of degree 3, 2, respectively. The following is our main result:

Theorem 1.1 (Theorems 4.1, 5.4). *Let X be a double or triple cover CY3, H the primitive ample divisor and E a slope stable sheaf with slope $\mu \in [-1,1]$. Then the inequality*

$$\frac{H\,\mathrm{ch}_2(E)}{H^3\,\mathrm{ch}_0(E)} \leq \Xi\left(\left|\frac{H^2\,\mathrm{ch}_1(E)}{H^3\,\mathrm{ch}_0(E)}\right|\right) \tag{2}$$

holds. Here, the function Ξ is defined as follows.

$$\Xi(t) := \begin{cases} t^2 - t & (t \in [0, 1/4]) \\ 3t/4 - 3/8 & (t \in [1/4, 1/2]) \\ t/4 - 1/8 & (t \in [1/2, 3/4]) \\ t^2 - 1/2 & (t \in [3/4, 1]). \end{cases}$$

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See Figure 1 for the graph of the function Ξ. Using this stronger BG inequality, we prove the following BG type inequality (involving ch_3) for $\nu_{\beta,\alpha}$-stable objects, which are certain two term complexes in the derived category. For the precise definition of $\nu_{\beta,\alpha}$-stability, see Section 2.

Theorem 1.2 (Theorem 2.3, Corollary 6.4). *Let X be a double or triple cover CY3, take real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha > \frac{1}{2}\beta^2 + \frac{1}{2}(\beta - \lfloor\beta\rfloor)(\lfloor\beta\rfloor + 1 - \beta)$. Let E be a $\nu_{\beta,\alpha}$-semistable object. Then the inequality*

$$Q^{\Gamma}_{\alpha,\beta}(E) \geq 0$$

holds. Here, we put $\Gamma := \frac{2}{9}H^2$ (resp. $\frac{1}{3}H^2$) when X is a triple (resp. double) cover CY3, and the quadratic form $Q^{\Gamma}_{\alpha,\beta}$ is defined as follows:

$$\begin{aligned} Q^{\Gamma}_{\alpha,\beta}(E) := (2\alpha - \beta^2)&\left(\overline{\Delta}_H(E) + 3\frac{\Gamma.H}{H^3}\left(H^3 \mathrm{ch}_0^{\beta}(E)\right)^2\right) \\ &+ 2\left(H\,\mathrm{ch}_2^{\beta}(E)\right)\left(2H\,\mathrm{ch}_2^{\beta}(E) - 3\Gamma.H\,\mathrm{ch}_0^{\beta}(E)\right) \\ &- 6\left(H^2\,\mathrm{ch}_1^{\beta}(E)\right)\left(\mathrm{ch}_3^{\beta}(E) - \Gamma\,\mathrm{ch}_1^{\beta}(E)\right). \end{aligned}$$

The above theorem enables us to construct an open subset in the space of Bridgeland stability conditions [BMS16, BMT14, BMSZ17]. For real numbers α, β, a, b, we define a group homomorphism $Z^{a,b}_{\beta,\alpha}\colon K(X) \to \mathbb{C}$ as

$$Z^{a,b}_{\beta,\alpha} := -\mathrm{ch}_3^{\beta} + bH\,\mathrm{ch}_2^{\beta} + aH^2\,\mathrm{ch}_1^{\beta} + i\left(H\,\mathrm{ch}_2^{\beta} - \frac{1}{2}\alpha^2 H^3\,\mathrm{ch}_0^{\beta}\right).$$

We denote by $\mathcal{A}^{\beta,\alpha}$ the double-tilted heart defined in [BMT14].

Theorem 1.3 (Theorem 7.2). *We have a continuous family $\left(Z^{a,b}_{\beta,\alpha}, \mathcal{A}^{\beta,\alpha}\right)$ of stability conditions parametrized by real numbers α, β, a, b satisfying*

$$\alpha > 0, \quad \alpha^2 + \left(\beta - \lfloor\beta\rfloor - \frac{1}{2}\right)^2 > \frac{1}{4}, \quad a > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha + \gamma,$$

where we put $\gamma := 2/9$ (resp. $1/3$) when X is a triple (resp. double) cover CY3. Acting by the group $\widetilde{\mathrm{GL}}^+(2;\mathbb{R})$, it forms an open subset in the space of stability conditions.

1.2. Strategy of the proof

In this subsection, we briefly explain how to prove Theorem 1.1. Let us first recall the arguments in [Li19a] for a quintic threefold $X_5 \subset \mathbb{P}^4$. We consider $(2,2,5)$, $(2,5)$, $(2,2)$ complete intersections

$$C_{2,2,5} \subset T_{2,5} \subset X_5, \quad C_{2,2,5} \subset S_{2,2}.$$

The stronger BG inequality on X_5 is proved in the following way:

1. First, we reduce the problem to proving the same inequality for stable sheaves on the surface $T_{2,5} \subset X_5$ by using the restriction technique.
2. Again using the restriction, the problem is further reduced to establishing a stronger Clifford type bounds on global sections for stable vector bundles on the curve $C_{2,2,5} \subset T_{2,5}$.
3. Regard the stable vector bundle on $C_{2,2,5}$ as a torsion sheaf on the surface $S_{2,2}$ via the inclusion $C_{2,2,5} \subset S_{2,2}$. Then a wall-crossing argument in the space of Bridgeland stability conditions on the surface $S_{2,2}$ gives the desired Clifford type bounds. The argument in this step first appeared in [Fey20].

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In step (3), the crucial fact is that the surface $S_{2,2}$ is del Pezzo, on which a stronger BG inequality holds.

For double/triple cover CY3s, the situation is quite similar. In fact, we have smooth complete intersection varieties

$$C_{2,2,6} \subset T_{2,6} \subset X_6, \quad C_{2,2,6} \subset S_{2,2} \quad \text{in } \mathbb{P}(1,1,1,1,2),$$
$$C_{2,4,8} \subset T_{2,8} \subset X_8, \quad C_{2,4,8} \subset S_{2,4} \quad \text{in } \mathbb{P}(1,1,1,1,4),$$

where both of the surfaces $S_{2,2} \subset \mathbb{P}(1,1,1,1,2)$ and $S_{2,4} \subset \mathbb{P}(1,1,1,1,4)$ are isomorphic to the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$, which is del Pezzo. Note that we consider $(2,4)$ complete intersection in $\mathbb{P}(1,1,1,1,4)$ instead of $(2,2)$, to avoid the singularity.

Hence, we are able to apply the methods in [Li19a] to our cases. At this moment, we do not know the way to treat these examples uniformly, so the author believes it is still worth writing down the complete proofs. In fact, it turns out that, in our cases, we need the modified term Γ in Theorem 1.2, unlike the quintic case.

1.3. Open problems

1. In Theorem 1.2, we expect we can take $\Gamma = 0$. For this, we need a further improvement of Theorem 1.1.
2. Stability conditions we construct in this paper are said to be ‘near the large volume limit’ in physics. For weighted hypersurfaces, we expect the existence of another kind of stability conditions, called *Gepner type*. Mathematically, it is the stability condition invariant under the certain autoequivalence of the derived category. See [Tod14, Tod17] for discussions on the construction of Gepner type stability conditions. To construct the heart corresponding to the Gepner type stability condition, the first task is to prove a stronger form of the BG inequality for stable sheaves with a specific slope equal $-1/2$. Unfortunately, Theorem 1.1 is not enough for this purpose.
3. One might ask whether we can treat other Calabi–Yau weighted hypersurfaces inside $\mathbb{P}(a_1, a_2, a_3, a_4, a_5)$ with more general weights (a_i). Unfortunately, quintic and double/triple cover CY3s are the only cases where $\mathbb{P}(a_1, a_2, a_3, a_4, a_5)$ contains a smooth Calabi–Yau hypersurface and a smooth del Pezzo (or K3) complete intersection surface at the same time. Indeed, it happens precisely when the weighted $\mathbb{P}^4$ has only isolated singularities and its canonical line bundle can be written as $L^{\otimes m}$, where L is a free line bundle and $m \geq 2$. These conditions are equivalent to the following numerical conditions.
   - For any i with $a_i > 1$ and for any $j \neq i$, a_i does not divide a_j,
   - $\sum a_i = m \cdot \mathrm{lcm}(a_i)$.

   An easy but lengthy calculation show that there are only three solutions. If we allow smooth Deligne–Mumford stacks, i.e., if we allow the weighted $\mathbb{P}^4$ to have nonisolated singularities, there are several other solutions.

1.4. Plan of the paper

The paper is organized as follows. In Section 2, we recall about the notion of tilt stability in the derived category and about the BG type inequality conjecture. Sections 3 and 4 are devoted to proving Theorem 1.1 for a triple cover CY3. The key ingredient is the stronger Clifford type bound proved in Section 3. In Section 5, we treat the case of a double cover CY3. In Section 6, we prove Theorem 1.2. Finally, in Section 7, we prove Theorem 1.3.

Notation and Convention. *In this paper, we always work over the complex number field* $\mathbb{C}$. *We will use the following notations.*

- *For an ample divisor* H *and a real number* $\beta \in \mathbb{R}$, *we denote by* $\mathrm{ch}^\beta = (\mathrm{ch}_0^\beta, \cdots, \mathrm{ch}_n^\beta) := e^{-\beta H}\,\mathrm{ch}$, *the* β*-twisted Chern character.*

◦ $\hom(E, F) := \dim \mathrm{Hom}(E, F)$, *and* $\mathrm{ext}^i(E, F) := \dim \mathrm{Ext}^i(E, F)$ *for objects* E, F *in the derived category, and an integer* i.

2. Preliminaries

2.1. BG type inequality conjecture

In this subsection, we recall the notion of tilt stability, and the BG type inequality conjecture. We mainly follow the notations in the paper [Li19a]. Let X be a smooth projective variety of dimension $n \geq 2$, H an ample divisor. We take real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha > \frac{1}{2}\beta^2$. We define a slope function μ_H as follows:

$$\mu_H := \frac{H^{n-1} \operatorname{ch}_1}{H^n \operatorname{ch}_0} \colon \operatorname{Coh}(X) \to \mathbb{R} \cup \{+\infty\}.$$

We have the notion of μ_H-stability on $\operatorname{Coh}(X)$ and the corresponding torsion pair on $\operatorname{Coh}(X)$:

$$\mathcal{T}_\beta := \langle T \in \operatorname{Coh}(X) : T \text{ is } \mu_H\text{-semistable with } \mu_H(T) > \beta \rangle,$$
$$\mathcal{F}_\beta := \langle F \in \operatorname{Coh}(X) : F \text{ is } \mu_H\text{-semistable with } \mu_H(F) \leq \beta \rangle.$$

Here, $\langle S \rangle$ denotes the extension closure of a set $S \subset \operatorname{Coh}(X)$ of objects in the category $\operatorname{Coh}(X)$. By the general theory of torsion pairs [HRS96], we obtain the new abelian category

$$\operatorname{Coh}^\beta(X) := \left\langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \right\rangle \subset D^b(X),$$

which is the heart of a bounded t-structure on $D^b(X)$. On the heart $\operatorname{Coh}^\beta(X)$, we define the following slope function:

$$\nu_{\beta,\alpha} := \frac{H^{n-2} \operatorname{ch}_2 - \alpha H^n \operatorname{ch}_0}{H^{n-1} \operatorname{ch}_1 - \beta H^n \operatorname{ch}_0} \colon \operatorname{Coh}^\beta(X) \to \mathbb{R} \cup \{+\infty\}.$$

Then as similar to the μ_H-stability on $\operatorname{Coh}(X)$, we can define the notion of $\nu_{\beta,\alpha}$-stability on $\operatorname{Coh}^\beta(X)$. We also call $\nu_{\alpha,\beta}$-stability as *tilt-stability*.

Definition 2.1. Let $E \in \operatorname{Coh}^0(X)$ be an object.

1. We define the *Brill–Noether (BN)* slope of E as

$$\nu_{BN}(E) := \frac{H^{n-2} \operatorname{ch}_2(E)}{H^{n-1} \operatorname{ch}_1(E)} \in \mathbb{R} \cup \{+\infty\}.$$

2. We say the object E is *Brill–Noether (BN) (semi)stable* if it is $\nu_{0,\alpha}$-(semi)stable for every sufficiently small real number $0 < \alpha \ll 1$.

We refer [Li19a, Section 2] for the basic properties of tilt stability and BN stability. Let us define the discriminant of an object $E \in D^b(X)$ as

$$\overline{\Delta}_H(E) := (H^{n-1} \operatorname{ch}_1(E))^2 - 2H^n \operatorname{ch}_0(E) H^{n-2} \operatorname{ch}_2(E).$$

The following is the main question we investigate in this paper.

Question 2.2 [BMS16, BMT14, BMSZ17]. Assume that $n = \dim X = 3$. Find a 1-cycle $\Gamma \in A_1(X)_\mathbb{R}$ satisfying $\Gamma.H \geq 0$, and the following property: Let E be a $\nu_{\beta,\alpha}$-semistable object. Then the inequality

$$Q^\Gamma_{\alpha,\beta}(E) \geq 0$$

holds. Here, the quadratic form $Q^{\Gamma}_{\alpha,\beta}$ is defined as follows:

$$Q^{\Gamma}_{\alpha,\beta}(E) := (2\alpha - \beta^2)\left(\overline{\Delta}_H(E) + 3\frac{\Gamma.H}{H^3}\left(H^3\operatorname{ch}_0^{\beta}(E)\right)^2\right) + 2\left(H\operatorname{ch}_2^{\beta}(E)\right)\left(2H\operatorname{ch}_2^{\beta}(E) - 3\Gamma.H\operatorname{ch}_0^{\beta}(E)\right) - 6\left(H^2\operatorname{ch}_1^{\beta}(E)\right)\left(\operatorname{ch}_3^{\beta}(E) - \Gamma\operatorname{ch}_1^{\beta}(E)\right).$$

The conjectural inequality above is called *the Bogomolov–Gieseker (BG) type inequality conjecture*, proposed in [BMS16, BMT14] with $\Gamma = 0$. It is known that the BG type inequality conjecture with $\Gamma = 0$ fails for some classes of threefolds, such as the blow-up of $\mathbb{P}^3$ at a point (cf. [Kos17, MS19, Sch17]). The question with the modified term Γ appeared in [BMSZ17] and proved affirmatively for all Fano threefolds.

The following reduction of Question 2.2 plays an important role in this paper.

Theorem 2.3 (cf. [Li19a, Theorem 3.2]). *Assume that $n = \dim X = 3$. Let Γ be a 1-cycle with $\Gamma.H \geq 0$. Suppose that for every BN stable object with $\nu_{BN}(E) \in [0, 1/2]$, the inequality $Q^{\Gamma}_{0,0}(E) \geq 0$ holds.*

Then the inequality in Question 2.2 holds for any choice of real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha > \frac{1}{2}\beta^2 + \frac{1}{2}(\beta - \lfloor\beta\rfloor)(\lfloor\beta\rfloor + 1 - \beta)$.

Proof. Exactly the same arguments as in [Li19a, Theorem 3.2] work since the following statements are true.

- Let $(\beta', \alpha') \in \mathbb{R}^2$ be a point on the line through $p_H(E)$ and (β, α) with $\alpha' > \frac{1}{2}\beta'^2$. Then $Q^{\Gamma}_{\alpha,\beta}(E) < 0$ implies $Q^{\Gamma}_{\alpha',\beta'}(E) < 0$. Here, we define a point $p_H(E) \in \mathbb{R}^2$ as

$$p_H(E) := \left(\frac{H^2\operatorname{ch}_1(E)}{H^3\operatorname{ch}_0(E)}, \frac{H\operatorname{ch}_2(E)}{H^3\operatorname{ch}_0(E)}\right).$$

- The quadratic form $Q^{\Gamma}_{\alpha,\beta}$ is seminegative definite on the kernel of $\overline{Z}_{\alpha,\beta} := H^2\operatorname{ch}_1^{\beta} + i(H\operatorname{ch}_2 - \alpha H^3\operatorname{ch}_0)$.

□

2.2. *Star-shaped functions and the BG type inequalities*

In this subsection, we explain the wall-crossing technique used to obtain the (stronger) BG inequality for tilt-stable objects. This idea will also appear in the proof of the BG type inequality conjecture involving ch_3. As in the previous subsection, we denote by X a smooth projective variety of dimension n, and H an ample divisor on X. We use the following notion.

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ is called *star-shaped* if the following condition holds: For all real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, the line segment connecting the points $(\beta, f(\beta))$ and $(0, \alpha)$ is above the graph of f.

Recall that for an object $E \in D^b(X)$ with $\operatorname{ch}_0(E) \neq 0$, we define

$$p_H(E) := \left(\frac{H^{n-1}\operatorname{ch}_1(E)}{H^n\operatorname{ch}_0(E)}, \frac{H^{n-2}\operatorname{ch}_2(E)}{H^n\operatorname{ch}_0(E)}\right).$$

We have the following result:

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Proposition 2.5 (cf. [BMS16, Li19b]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a star-shaped function. Assume that, for every μ_H-semistable torsion free sheaf E, the inequality*

$$\frac{H^{n-2}\operatorname{ch}_2(E)}{H^n \operatorname{ch}_0(E)} \le f\left(\frac{H^{n-1}\operatorname{ch}_1(E)}{H^n \operatorname{ch}_0(E)}\right)$$

holds. Then for every $\alpha > 0$ and a $\nu_{0,\alpha}$-semistable object E with $\operatorname{ch}_0(E) \neq 0$, its Chern character satisfies the same inequality.

Proof. Assume for a contradiction that there exists a tilt-semistable object E violating the required inequality. By [BMS16, Theorem 3.5], the object E satisfies the usual BG inequality $\overline{\Delta}_H(E) \geq 0$. Hence, we may assume that it has the minimum discriminant $\overline{\Delta}_H(E)$ among all tilt-semistable objects violating the inequality.

Assume that E becomes strictly ν_{0,α_0}-semistable for some $\alpha_0 > 0$. Then there exists a Jordan–Hölder factor F of E such that $p_H(F)$ is on the line segment connecting $p_H(E)$ and $(0, \alpha_0)$. Since the function f is star-shaped, the object F also violates the required inequality. Moreover, by [BMS16, Corollary 3.10] we have $\overline{\Delta}_H(F) < \overline{\Delta}_H(E)$, which contradicts the minimality assumption on the discriminant.

Now we can assume that E is $\nu_{0,\alpha}$-semistable for all $\alpha \gg 0$. Hence, by [BMS16, Lemma 2.7], the object E satisfies one of the following conditions:

1. $E \in \operatorname{Coh}(X)$ and it is μ_H-semistable with $\operatorname{ch}_0(E) > 0$.
2. $\mathcal{H}^{-1}(E)$ is μ_H-semistable, and $\dim \operatorname{Supp} \mathcal{H}^0(E) \leq n-2$.

In both cases, we get the contradiction by our assumption that μ_H-semistable torsion free sheaves satisfy the desired inequality. □

2.3. Triple cover CY3

Let us consider a general hypersurface

$$X := X_6 \subset P := \mathbb{P}(1,1,1,1,2)$$

of degree 6 inside the weighted projective space. Then X is a smooth projective Calabi–Yau threefold, which we call *triple cover CY3*. We will use general $(2,2,6)$, $(2,6)$, $(2,2)$-complete intersections

$$C_{2,2,6} \subset T_{2,6} \subset X_6, \quad C_{2,2,6} \subset S_{2,2}$$

in P. Since the line bundle $\mathcal{O}_P(2)$ is free, they are smooth. The following are some of the numerical invariants of $C := C_{2,2,6}$, $T := T_{2,6}$, $S := S_{2,2}$, and X.

- $-K_P = 6H_P$, $H_P^4 = \frac{1}{2}$.
- $g(C) = 25$,
- $-K_S = 2H_S$, $(-K_S)^2 = 8$. In particular, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.
- $C = 3(-K_S)$ as divisors in S.
- $\operatorname{td}_S = (1, H_S, 1)$.
- $K_T = 2H_T$, $H_T^2 = 6$, $\operatorname{td}_T = (1, -H_T, 11)$.
- $\operatorname{td}_X = (1, 0, \frac{7}{6}H_X^2, 0)$, $H_X^3 = 3$.

All the computations are straightforward. For example, to compute $\operatorname{td}_{X,2}$, it is enough to compute $\chi(\mathcal{O}_X(1))$, which can be calculated using the exact sequence

$$0 \to \mathcal{O}_P(-5) \to \mathcal{O}_P(1) \to \mathcal{O}_X(1) \to 0.$$

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3. Clifford type theorem

Recall from the last subsection that we denote by

$$C = C_{2,2,6} \subset S = S_{2,2} \subset P = \mathbb{P}(1,1,1,1,2)$$

the weighted complete intersections. We have $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $C \in |\mathcal{O}_S(6,6)|$. In this section, we will prove the following proposition:

Proposition 3.1. *Let F be a slope stable vector bundle on C of rank r, slope μ. Put $t := \mu/12$. Assume that $t \in [0, 1/2] \cup [3/2, 2]$. The following inequalities hold:*

1. *When $t \in [0, 1/6)$, we have $h^0(F)/r \leq \frac{12t+24}{25}$.*
2. *When $t \in [1/6, 1/4)$, we have $h^0(F)/r \leq \max\left\{\frac{8t+8}{9}, \frac{10}{19}t + \frac{145}{152}\right\}$.*
3. *When $t \in [1/4, 1/2]$, we have $h^0(F)/r \leq \max\left\{4t, \frac{33}{38}t + \frac{69}{76}\right\}$.*
4. *When $t \in [3/2, 11/6]$, we have $h^0(F)/r \leq \max\left\{4t, \frac{231}{32}t - \frac{375}{64}\right\}$.*
5. *When $t \in (11/6, \sqrt{14}/2]$, we have $h^0(F)/r \leq \frac{233t-191}{32}$.*
6. *When $t \in [\sqrt{14}/2, 23/12]$, we have $h^0(F)/r \leq \frac{192t-168}{25}$.*
7. *When $t \in [23/12, 2]$, we have $h^0(F)/r \leq 12t - 15$.*

Figure 2 below indicates how the inequalities in Proposition 3.1 look like.

Remark 3.2. The parameter $t = \mu/12$ naturally appears as the BN slope of the sheaf $\iota_* F$, where $\iota\colon C \hookrightarrow S$ is an embedding. Indeed, we have $\nu_{BN}(\iota_* F) = t - 3$, as we will see in the proof of Lemma 3.7 below.

It is also compatible with the slope function on T in the following sense. For a vector bundle F on T, we have $t(F|_C) = \mu_{H_T}(F)$.

Our strategy of the proof of Proposition 3.1 is to use Bridgeland stability conditions on the surface S, with the following three steps.

1. Regard F as a torsion sheaf $\iota_* F \in \mathrm{Coh}(S)$, which is $\nu_{0,\alpha}$-stable for $\alpha \gg 0$.
2. Estimate the first possible wall for $\iota_* F$ on the line $\beta = 0$ in (α, β) plane, using the stronger form of the BG inequality on S.
3. Bound global sections of BN-stable objects on S.

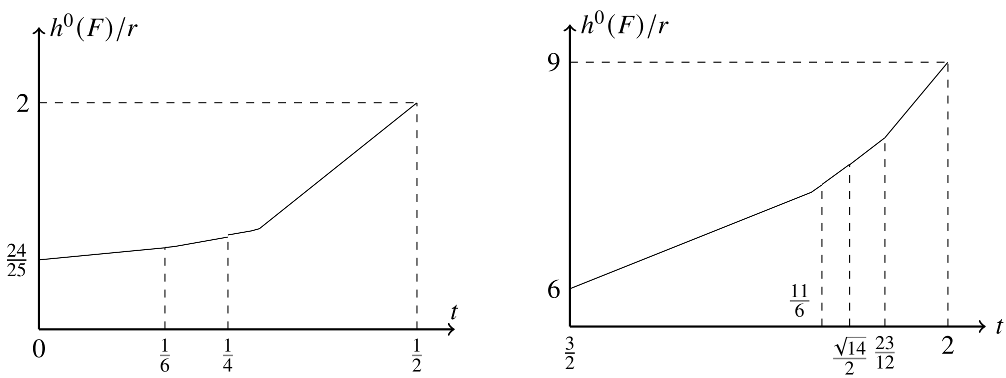

Figure 2. *The strong Clifford type bounds on C.*

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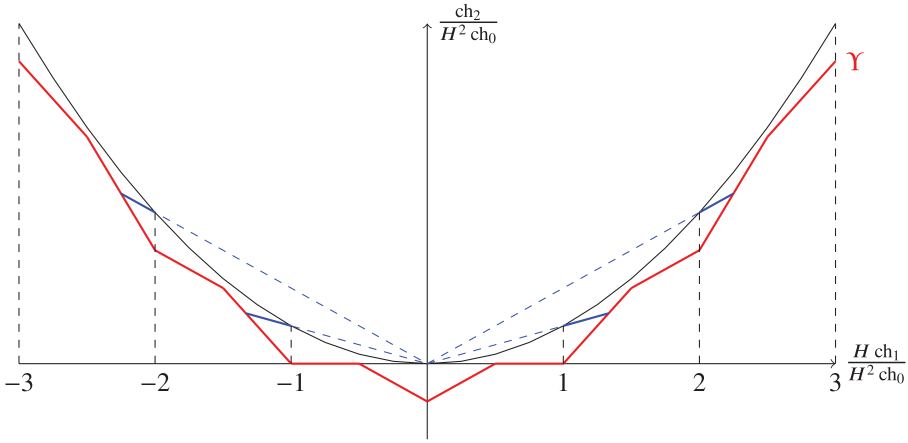

Figure 3. *The strong BG inequality Υ (red curve) on the quadric surface. Blue lines show the modified curve $\widetilde{\Upsilon}$.*

We define a function Υ on $\mathbb{R}$ as

$$\Upsilon(x) := \begin{cases} \frac{1}{2}x^2 - \frac{1}{2}(1-\{x\})^2 & (\{x\} \in (0, 1/2]) \\ \frac{1}{2}x^2 - \frac{1}{2}\{x\}^2 & (\{x\} \in [1/2, 1)) \\ \frac{1}{2}x^2 & (\{x\} = 0). \end{cases}$$

Here, $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$. See Figure 3 above for the shape of Υ. The following stronger BG inequality on the quadric surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ is well-known. We include a proof here since it demonstrates the technique which we will frequently use in this section.

Lemma 3.3. *Let F be a slope semistable torsion free sheaf on S. Then we have an inequality*

$$\frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)} \le \Upsilon(\mu_H(F)). \tag{3}$$

Proof. Since we have $\Upsilon(x+1) = \Upsilon(x) + x + 1/2$, the claim is invariant under tensoring with the line bundle $\mathcal{O}_S(H)$. Hence, we may assume $\mu_H(F) \in (0, 1)$. By the stability of F and the Serre duality, we have

$$\hom(\mathcal{O}(1), F) = 0, \quad \mathrm{ext}^2(\mathcal{O}(1), F) = \hom(F, \mathcal{O}(-1)) = 0$$

and hence $0 \ge -\mathrm{ext}^1(\mathcal{O}(1), F) = \chi(\mathcal{O}(1), F)$. By computing the right-hand side using the Riemann–Roch theorem, we get the inequality

$$\frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)} \le 0.$$

On the other hand, again by the stability of F and the Serre duality, we also have

$$\hom(F, \mathcal{O}) = 0, \quad \mathrm{ext}^2(F, \mathcal{O}) = \hom(\mathcal{O}, F(-2)) = 0,$$

which imply the inequality $0 \ge -\mathrm{ext}^1(F, \mathcal{O}) = \chi(F, \mathcal{O})$. Hence, we obtain

$$\frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)} \le \mu_H(F) - \frac{1}{2}.$$

Taking the minimum, the inequality (3) holds. □

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Remark 3.4. In [Rud94], Rudakov proved an inequality stronger than equation (3). However, our inequality is already optimal at $\mu_H = 1/2$ (consider $F = \mathcal{O}_S(1,0)$). Because of this fact, we cannot improve our inequality in Theorem 1.1 at $\mu_H = 1/2$, even if we use the result in [Rud94].

We define a function $\widetilde{\Upsilon}\colon \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{\Upsilon}(x) := \begin{cases} \Upsilon(x) & (|x| \in [0,1]) \\ \max\left\{\Upsilon(x), \frac{1}{2}\lfloor |x| \rfloor x\right\} & (|x| \geq 1). \end{cases}$$

Here, $\lfloor |x| \rfloor$ denotes the integral part of the absolute value of a real number $x \in \mathbb{R}$. Note that the function $\widetilde{\Upsilon}$ is star-shaped and we have $\Upsilon(x) \leq \widetilde{\Upsilon}(x)$ for all $x \in \mathbb{R}$; see Figure 3.

We have the following consequences of Lemma 3.3.

Lemma 3.5. *The following statements hold:*

1. *Fix a positive real number $\alpha > 0$. Let $F \in \mathrm{Coh}^0(S)$ be a $\nu_{0,\alpha}$-semistable object with $\mathrm{ch}_0(F) \neq 0$. Then the Chern character of F satisfies the inequality*

$$\frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)} \leq \widetilde{\Upsilon}(\mu_H(F)). \tag{4}$$

2. *For all real numbers $\beta, \alpha \in \mathbb{R}$ with $\alpha > \Upsilon(\beta)$, the pair $\left(Z_{\beta,\alpha}, \mathrm{Coh}^{\beta}(S)\right)$ defines a stability condition on $D^b(S)$. Here, the group homomorphism $Z_{\beta,\alpha}\colon K(S) \to \mathbb{C}$ is defined as*

$$Z_{\beta,\alpha} := -\mathrm{ch}_2 + \alpha H^2\,\mathrm{ch}_0 + i\left(H\,\mathrm{ch}_1 - \beta H^2\,\mathrm{ch}_0\right).$$

Proof.

(1) The first assertion follows from Lemma 3.3 and Proposition 2.5.
(2) For the second assertion, we can apply the arguments in [AB13, Bri08] by replacing the classical BG inequality with the stronger one, equation (3).

□

In the next two lemmas, we control the position of the first possible wall for $\iota_* F$, where F is a stable bundle on C, and then bound the slopes of the Harder–Narasimhan (HN) factors of $\iota_* F$ with respect to BN stability.

Lemma 3.6. *Let F be a slope stable vector bundle on C with rank r, slope μ. Let $(\beta_1, \alpha_1), (\beta_2, \alpha_2)$, $\beta_1 < 0 < \beta_2$, be the end points of a wall for $\iota_* F$ with respect to $\nu_{\beta,\alpha}$-stability. Then we have $\beta_2 - \beta_1 \leq 6$.*

Proof. By the Grothendieck–Riemann–Roch theorem, we have

$$\mathrm{ch}(\iota_* F) = (0, 6rH, r(\mu - 36)). \tag{5}$$

Suppose that there exists a positive integer α and a destabilizing sequence

$$0 \to F_2 \to \iota_* F \to F_1 \to 0$$

in $\mathrm{Coh}^0(S)$ for $\nu_{0,\alpha}$-stability. Denote by W the corresponding wall. Note that F_2 is a coherent sheaf. Let $T \subset F_2$ be a torsion part, and put $Q := F_2/T$. We have the following diagram in the tilted category $\mathrm{Coh}^0(S)$:

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & T & = & T & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_2 & \longrightarrow & \iota_* F & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q & \longrightarrow & \iota_* F/T & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & .
\end{array}$$

By taking the Coh(S)-cohomology of the bottom row in the above diagram, we get the exact sequence

$$0 \to Q/\mathcal{H}^{-1}(F_1) \to \iota_* F/T \to F_1 \to 0$$

in Coh(S). In particular, the sheaf $Q/\mathcal{H}^{-1}(F_1)$ is scheme-theoretically supported on the curve C. Hence, we have a surjection $Q|_C \twoheadrightarrow Q/\mathcal{H}^{-1}(F_1)$ and so get an inequality

$$6H^2 \operatorname{ch}_0(Q) = H \operatorname{ch}_1(Q|_C) \geq H \operatorname{ch}_1(Q/\mathcal{H}^{-1}(F_1)).$$

Note also that we have $\operatorname{ch}_0(Q) = \operatorname{ch}_0(\mathcal{H}^{-1}(F_1))$. Now we have

$$\mu_H(Q) - \mu_H(\mathcal{H}^{-1}(F_1)) = \frac{H \operatorname{ch}_1(Q/\mathcal{H}^{-1}(F_1))}{H^2 \operatorname{ch}_0(Q)} \leq 6. \tag{6}$$

Now let $(\beta_1, \alpha_1), (\beta_2, \alpha_2)$ be the end points of the wall W with $\beta_1 < 0 < \beta_2$. By Bertram's nested wall theorem (see, e.g., [Li19a, Lemma 2.9], [Maci14]), we know that for $0 < \epsilon \ll 1$, we have

$$Q \in \operatorname{Coh}^{\beta_2 - \epsilon}(S), \quad \mathcal{H}^{-1}(F_1)[1] \in \operatorname{Coh}^{\beta_1 + \epsilon}(S),$$

which in particular imply

$$\mu_H(Q) > \beta_2 - \epsilon, \quad \mu_H(\mathcal{H}^{-1}(F_1)) \leq \beta_1 + \epsilon.$$

Combining with the inequality (6), we have the desired inequality

$$\beta_2 - \beta_1 \leq 6.$$

□

Lemma 3.7. *Let F be a slope stable vector bundle on C with rank r, slope $\mu \in (0, 24)$. Let $t := \mu/12$. The following statements hold:*

1. *If $t \in \left(0, 2 - \frac{\sqrt{14}}{2}\right]$, then the sheaf $\iota_* F$ is BN-stable.*
2. *We have*

$$\nu^+_{BN}(\iota_* F) \leq \begin{cases} 1 - \frac{1}{2t} & (t \in (2 - \sqrt{14}/2, 1/2] \cup [3/2, \sqrt{14}/2]) \\ \frac{-9t+11}{-8t+7} & (t \in [\sqrt{14}/2, 23/12]) \\ 3t - 5 & (t \in [23/12, 2]). \end{cases}$$

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3. *We have*

$$\nu_{BN}^{-}(\iota_* F) \geq \begin{cases} \frac{-5(2t-7)}{2(t-6)} & (t \in (2-\sqrt{14}/2, 1/2] \cup [3/2, 11/6]) \\ -2 & (t \in [11/6, 2]). \end{cases}$$

Proof. Let W be a wall for $\iota_* F$, and let $(\beta_1, \alpha_1), (\beta_2, \alpha_2)$ be the end points of the wall W with $\beta_1 < 0 < \beta_2$. Recall that the wall W is a line segment with slope $\nu_{BN}(\iota_* F) = \mu/12 - 3 = t - 3$ (see equation (5) for the second equality). Since the curve Υ is not continuous when $\beta \in \mathbb{Z}$, the points (β_i, α_i) are either on the graph of Υ or on the vertical lines

$$L_n := \left\{ (n, y) : \frac{n^2 - 1}{2} < y < \frac{n^2}{2} \right\}, \quad n \in \mathbb{Z}.$$

When both of the end points (β_i, α_i) are on the curve Υ, we say that the wall W is of *Type A*; otherwise, we say it is of *Type B*.

First, assume that W is of Type A. By Lemma 3.6, we know that $\beta_2 - \beta_1 \leq 6$. Hence, the slope of the line through $(\beta_2, \Upsilon(\beta_2))$ and $(\beta_2 - 6, \Upsilon(\beta_2 - 6))$ is smaller than or equal to that of W, i.e., $\beta_2 - 3 \leq t - 3$. We conclude that every Type A wall is below the line $y = (t-3)(x-t) + \Upsilon(t)$.

On the other hand, for a given point $p = (\beta, \alpha) \in L_n$, let W_p be the line passing through the point p with slope $t - 3$. It is easy to compute the intersection points of W_p and $\Upsilon \cup \bigcup_{n \in \mathbb{Z}} L_n$. Together with the constraint $\beta_2 - \beta_1 \leq 6$, we can find the first possible wall of Type B.

Using these observations, we can list up the equation of the first possible wall:

- When $t \in [0, 1/2]$, the following is the first possible wall

$$y = (t-3)(x-t) + t - 1/2.$$

  Note that, if $t \in [0, 2-\sqrt{14}/2]$, it is negative at $x = 0$, hence $\iota_* F$ is BN stable.
- When $t \in [3/2, 2]$, one of the following is the first possible wall

$$y = (t-3)(x-t) + t - 1/2, \quad y = (t-3)(x+4) + 8.$$

  The first one is the line passing through the points $(t, \Upsilon(t))$ and $(t-6, \Upsilon(t-6))$. The second one is the line with slope $t-3$, passing through the point $(-4, \Upsilon(-4))$. See Figure 4 below.

Let L be the first possible wall described above, and let $(\beta_{\max}, \alpha_{\max}), (\beta_{\min}, \alpha_{\min})$ be the intersection points of L with the curve Υ with $\beta_{\min} < \beta_{\max}$. Then any wall W should be below the line L. Now consider the maximal destabilizing subobject $E_1 \subset \iota_* F$ with respect to the BN stability. We have three numerical constraints on E_1:

- Since $\iota_* F$ is $\nu_{\alpha,0}$-stable for α sufficiently large, we have $\mathrm{ch}_0(E_1) > 0$.
- E_1 satisfies the BG type inequality (4).
- The point $p_H(E_1)$ is below the line L.

Among all points satisfying the above three conditions, its slope becomes maximum at the point $(\alpha_{\max}, \beta_{\max})$, hence we get the bound

$$\nu_{BN}(E_1) \leq \frac{\alpha_{\max}}{\beta_{\max}}.$$

Now the straightforward computation shows the result. Similarly, we can get the bound $\nu_{BN}^{-}(\iota_* F) \geq \alpha_{\min}/\beta_{\min}$. □

The following lemma gives the upper bound on the number of global sections for BN stable objects.

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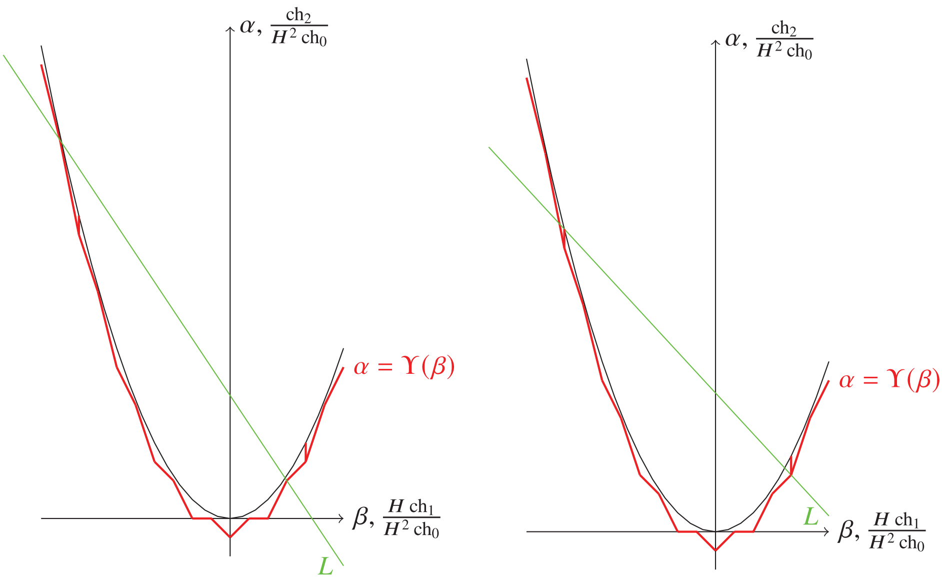

Figure 4. *The first possible wall L when $t = 3/2$ (left) and $t = 23/12$ (right).*

Lemma 3.8. *Let $F \in \mathrm{Coh}^0(S)$ be a BN stable object. Then the following inequalities hold:*

- *When $-1 < \nu_{BN}(F) < +\infty$, we have*

$$\hom(\mathcal{O}_S, F) = \mathrm{ch}_0(F) + H\,\mathrm{ch}_1(F) + \mathrm{ch}_2(F).$$

- *When $\nu_{BN}(E) \in (-n-1, -n)$, $n \in \mathbb{Z}_{>0}$, we have*

$$\hom(\mathcal{O}_S, F) \leq \mathrm{ch}_0(F) + \frac{1}{2n+1}H\,\mathrm{ch}_1(F) + \frac{1}{(2n+1)^2}\mathrm{ch}_2(F).$$

- *When $\nu_{BN}(E) = -n$, $n \in \mathbb{Z}_{>0}$, we have*

$$\hom(\mathcal{O}_S, F) \leq \mathrm{ch}_0(F) + \frac{1}{4n}H\,\mathrm{ch}_1(F).$$

Proof. First, assume that $\nu_{BN}(F) > -1$. Noting $\nu_{BN}(\mathcal{O}_S[1]) = +\infty$ and $\nu_{BN}(\mathcal{O}_S(-2)[1]) = -1$, we have the following vanishings for any $i \geq 0$:

$$\hom(\mathcal{O}_S, F[1+i]) = \hom(F, \mathcal{O}_S(-2)[1-i]) = 0,$$
$$\hom(\mathcal{O}_S, F[-1-i]) = \hom(\mathcal{O}_S[1+i], F) = 0.$$

Hence, by the Riemann–Roch, we get

$$\hom(\mathcal{O}_S, F) = \chi(F) = \int_S \mathrm{ch}(F).(1, H, 1)$$
$$= \mathrm{ch}_0(F) + H\,\mathrm{ch}_1(F) + \mathrm{ch}_2(F).$$

Next, consider the case $\nu_{BN}(E) \in (-n-1, -n)$, $n \in \mathbb{Z}_{>0}$. Let

$$(x_F, y_F) := p_H(F) = \left(\frac{H\,\mathrm{ch}_1(F)}{H^2\,\mathrm{ch}_0(F)}, \frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)}\right).$$

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Since we assumed $y_F/x_F = \nu_{BN}(F) \leq -1 < 0$, the line through $(0,0)$ and (x_F, y_F) intersects with the region $y \geq 1/2x^2, x < 0$.

Take such a point (β, α). Then we know that the objects $F, \mathcal{O}_S \in \mathrm{Coh}^\beta(S)$ are $\nu_{\beta,\alpha}$-semistable with

$$\nu_{\beta,\alpha}(F) = \nu_{\beta,\alpha}(\mathcal{O}_S) = \alpha/\beta = \nu_{BN}(F).$$

Let us consider the exact triangle

$$\mathrm{Hom}(\mathcal{O}_S, F) \otimes \mathcal{O}_S \xrightarrow{ev} F \to \widetilde{F} := \mathrm{Cone}(ev).$$

Since the only Jordan–Hölder factor of $\mathrm{Hom}(\mathcal{O}_S, F) \otimes \mathcal{O}_S$ with respect to $\nu_{\beta,\alpha}$-stability is $\mathcal{O}_S$, the evaluation map ev must be injective in the category $\mathrm{Coh}^\beta(S)$. Hence, it follows that $\widetilde{F} \in \mathrm{Coh}^\beta(S)$, and it is $\nu_{\beta,\alpha}$-semistable with $\nu_{\beta,\alpha}(\widetilde{F}) = \nu_{\beta,\alpha}(F) = \nu_{BN}(F)$. Now choose β sufficiently close to zero so that $\mathcal{O}_S(-2n)[1] \in \mathrm{Coh}^\beta(S)$. As before, we have the vanishing statements

$$\begin{aligned}\hom(\mathcal{O}_S(-2n), \widetilde{F}[1+i]) &= \hom(\widetilde{F}, \mathcal{O}_S(-(2n+2))[1-i]) = 0,\\ \hom(\mathcal{O}_S(-2n), \widetilde{F}[-1-i]) &= \hom(\mathcal{O}_S(-2n)[1+i], \widetilde{F}) = 0\end{aligned}$$

for $i \geq 0$. Hence, we have

$$\begin{aligned}0 &\leq \hom(\mathcal{O}_S(-2n), \widetilde{F})\\ &= \chi(\mathcal{O}_S(-2n), \widetilde{F})\\ &= \mathrm{ch}_2(F) + (2n+1)H\,\mathrm{ch}_1(F) + (2n+1)^2(\mathrm{ch}_0(F) - \hom(\mathcal{O}_S, F)),\end{aligned}$$

and so

$$\hom(\mathcal{O}_S, F) \leq \mathrm{ch}_0(F) + \frac{H\,\mathrm{ch}_1(F)}{2n+1} + \frac{\mathrm{ch}_2(F)}{(2n+1)^2}$$

as required.

Finally, assume that $\nu_{BN}(F) = -n$, $n \in \mathbb{Z}_{>0}$. Then the same argument shows that $\chi(\mathcal{O}_S(-2n+1), \widetilde{F}) \geq 0$, and we get the inequality

$$\hom(\mathcal{O}_S, F) \leq \mathrm{ch}_0(F) + \frac{H\,\mathrm{ch}_1(F)}{2n} + \frac{\mathrm{ch}_2(F)}{(2n)^2} = \mathrm{ch}_0(F) + \frac{1}{4n}H\,\mathrm{ch}_1(F).$$ □

Let us define a function $\Omega\colon \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ as

$$\Omega(x,y) := \begin{cases} y + x & (x/y > -1)\\ \frac{y}{2n+1} + \frac{x}{(2n+1)^2} & (x/y \in (-n-1, -n), n \in \mathbb{Z}_{>0})\\ \frac{1}{4n}y & (x/y = -n, n \in \mathbb{Z}_{>0}).\end{cases}$$

Lemma 3.9. *Let $O \in \mathbb{R}^2$ be the origin, and let $P = (x_p, y_p), Q = (x_q, y_q) \in \mathbb{R} \times \mathbb{R}_{>0}$ be points satisfying $x_p/y_p < x_q/y_q$ and $y_p > y_q$.*

Among all the sequences $O = P_0, P_1, \cdots, P_{m-1}, P_m = P$ of points in the triangle OPQ such that $P_0P_1 \cdots P_m$ form convex polygons, the sum

$$\sum_{i=1}^{m} \Omega(\overrightarrow{P_{i-1}P_i})$$

can achieve the maximum only when $m \leq 2$.

Moreover, when $m = 2$, the point $P_1 = (x_1, y_1)$ can be chosen to satisfy one of the following conditions:

- $P_1 = Q$,
- *P_1 is on the line segment OQ (resp. PQ) such that the slope of P_1P (resp. OP_1) is $-1/n$ for some $n \in \mathbb{Z}_{>0}$.*
- *The lines OP_1 and P_1P have slopes $-1/m, -1/n$ for some integers $m, n \in \mathbb{Z}_{>0}$.*

Proof. The proof is elementary and almost identical with that of [Li19a, Lemma 4.11]. The key observations are the following:

- The function Ω is linear with respect to both variables x and y as long as the slope x/y is fixed.
- The function Ω is upper semicontinuous.

We refer to [Li19a, Lemma 4.11] for the details. □

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Let F be a slope stable vector bundle on C of rank r and slope μ. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = \iota_* F$$

be the HN filtration with respect to the BN stability, and define $P_i := (\mathrm{ch}_2(E_i), H\,\mathrm{ch}_1(E_i))$. We have an inequality

$$h^0(F) \leq \sum_{i=1}^{m} \Omega(\overrightarrow{P_{i-1}P_i}) \tag{7}$$

by Lemma 3.8 (cf. [Li19a, Equation (21)]). We will bound the RHS in the above inequality. Let us put

$$P = (x_p, y_p) := (\mathrm{ch}_2(\iota_* F), H\,\mathrm{ch}_1(\iota_* F)) = (r(\mu - 36), 12r),$$

and $Q = (x_q, y_q)$ to be a point such that x_q/y_q is the upper bound for $\nu^+_{BN}(\iota_* F)$, and $(x_p - x_q)/(y_p - y_q)$ is the lower bound for $\nu^-_{BN}(\iota_* F)$, given in Lemma 3.7. We know that the HN polygon of $\iota_* F$ with respect to the BN stability is inside the triangle OPQ. Hence, by Lemma 3.9, we may assume $m = 2$ and the point P_1 satisfies one of the following conditions:

- $P_1 = Q$,
- P_1 is on the line segment OQ (resp. PQ) such that the slope of P_1P (resp. OP_1) is $-1/n$ for some $n \in \mathbb{Z}_{>0}$.
- The lines OP_1 and P_1P have slopes $-1/m, -1/n$ for some integers $m, n \in \mathbb{Z}_{>0}$.

We now argue case by case.

(0) When $t \in (0, 2-\sqrt{14}/2]$, the sheaf $\iota_* F$ is BN stable. The BN slope is $\nu_{BN}(\iota_* F) = t-3 \in (-3, -2)$, hence by Lemma 3.8, we have

$$h^0(F)/r \leq \frac{12}{5} + \frac{12(t-3)}{25} = \frac{12t + 24}{25}.$$

(1) Assume $t \in (2 - \sqrt{14}/2, 1/6)$. By Lemma 3.7, we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-3} \in (-1/2, -1/3)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{2t}{2t-1} \in (-1/2, -1/3)$.
- The slope of $\overrightarrow{QP}$ is $\frac{2(t-6)}{-5(2t-7)} \in (-1/2, -1/3)$.

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Hence, we may assume $P_1 = Q = (x_q, y_q)$. We get

$$\begin{aligned} h^0(F) &\leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) \\ &= \frac{y_q}{5} + \frac{x_q}{25} + \frac{y_p - y_q}{5} + \frac{x_p - x_q}{25} = \frac{12t+24}{25}r. \end{aligned}$$

(2) Assume $t \in [1/6, 1/4)$. Then the slopes of the triangle OPQ are the same as the case (1). The only difference is that the slope of $\overrightarrow{OQ}$ sits inside the interval $(-1, -1/2]$, instead of $(-1/2, -1/3)$. Hence, we may take P_1 as Q or the point A on the line segment QP with slope $-1/2$. The coordinates are given as

$$Q = ((2t-1)r, 2tr), \quad A = \left(\frac{24(2t^2-8t+1)}{-6t+11}r, \frac{12(2t^2-8t+1)}{-6t+11}r\right).$$

First, consider the case of $P_1 = Q \neq A$. We have

$$\begin{aligned} \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) &= \frac{y_q}{3} + \frac{x_q}{9} + \frac{y_p - y_q}{5} + \frac{x_p - x_q}{25} \\ &= \frac{8t+8}{9}r. \end{aligned}$$

When $P_1 = A$, we have

$$\begin{aligned} \Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) &= \frac{y_a}{8} + \frac{y_p - y_a}{5} + \frac{x_p - x_a}{25} \\ &= \frac{1}{200}y_a + \frac{12t+24}{25}r. \end{aligned}$$

As a function on $t \in [1/6, 1/4]$, we have an inequality

$$y_a(t) \leq \left(\frac{176}{19}t - \frac{23}{19}\right)r$$

since the equality holds for $t = 1/4, 1/6$, and $y_a''(t) > 0$ for $t < 11/6$. Hence, we obtain

$$\begin{aligned} \Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) &\leq \frac{1}{200}\left(\frac{176}{19}t - \frac{23}{19}\right)r + \frac{12t+24}{25}r \\ &= \left(\frac{10}{19}t + \frac{145}{152}\right)r. \end{aligned}$$

We conclude that

$$h^0(F)/r \leq \max\left\{\frac{8t+8}{9}, \frac{10}{19}t + \frac{145}{152}\right\}.$$

(3) Assume $t \in [1/4, 1/2]$. Again the only difference with the cases (1), (2) is that the slope of OQ is smaller than or equal to -1 (or $+\infty$ when $t = 1/2$) in the present case. Hence, we may choose P_1 to be Q, or the points A, B on the line segment QP with slope $-1/2, -1$, respectively. The coordinate of Q, A are the same as in equation (2), and we have

$$B = (x_b, y_b) = \left(-\frac{12(2t^2-8t+1)}{8t-23}r, \frac{12(2t^2-8t+1)}{8t-23}r\right).$$

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We get

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = y_q + x_q + \frac{y_p - y_q}{5} + \frac{x_p - x_q}{25} = 4tr,$$
$$\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) = \frac{1}{4}y_b + \frac{y_p - y_b}{5} + \frac{x_p - x_b}{25} = \frac{9}{100}y_b + \frac{12t+24}{25}r.$$

As a function of $t \in [1/4, 1/2]$, we have the inequality

$$y_b(t) \le \left(\frac{82}{19}t - \frac{11}{19}\right)r$$

and hence

$$\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) \le \frac{33}{38}tr + \frac{69}{76}r.$$

We can directly compute that

$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) = \frac{1}{200}y_a + \frac{12t+24}{25} \le \frac{33}{38}tr + \frac{69}{76}r.$$

We conclude that

$$h^0(F)/r \le \max\left\{4t, \frac{33}{38}t + \frac{69}{76}\right\}.$$

(4) Assume $t \in [3/2, 11/6]$. In this case, we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-3} \in (-1, -1/2)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{2t}{2t-1} > 0$,
- The slope of $\overrightarrow{QP}$ is $\frac{2(t-6)}{-5(2t-7)} \in [-1/2, -1/3)$.

There are four choices of the point P_1, say Q, A, B, C, where A is the point on the line PQ with slope -1, B is the point on the line OQ such that the slope of BP is $-1/2$ and C is the intersection point of two lines OA and BP. Explicitly, we have

$$Q = ((2t-1)r, 2tr), \quad A = \left(-\frac{12(2t^2-8t+1)}{8t-23}r, \frac{12(2t^2-8t+1)}{8t-23}r\right),$$
$$B = \left(\frac{12(2t-1)(t-1)}{6t-1}r, \frac{24t(t-1)}{6t-1}r\right), \quad C = (-12(t-1)r, 12(t-1)r).$$

Hence, we get

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = \frac{4}{5}y_q + \frac{24}{25}x_q + \frac{12t+24}{25}r = 4tr,$$
$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) = \frac{9}{100}y_a + \frac{12t+24}{25}r,$$
$$\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) = y_b + x_b + \frac{y_p - y_b}{8} = \frac{7}{8}y_b + x_b + \frac{3}{2}r,$$
$$\Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) = \frac{y_c}{4} + \frac{y_p - y_c}{8} = \frac{3}{2}tr.$$

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First, we can show that

$$\frac{9}{100}y_a < \frac{4}{5}y_q + \frac{24}{25}x_q$$

for $t \in [3/2, 11/6]$. On the other hand, it is easy to see

$$\Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) \leq \Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) = \frac{90t^2 - 96t + 21}{2(6t-1)}r \leq \left(\frac{231}{32}t - \frac{375}{64}\right)r.$$

Hence, we can conclude that

$$h^0(F)/r \leq \max\left\{4t, \frac{231}{32}t - \frac{375}{64}\right\}.$$

(5) Assume $t \in (11/6, \sqrt{14}/2]$. Then we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-3} \in (-1, -1/2)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{2t}{2t-1} > 0$,
- The slope of $\overrightarrow{QP}$ is $-1/2$.

Hence, we can choose the point P_1 to be B or C appeared in the case (4) above. For $t \in [11/6, \sqrt{14}/2]$, we have

$$\Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) \leq \Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) = \frac{90t^2 - 96t + 21}{2(6t-1)}r \leq \left(\frac{233}{32}t - \frac{191}{32}\right)r.$$

(6) Assume $t \in [\sqrt{14}/2, 23/12]$. In this case, we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-3} \in (-1, -1/2)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{8t-7}{9t-11} > 0$,
- The slope of $\overrightarrow{QP}$ is $-1/2$.

We may choose the point P_1 as Q or A, where A is the point on the line PQ with slope -1. Explicitly, we have

$$Q = \left(\frac{12}{25}(9t-11)r, \frac{12}{25}(8t-7)r\right), \quad A = (-12(t-1)r, 12(t-1)r),$$

and hence

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = y_q + x_q + \frac{y_p - y_q}{8} = \frac{192t - 168}{25}r,$$
$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) = \frac{y_a}{4} + \frac{y_p - y_a}{8} = \frac{3}{2}tr.$$

We can see that $\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) \geq \Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP})$, hence we conclude that

$$h^0(F)/r \leq \frac{192t - 168}{25}.$$

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(7) Assume $t \in [23/12, 2)$. Then we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-3} \in (-1, -1/2)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{1}{3t-5} > 0$,
- The slope of $\overrightarrow{QP}$ is $-1/2$.

Hence, $P_1 = Q$ or A, where A is the point on the line segment PQ with slope -1, i.e.,

$$Q = ((12t - 20)r, 4r), \quad A = ((-12t + 12)r, (12t - 12)r).$$

We can calculate as

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = (12t - 15)r,$$
$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) = \frac{3}{2}tr$$

hence taking the maximum, we conclude that

$$h^0(F)/r \leq 12t - 15.$$

□

4. Stronger BG inequality

Using the Clifford type bound obtained in Proposition 3.1, we prove the following stronger version of the (classical) BG inequality on a triple cover CY3 $X := X_6 \subset \mathbb{P}(1, 1, 1, 1, 2)$.

Theorem 4.1. *Let X be a triple cover CY3. Let $F \in D^b(X)$ be a $\nu_{\alpha,0}$-semistable object for some $\alpha > 0$, with $\mu_H(F) \in [-1, 1]$. Then we have the following inequality*

$$\frac{H \operatorname{ch}_2(F)}{H^3 \operatorname{ch}_0(F)} \leq \Xi(|\mu_H(F)|), \tag{8}$$

where

$$\Xi(t) := \begin{cases} t^2 - t & (t \in [0, 1/4]) \\ 3t/4 - 3/8 & (t \in [1/4, 1/2]) \\ t/4 - 1/8 & (t \in [1/2, 3/4]) \\ t^2 - 1/2 & (t \in [3/4, 1]). \end{cases}$$

Proof. Assume for contradiction that there is a tilt semistable object F violating the inequality (8). We may assume $\mu_H(F) \geq 0$ by replacing F with $F^\vee$ if necessary. First, observe that the following conditions hold:

- Let $p = (a, b)$ be an arbitrary point with $a \in [0, 1], b > \Xi(a)$, and take a real number $\alpha > 0$ (resp. $\alpha' > 1/2$). Then the line segment connecting the points p and $(0, \alpha)$ (resp. $(1, \alpha')$) is above the graph of Ξ.
- Let L be the line through $p_H(F)$ and $p_H(F(-2H)[1])$. Then L passes through points $(0, \alpha_0)$, $(-1, \alpha_0')$ with $\alpha_0 > 0, \alpha_0' > 1/2$. Putting $(a, b) := p_H(F)$, the conditions are equivalent to the inequalities

$$b > a^2 - a, \quad b > a^2 - 1/2.$$

Under these conditions, we can apply the arguments in [Li19a, Proposition 5.2, Corollary 5.4]. As a result, by restricting to the surface $T = T_{2,6} \subset X_6$, we obtain a tilt-stable object F on T with $\mu_H(F) \in (0, 1)$ and

$$\frac{\operatorname{ch}_2(F)}{H^2 \operatorname{ch}_0(F)} > \Xi\left(\frac{H \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)}\right). \tag{9}$$

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Furthermore, by the first paragraph in the proof of [Li19a, Proposition 5.2], we may assume that

- $\mu_{H_T}(F) \in (0, 1/2]$,
- F is μ_{H_T}-stable *coherent sheaf*,
- $F|_C, F^\vee(2H_T)|_C$ are slope stable.

Using the Riemann–Roch and the vanishings

$$\hom(\mathcal{O}_T, F(-2H_T)) = 0 = \hom(\mathcal{O}_T, F^\vee)$$

(both follows from slope stability of F and the assumption on its slope), we have

$$\begin{aligned} \mathrm{ch}_2(F) - H_T\, \mathrm{ch}_1(F) + 11\, \mathrm{ch}_0(F) &= \chi(F) \\ &\leq h^0(F|_C) + h^0(F^\vee(2H_T)|_C). \end{aligned} \tag{10}$$

Note that we have

$$\begin{aligned} \mathrm{ch}(F|_C) &= (\mathrm{ch}_0(F), 2H\, \mathrm{ch}_1(F)), \\ \mathrm{ch}(F^\vee(2H_T)|_C) &= (\mathrm{ch}_0(F), 4H^2\, \mathrm{ch}_0(F) - 2H\, \mathrm{ch}_1(F)). \end{aligned}$$

Applying Proposition 3.1 to the RHS of equation (10), we get

$$\frac{\mathrm{ch}_2(F)}{H^2\, \mathrm{ch}_0(F)} \leq \begin{cases} -\frac{23}{25}\mu_H(F) - \frac{13}{75} & (\mu_H(F) \in (0, 1/12]) \\ -\frac{1}{5}\mu_H(F) - \frac{7}{30} & (\mu_H(F) \in [1/12, 2 - \sqrt{14}/2]) \\ -\frac{641}{4800}\mu_H(F) - \frac{1157}{4800} & (\mu_H(F) \in [2 - \sqrt{14}/2, 1/6)) \\ -\frac{421}{3648}\mu_H(F) - \frac{595}{2432} & (\mu_H(F) \in [1/6, 89/496]) \\ -\frac{95}{1728}\mu_H(F) - \frac{883}{3456} & (\mu_H(F) \in [89/496, 37/206]) \\ \frac{13}{27}\mu_H(F) - \frac{19}{54} & (\mu_H(F) \in [37/206, 1/4)) \\ \frac{109}{228}\mu_H(F) - \frac{53}{152} & (\mu_H(F) \in [1/4, 69/238]) \\ \mu_H(F) - \frac{1}{2} & (\mu_H(F) \in [69/238, 1/2]). \end{cases} \tag{11}$$

In all cases, the inequalities (11) contradict the inequality (9). □

5. The case of double cover

In this section, we consider the double cover X of $\mathbb{P}^3$ branched along a smooth hypersurface of degree 8; X is another example of Calabi–Yau threefolds. As in the previous sections, we treat X as a weighted hypersurface in $P = \mathbb{P}(1, 1, 1, 1, 4)$ of degree 8. Let

$$C_{2,4,8} \subset T_{2,8} \subset X_8, \quad C_{2,4,8} \subset S_{2,4}$$

be smooth $(2, 4, 8)$-, $(2, 8)$-, $(2, 4)$-complete intersections in P. The following is the list of their numerical invariants we need.

- $-K_P = 8H$, $H_P^4 = 1/4$,
- $g(C) = 49$,
- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$,
- $K_T = 2H_T$, $H_T^2 = 4$, $\mathrm{td}_T = (1, -H_T, 10)$,
- $\mathrm{td}_X = (1, 0, \frac{11}{6}H_X^2, 0)$, $H_X^3 = 2$.

Remark 5.1. To make the surface S smooth, we take a $(2, 4)$-complete intersection instead of a $(2, 2)$-complete intersection.

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5.1. Clifford type bound

In this subsection, we will prove the Clifford type theorem for the curve $C = C_{2,4,8}$, using the embedding ι into the quadric surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 5.2. *Let F be a slope stable vector bundle on C with rank r slope μ. Put $t := \frac{\mu}{16}$, and assume $t \in [0, 1/2] \cup [3/2, 2]$. Then we have the following statements hold:*

1. *When $t \in \left[0, \frac{5-\sqrt{23}}{2}\right]$, the sheaf $\iota_* F$ is BN stable.*
2. *We have*

$$\nu^+_{BN}(\iota_* F) \leq \begin{cases} 1 - \frac{1}{2t} & (t \in [\frac{5-\sqrt{23}}{2}, 1/2] \cup [3/2, \frac{\sqrt{23}-1}{2}]) \\ \frac{-13t+16}{-12t+11} & (t \in [\frac{\sqrt{23}-1}{2}, 31/16]) \\ 4t - 7 & (t \in [31/16, 2]). \end{cases}$$

3. *We have*

$$\nu^-_{BN}(\iota_* F) \geq \begin{cases} \frac{-7(2t-9)}{2(t-8)} & (t \in (\frac{5-\sqrt{23}}{2}, 1/2] \cup [3/2, 15/8]) \\ -3 & (t \in [15/8, 2]). \end{cases}$$

Proof. The proof is almost identical to that of Lemma 3.7. Hence, we just give an outline of the proof. Let us consider the embedding $\iota\colon C \hookrightarrow S$. For $F \in \mathrm{Coh}(C)$, we have $\mathrm{ch}(\iota_* F) = (0, 8rH, r(\mu - 64))$. Let W be a wall for $\iota_* F$ with respect to the $\nu_{\alpha,0}$-stability, and let β_1, β_2 be the β-coordinates of the end points of the wall W with $\beta_1 < 0 < \beta_2$. Then we can show that $\beta_2 - \beta_1 \leq 8$. We have $\nu_{BN}(\iota_* F) = t - 4$, and we can get the bounds of the first possible wall as follows:

- When $t \in [0, 1/2]$, the equation of the first possible wall is

$$y = (t-4)(x-t) + t - 1/2,$$

  which is the line passing through the points $(t, \Upsilon(t)), (t-8, \Upsilon(t-8))$. We can see that $y(0) \leq 0$ for $t \in [0, \frac{5-\sqrt{23}}{2}]$, hence the sheaf $\iota_* F$ is BN stable.
- When $t \in [3/2, 2]$, we have two possibilities of the first wall:

$$y = (t-4)(x-t) + t - 1/2, \quad y = (t-4)(x+6) + 18.$$

  The first equation is the line passing through the points $(t, \Upsilon(t)), (t-8, \Upsilon(t-8))$, and the second one is the line passing through the point $(-6, \Upsilon(-6))$ with slope $t - 4$.

As similar to Lemma 3.7, we get the bound on $\nu^{\pm}_{BN}(\iota_* F)$ by computing the end points of the first possible walls listed above. □

We get the following Clifford type bound:

Proposition 5.3. *Let F be a slope stable vector bundle on C of rank r, slope $\mu \in (0, 8] \cup [24, 32)$. Put $t := \mu/16$. The following inequalities hold:*

1. *When $t \in (0, 1/8)$, we have $h^0(F)/r \leq \frac{16(t+3)}{49}$.*
2. *When $t \in [1/8, 1/6)$, we have $h^0(F)/r \leq \max\left\{\frac{12t+24}{25}, \frac{85t}{246} + \frac{481}{492}\right\}$.*
3. *When $t \in [1/6, 1/4)$, we have $h^0(F)/r \leq \max\left\{\frac{8t+8}{9}, \frac{17t}{38} + \frac{147}{152}\right\}$.*
4. *When $t \in [1/4, 1/2]$, we have $h^0(F)/r \leq \max\left\{4t, \frac{63t}{82} + \frac{153}{164}\right\}$.*
5. *When $t \in [3/2, 15/8]$, we have $h^0(F)/r \leq 4t$.*

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6. *When* $t \in (15/8, \frac{\sqrt{23}-1}{2}]$, *we have* $h^0(F)/r \leq \frac{133t-114}{18}$.
7. *When* $t \in [\frac{\sqrt{23}-1}{2}, 31/16]$, *we have* $h^0(F)/r \leq \frac{236}{49}t - \frac{148}{21}$.
8. *When* $t \in [31/16, 2)$, *we have* $h^0(F)/r \leq 16t - 23$.

Proof. As in the proof of Proposition 3.1, the problem is reduced to computing $\Omega(\overrightarrow{OP_1}) + \Omega(\overrightarrow{P_1P})$ for appropriate candidate points P_1 in the triangle OPQ. Here, the points P, Q are defined as before, namely, $P := (\mathrm{ch}_2(\iota_*F), H\,\mathrm{ch}_1(\iota_*F)) = (16(t-4)r, 16r)$, and $Q = (x_q, y_q)$ is the point such that x_q/y_q is the upper bound for $\nu^+_{BN}(\iota_*F)$, and $(x_p - x_q)/(y_p - y_q)$ is the lower bound for $\nu^-_{BN}(\iota_*F)$, given in Lemma 5.2.

(0) First, assume that $t \in \left[0, \frac{5-\sqrt{23}}{2}\right]$. In this case, the sheaf ι_*F is BN stable with BN slope $\nu_{BN}(\iota_*F) = t - 4 \in (-4, -3)$. Hence, we have

$$h^0(F)/r \leq \frac{16}{7} + \frac{16(t-4)}{49} = \frac{16(t+3)}{49}$$

by Lemma 3.8.

(1) Assume $t \in \left[\frac{5-\sqrt{23}}{2}, \frac{1}{8}\right)$. By Lemma 5.2, we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-4} \in (-1/3, -1/4)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{2t}{2t-1} \in (-1/3, -1/4)$,
- The slope of $\overrightarrow{PQ}$ is $\frac{2(t-8)}{-7(2t-9)} \in (-1/3, -1/4)$.

Hence, we can assume $P_1 = Q$ and get

$$h^0(F)/r \leq \left(\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP})\right)/r = \frac{16}{7} + \frac{16(t-4)}{49} = \frac{16(t+3)}{49}.$$

(2) Assume $t \in [1/8, 1/6)$. Then the slope bounds on $\nu_{BN}(\iota_*F)$ are the same as the case (1), but the slope of $\overrightarrow{OQ}$ is in the interval $(-1/2, -1/3]$. Hence, we may take P_1 to be Q or A, where A is the point on the line PQ with slope $-1/3$. The coordinates are given as

$$Q = ((2t-1)r, 2tr), \quad A = \left(\frac{48(2t^2-10t+1)}{-8t+15}r, \frac{16(2t^2-10t+1)}{8t-15}r\right).$$

Hence, we have

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = \frac{y_q}{5} + \frac{x_q}{25} + \frac{y_p - y_q}{7} + \frac{x_p - x_q}{49} = \frac{12t+24}{25}r,$$
$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) = \frac{y_a}{12} + \frac{y_p - y_a}{7} + \frac{x_p - x_a}{49} = \frac{1}{588}y_a + \frac{16(t+3)}{49}r.$$

As a function on $t \in [1/8, 1/6)$, we have

$$y_a(t) \leq \frac{458t - 47}{41}r,$$

and hence

$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) \leq \frac{85t}{246}r + \frac{481}{492}r.$$

We conclude that

$$h^0(F)/r \leq \max\left\{\frac{12t+24}{25}, \frac{85t}{246} + \frac{481}{492}\right\}.$$

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(3) Assume that $t \in [1/6, 1/4)$. Then the slopes of $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{QP}$ are same as in the case (1), but the slope of $\overrightarrow{OQ} \in (-1, -1/2]$ instead. There are three possibilities of the point P_1, namely, Q, A and B. Here, A, B are the points on the line PQ with slope $-1/3, -1/2$, respectively. The coordinates of Q, A are same as in the case (2), and

$$B = \left(\frac{32(2t^2 - 10t + 1)}{-10t + 31} r, \frac{16(2t^2 - 10t + 1)}{10t - 31} r \right).$$

We therefore get

$$\begin{aligned}
\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) &= \frac{y_q}{3} + \frac{x_q}{9} + \frac{y_p - y_q}{7} + \frac{x_p - x_q}{49} = \frac{8(t+1)}{9} r, \\
\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) &= \frac{y_a}{12} + \frac{y_p - y_a}{7} + \frac{x_p - x_a}{49} = \frac{1}{588} y_a + \frac{16(t+3)}{49} r, \\
\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) &= \frac{y_b}{8} + \frac{y_p - y_b}{7} + \frac{x_p - x_b}{49} = \frac{9}{392} y_b + \frac{16(t+3)}{49} r.
\end{aligned}$$

We can see that $\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}) \le \Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP})$. Furthermore, as a function on $t \in [1/6, 1/4)$, we have $y_b(t) \le \frac{100}{19} tr - \frac{31}{57} r$, and hence we have

$$\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) \le \frac{17}{38} tr + \frac{147}{152} r.$$

(4) Assume that $t \in [1/4, 1/2]$. In this case, the slope of $\overrightarrow{OQ}$ is bigger than or equal to -1. Hence, we may take $P_1 = Q, A, B$ or C, where A, B are defined as in (3), and C is the point on the line PQ with slope -1. We have

$$C = \left(-\frac{16(2t^2 - 10t + 1)}{12t - 47} r, \frac{16(2t^2 - 10t + 1)}{12t - 47} r \right).$$

We have

$$\begin{aligned}
\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) &= y_q + x_q + \frac{y_p - y_q}{7} + \frac{x_p - x_q}{49} = 4tr, \\
\Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) &= \frac{y_c}{4} + \frac{y_p - y_c}{7} + \frac{x_p - x_c}{49} = \frac{25}{196} y_c + \frac{16(t+3)}{49} r,
\end{aligned}$$

and $\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}), \Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{CB})$ are as in (3). Hence, we can show that

$$\Omega(\overrightarrow{OA}) + \Omega(\overrightarrow{AP}), \Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}) \le \Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}).$$

On the other hand, as a function on $t \in [1/4, 1/2]$, we have $y_c(t) \le \frac{142}{41} tr - \frac{15}{41} r$, and so

$$\Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) \le \frac{63}{82} tr + \frac{153}{164} r.$$

(5) Assume that $t \in [3/2, 15/8]$. In this case, we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-4} \in (-1/2, -1/3)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{2t}{2t-1} > 0$,
- The slope of $\overrightarrow{PQ}$ is $\frac{2(t-8)}{-7(2t-9)} \in [-1/3, -1/4)$.

Hence, we may choose P_1 as Q, B, C, D, E, F, where B, C are defined as in (4), D is the point on the line OQ such that the slope of DP is $-1/3$ and E (resp. F) are the intersection points of the lines DP and

OB (resp. OC). Hence, the computations of $\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}), \Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP})$, and $\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP})$ are exactly the same as in (4), and we can see that

$$\Omega(\overrightarrow{OB}) + \Omega(\overrightarrow{BP}), \Omega(\overrightarrow{OC}) + \Omega(\overrightarrow{CP}) \leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}).$$

For D, E and F, the coordinates are given as

$$D = \left(\frac{16(2t-1)(t-1)}{8t-1}r, \frac{32t(t-1)}{8t-1}r\right), \quad E = (-(32t-32)r, (16t-16)r),$$
$$F = (-(8t-8)r, (8t-8)r).$$

We get

$$\Omega(\overrightarrow{OD}) + \Omega(\overrightarrow{DP}) = y_d + x_d + \frac{y_p - y_d}{12},$$
$$\Omega(\overrightarrow{OE}) + \Omega(\overrightarrow{EP}) = \frac{y_e}{8} + \frac{y_p - y_e}{12} = \frac{2}{3}(t+1)r,$$
$$\Omega(\overrightarrow{OF}) + \Omega(\overrightarrow{FP}) = \frac{y_f}{4} + \frac{y_p - y_f}{12} = \frac{4}{3}tr.$$

We can also see that $\Omega(\overrightarrow{OD}) + \Omega(\overrightarrow{DP}) \leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = 4tr$, and hence we conclude that

$$h^0(F)/r \leq 4t.$$

(6) Assume $t \in \left(15/8, \frac{\sqrt{23}-1}{2}\right]$. The only difference with (5) is that the slope of $\overrightarrow{QP}$ is equal to $-1/3$ in the present case. Hence, we may choose P_1 to be Q, E, or F appeared in (5). It is easy to see that

$$\Omega(\overrightarrow{OE}) + \Omega(\overrightarrow{EP}) = \Omega(\overrightarrow{OF}) + \Omega(\overrightarrow{FP}) \leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}),$$
$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = \frac{4(46t^2 - 50t + 11)}{3(8t-1)}r \leq \frac{133t - 114}{18}r.$$

(7) Assume that $t \in \left[\frac{\sqrt{23}-1}{2}, \frac{31}{16}\right]$. Then we have

- The slope of $\overrightarrow{OP}$ is $\frac{1}{t-4} \in (-1/2, -1/3)$,
- The slope of $\overrightarrow{OQ}$ is $\frac{12t-11t}{13t-16} > 0$,
- The slope of $\overrightarrow{PQ}$ is $-1/3$.

Hence we may choose P_1 to be Q, E or F, where the points E, F are defined as in (5). We have

$$Q = \left(\frac{16(13t-16)}{49}r, \frac{16(12t-11)}{49}r\right)$$

and hence

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = y_q + x_q + \frac{y_p - y_q}{12} = \frac{236}{49}tr - \frac{148}{21}r.$$

On the other hand, from the computations in (5), we see that

$$\Omega(\overrightarrow{OE}) + \Omega(\overrightarrow{EP}) = \Omega(\overrightarrow{OF}) + \Omega(\overrightarrow{FP}) = \frac{4}{3}tr \leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}).$$

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We conclude that

$$h^0(F)/r \leq \frac{236}{49}t - \frac{148}{21}.$$

(8) Assume $t \in [31/16, 2)$. Then we have

- ◦ The slope of $\overrightarrow{OP}$ is $\frac{1}{t-4} \in (-1/2, -1/3)$,
- ◦ The slope of $\overrightarrow{OQ}$ is $\frac{1}{4t-7} > 0$,
- ◦ The slope of $\overrightarrow{PQ}$ is $-1/3$.

Hence, we may choose P_1 to be Q, E or F, where the points E, F are defined as in (5). We have

$$Q = (4(4t-7)r, 4r)$$

and hence

$$\Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}) = y_q + x_q + \frac{y_p - y_q}{12} = (16t - 23)r.$$

As in (7), we see that

$$\Omega(\overrightarrow{OE}) + \Omega(\overrightarrow{EP}) = \Omega(\overrightarrow{OF}) + \Omega(\overrightarrow{FP}) = \frac{4}{3}tr \leq \Omega(\overrightarrow{OQ}) + \Omega(\overrightarrow{QP}),$$

and we can conclude that

$$h^0(F)/r \leq 16t - 23.$$

□

5.2. *Strong (classical) BG inequality*

Using Proposition 5.3, we get the following (classical) BG type inequality on a double cover CY3 X.

Theorem 5.4. *Let X be a double cover CY3. Let $F \in D^b(X)$ be a $\nu_{\alpha,0}$-semistable object for some $\alpha > 0$, with $\mu_H(E) \in (-1, 1)$. Then we have the inequality*

$$\frac{H\,\mathrm{ch}_2(F)}{H^3\,\mathrm{ch}_0(F)} \leq \Xi\left(\left|\frac{H^2\,\mathrm{ch}_1(F)}{H^3\,\mathrm{ch}_0(F)}\right|\right).$$

Here, the function Ξ is defined as in Theorem 4.1, i.e.,

$$\Xi(t) = \begin{cases} t^2 - t & (t \in [0, 1/4]) \\ 3t/4 - 3/8 & (t \in [1/4, 1/2]) \\ t/4 - 1/8 & (t \in [1/2, 3/4]) \\ t^2 - 1/2 & (t \in [3/4, 1]). \end{cases}$$

Proof. As in the proof of Theorem 4.1, the problem is reduced to proving the same statement for tilt-semistable objects on T. Assume that there exists a tilt semistable object F on T violating the inequality in the statement. As before, we may assume that $\mu(F) \in (0, 1/2]$ and $F|_C$ is slope semistable. Then we have (cf. equation (10))

$$\begin{aligned} \mathrm{ch}_2(F) - H_T\,\mathrm{ch}_1(F) + 10\,\mathrm{ch}_0(F) &= \chi(F) \\ &\leq h^0(F|_C) + h^0(F^\vee(2H_T)|_C), \end{aligned} \tag{12}$$

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and

$$\begin{aligned}\mathrm{ch}(F|_C) &= (\mathrm{ch}_0(F), 4H\,\mathrm{ch}_1(F)),\\ \mathrm{ch}(F^\vee(2H)|_C) &= (\mathrm{ch}_0(F), 4(2H^2\,\mathrm{ch}_0(F) - H\,\mathrm{ch}_1(F))).\end{aligned}$$

By applying Proposition 5.3 to the right-hand side of the inequality (12), we have

$$\frac{\mathrm{ch}_2(F)}{H^2\,\mathrm{ch}_0(F)} \leq \begin{cases} -\frac{143}{49}\mu_H(F) - \frac{23}{98} & (\mu_H(F) \in (0, 1/16]) \\ -\frac{6}{49}\mu_H(F) - \frac{1081}{588} & (\mu_H(F) \in [1/16, \frac{5-\sqrt{23}}{2}]) \\ -\frac{2701}{3528}\mu_H(F) - \frac{127}{882} & (\mu_H(F) \in [\frac{5-\sqrt{23}}{2}, 1/8)) \\ \frac{85}{984}\mu_H(F) - \frac{503}{1968} & (\mu_H(F) \in [1/8, 217/1654]) \\ \frac{3}{25}\mu_H(F) - \frac{13}{50} & (\mu_H(F) \in [217/1654, 1/6)) \\ \frac{17}{152}\mu_H(F) - \frac{157}{608} & (\mu_H(F) \in [1/6, 107/604]) \\ \frac{2}{9}\mu_H(F) - \frac{5}{18} & (\mu_H(F) \in [107/604, 1/4)) \\ \frac{63}{328}\mu_H(F) - \frac{175}{656} & (\mu_H(F) \in [1/4, 153/530]) \\ \mu_H(F) - \frac{1}{2} & (\mu_H(F) \in [153/530, 1/2]), \end{cases}$$

which is a cotradiction. □

6. BG type inequality conjecture

In this section, we will prove that the strong BG inequality in Theorem 1.1 implies Theorem 1.2. We work in the following general set up. Let X be a smooth projective Calabi–Yau threefold, H a nef and big divisor on X. Let us put $d := H^3$, $e := H.\,\mathrm{td}_{X,2}$. We define the positive real number $\delta_X = \delta_X(H)$ as follows:

$$\delta_X := \max\left\{\frac{4}{d}, \frac{e}{d}, \frac{26}{3d} - \frac{e}{d} - \frac{1}{3}, \frac{16-3e}{3d}\right\}.$$

Note that we always have $\frac{57-7e}{13d} < \max\{\frac{4}{d}, \frac{16-3e}{3d}\} \leq \delta_X$.

It is easy to compute the number δ_X in the following cases:

- When X is a triple cover CY3, we have $\delta_X = 25/18$.
- When X is a double cover CY3, we have $\delta_X = 13/6$.

Remark 6.1. As pointed out by the referee, there exist some inequalities between d and e; see [KW14, Proposition 2.2]. For example, if H is very ample, then the inequality $3e \leq d + 9$ holds by [KW14, Proposition 2.2 (1)]. This implies

$$\frac{26}{3d} - \frac{e}{d} - \frac{1}{3} \geq \frac{17}{3d} > \frac{4}{d}$$

and hence the definition of δ_X simplifies.

We put the following assumption:

Assumption 6.2. *Every object $E \in D^b(X)$, which is $\nu_{0,\alpha}$-semistable for some $\alpha > 0$, satisfies the strong BG inequality (2).*

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Proposition 6.3. *Let X be a smooth projective Calabi–Yau threefold, H a nef and big divisor on X. For a real number $\delta \geq \delta_X$, define a 1-cycle Γ as $\Gamma := \delta H^2 - \mathrm{td}_{X,2}$. Assume that Assumption 6.2 holds. Then every BN stable object $E \in D^b(X)$ with $\nu_{BN}(E) \in [0, 1/2]$ satisfies the inequality*

$$Q^\Gamma(E) := Q^\Gamma_{0,0}(E) = 2(H\,\mathrm{ch}_2(E))(2H\,\mathrm{ch}_2(E) - 3\Gamma H\,\mathrm{ch}_0(E)) \\ - 6(H^2\,\mathrm{ch}_1(E))(\mathrm{ch}_3(E) - \Gamma\,\mathrm{ch}_1(E)) \geq 0.$$

Proof. First, assume that $\nu_{BN}(E) \in (0, 1/2]$. Let us consider the universal extension

$$E \to \widetilde{E} \to \mathrm{Hom}(\mathcal{O}_X, E) \otimes \mathcal{O}_X[1].$$

By [Li19a, Lemma 2.12], $\widetilde{E}$ is BN semistable with $\nu_{BN}(\widetilde{E}) = \nu_{BN}(E)$. By Assumption 6.2, we can see that $\mu_H(\widetilde{E}) \notin (-1/4, 0]$. Indeed, if otherwise, we have

$$\frac{H\,\mathrm{ch}_2(\widetilde{E})}{H^3\,\mathrm{ch}_0(\widetilde{E})} \leq \mu_H(\widetilde{E})^2 + \mu_H(\widetilde{E}) < \frac{1}{2}\mu_H(\widetilde{E}).$$

Dividing both sides by $\mu_H(\widetilde{E})(< 0)$, we get $\nu_{BN}(\widetilde{E}) > 1/2$, a contradiction. When $\mu_H(\widetilde{E}) \in [-1/2, -1/4]$, using Assumption 6.2 to the object $\widetilde{E}$, we have

$$\frac{H\,\mathrm{ch}_2(\widetilde{E})}{H^3\,\mathrm{ch}_0(\widetilde{E})} \leq -\frac{3}{4}\mu_H(\widetilde{E}) - \frac{3}{8}. \tag{13}$$

Note that we have $\mathrm{ch}_0(\widetilde{E}) = \mathrm{ch}_0(E) - \hom(\mathcal{O}_X, E)$ and $\mathrm{ch}_i(\widetilde{E}) = \mathrm{ch}_i(E)$ for $i = 1, 2$. Note also that we have $H^2\,\mathrm{ch}_1(E) \geq 0$ since $E \in \mathrm{Coh}^0(X)$. Together with the assumption $\mu_H(\widetilde{E}) < 0$, we have $\mathrm{ch}_0(\widetilde{E}) < 0$. From these observations, the inequality (13) is equivalent to the inequality

$$\hom(\mathcal{O}_X, E) \leq \frac{8}{3d}H\,\mathrm{ch}_2(E) + \frac{2}{d}H^2\,\mathrm{ch}_1(E) + \mathrm{ch}_0(E). \tag{14}$$

On the other hand, the inequality $\mu_H(\widetilde{E}) < -1/2$ is equivalent to

$$\hom(\mathcal{O}_X, E) \leq \frac{2}{d}H^2\,\mathrm{ch}_1(E) + \mathrm{ch}_0(E),$$

which is stronger than equation (14) since we have $H\,\mathrm{ch}_2(E) > 0$ by our assumption $\nu_{BN}(E) > 0$. If $\mu_H(\widetilde{E}) > 0$, the same inequality (14) obviously holds since $\mathrm{ch}_0(E) - \hom(\mathcal{O}_X, E) = \mathrm{ch}_0(\widetilde{E}) > 0$ and $H^2\,\mathrm{ch}_1(E) > 0$. Hence, the inequality (14) always holds.

On the other hand, by using the BN stability of E and $\mathcal{O}_X[1]$, we have

$$\hom(\mathcal{O}_X, E) \geq \chi(E) = \mathrm{ch}_3(E) + \mathrm{td}_{X,2}\,\mathrm{ch}_1(E). \tag{15}$$

Combining the inequalities (14) and (15), we get

$$\mathrm{ch}_3(E) + \mathrm{td}_{X,2}\,\mathrm{ch}_1(E) \leq \frac{1}{d}H^3\,\mathrm{ch}_0(E) + \frac{2}{d}H^2\,\mathrm{ch}_1(E) + \frac{8}{3d}H\,\mathrm{ch}_2(E),$$

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and hence

$$\begin{aligned}Q^{\Gamma}(E) &\geq 4(H\operatorname{ch}_2(E))^2 - 6\cdot\frac{\delta d-e}{d}H\operatorname{ch}_2(E)H^3\operatorname{ch}_0(E) + 6\delta\Big(H^2\operatorname{ch}_1(E)\Big)^2 \\ &\quad - 6H^2\operatorname{ch}_1(E)\left(\frac{1}{d}H^3\operatorname{ch}_0(E) + \frac{2}{d}H^2\operatorname{ch}_1(E) + \frac{8}{3d}H\operatorname{ch}_2(E)\right) \\ &= 4b^2 - \frac{16}{d}ab + 6\left(\delta - \frac{2}{d}\right)a^2 - 6\left(\delta - \frac{e}{d}\right)rb - \frac{6}{d}ra.\end{aligned} \tag{16}$$

Here, we put $(r, a, b) := (H^3\operatorname{ch}_0(E), H^2\operatorname{ch}_1(E), \operatorname{ch}_2(E))$, to simplify the notation. Note that we have $a \geq 0$ since $E \in \operatorname{Coh}^0(X)$. Moreover, we also have $b \geq 0$ from the assumption $\nu_{BN}(E) \geq 0$. Note also that by definition of δ_X, we have $\delta - 2/d, \delta - e/d \geq 0$.

By Assumption 6.2, we know that $\mu_H(E) \notin [0, 1/2]$. When $\mu_H(E) \notin [1/2, 1]$, we have $r < a$. Together with the inequality (16), we have

$$\begin{aligned}Q^{\Gamma}(E) &\geq 4b^2 - \frac{16}{d}ab + 6\left(\delta - \frac{2}{d}\right)a^2 - 6\left(\delta - \frac{e}{d}\right)ab - \frac{6}{d}a^2 \\ &= A_1a^2 - B_1ab + 4b^2,\end{aligned}$$

where we put $A_1 := 6(\delta - 3/d), B_1 := 6(\delta - e/d) + 16/d > 0$. We can further compute as

$$A_1a^2 - B_1ab + 4b^2 = (a-2b)(A_1a + (2A_1 - B_1)b) + (4A_1 - 2B_1 + 4)b^2. \tag{17}$$

By the assumption $0 < \nu_{BN}(E) \leq 1/2$, we have $0 < 2b \leq a$. Moreover, we also have $\delta \geq \delta_X \geq \frac{26}{3d} - \frac{e}{d} - \frac{1}{3}$ by definition. From these we can conclude that the right-hand side of the equality (17) is nonnegative, and hence we have $Q^{\Gamma}(E) \geq 0$ as required.

When $\mu_H(E) \in [1/2, 3/4]$, by Assumption 6.2, we have

$$-r \geq -2a + 8b.$$

Combining with the inequality (16), we have

$$\begin{aligned}Q^{\Gamma}(E) &\geq 4b^2 - \frac{16}{d}ab + 6\left(\delta - \frac{2}{d}\right)a^2 + 6\left(\delta - \frac{e}{d}\right)(-2a+8b)b + \frac{6}{d}(-2a+8b)a \\ &= \left(4 + 48\left(\delta - \frac{e}{d}\right)\right)b^2 + \left(\frac{32}{d} - 12\left(\delta - \frac{e}{d}\right)\right)ab + 6\left(\delta - \frac{e}{d}\right)a^2 \\ &=: C_2b^2 - B_2ab + A_2a^2 \\ &= (a-2b)(A_2a + (2A_2 - B_2)b) + (4A_2 - 2B_2 + C_2)b^2,\end{aligned}$$

where the real numbers A_2, B_2, C_2 are defined so that the second equality holds. Since $\delta \geq 4/d, 3/d$, we can see that $4A_2 - 2B_2 + C_2 \geq 0$. Moreover, using the inequalities $0 < 2b \leq a$ and $\delta \geq -e/d + 16/3d$, we also obtain $A_2a + (2A_2 - B_2)b \geq 0$. Hence, we have $Q^{\Gamma}(E) \geq 0$.

Next, consider the case when $\mu(E) \in [3/4, 1]$. By Assumption 6.2, we have $b/r \leq 7a/4r - 5/4$, equivalently,

$$-r \geq -\frac{7}{5}a + \frac{4}{5}b.$$

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Together with the inequality (16), we have

$$\begin{aligned}
5Q^\Gamma(E) &\geq 20b^2 - \frac{80}{d}ab + 30\left(\delta - \frac{2}{d}\right)a^2 \\
&\quad + 6\left(\delta - \frac{e}{d}\right)b(-7a+4b) + \frac{6}{d}a(-7a+4b) \\
&= \left(20 + 24\left(\delta - \frac{e}{d}\right)\right)b^2 - \left(42\left(\delta - \frac{e}{d}\right) - \frac{66}{d}\right)ab + \left(30\delta - \frac{102}{d}\right)a^2 \\
&=: C_3b^2 - B_3ab + A_3a^2 \\
&= (a-2b)(A_3a + (2A_3 - B_3)b) + (4A_3 - 2B_3 + C_3)b^2.
\end{aligned}$$

Using the inequalities $a \geq 2b$ and $\delta \geq -e/d + 46/10d - 1/3, (57-7e)/13d$, we can show that $Q^\Gamma(E) \geq 0$.

The remaining case is when $\nu_{BN}(E) = 0$. The issue is that we do not know whether $\widetilde{E}$ is BN semistable or not. If it is $\nu_{\alpha,0}$-semistable for some $\alpha > 0$, as in the case of $\nu_{BN}(E) > 0$, we have the inequality

$$\hom(\mathcal{O}_X, E) \leq \frac{2}{d}H^2 \operatorname{ch}_1(E) + \operatorname{ch}_0(E). \tag{18}$$

Assume that $\widetilde{E}$ is $\nu_{\alpha,0}$-unstable for all $\alpha > 0$. Then by the proof of [Li19a, Proposition 3.3], for each $0 < \delta \ll 1$, there exists $\alpha_i > 0$ and a filtration of E such that each factor E_i is $\nu_{\alpha_i,0}$-semistable with $\nu_{BN}(E_i) < \delta$. By Assumption 6.2, we must have

$$\mu_H(E_i) \notin \left[-\frac{3}{8\delta + 6}, 0\right].$$

Taking a limit $\delta \to +0$, we get $\mu(\widetilde{E}) \notin [-1/2, 0]$, hence the inequality (18) holds. Furthermore, by using the derived dual (cf. proof of [Li19a, Proposition 3.3]), we also have

$$\hom(\mathcal{O}_X, E[2]) \leq \frac{2}{3}H^2 \operatorname{ch}_1(E) - \operatorname{ch}_0(E).$$

Hence, we get

$$\begin{aligned}
\operatorname{ch}_3(E) + \operatorname{td}_{X,2}\operatorname{ch}_1(E) = \chi(E) &\leq \hom(\mathcal{O}_X, E) + \hom(\mathcal{O}_X, E[2]) \\
&\leq \frac{4}{d}H^2 \operatorname{ch}_1(E),
\end{aligned}$$

from which we deduce $Q^\Gamma(E) \geq 0$, as we assume $\delta \geq 4/d$. □

Corollary 6.4. *Let X be a triple (resp. double) CY3. We put $\gamma := 2/9$ (resp. $1/3$) and $\Gamma := \gamma H^2$. Let E be a BN stable object on X with $\nu_{BN}(E) \in [0, 1/2]$. Then we have*

$$\begin{aligned}
Q^\Gamma(E) := Q^\Gamma_{0,0}(E) &= 2(H\operatorname{ch}_2(E))(2H\operatorname{ch}_2(E) - 3\Gamma H\operatorname{ch}_0(E)) \\
&\quad - 6(H^2\operatorname{ch}_1(E))(\operatorname{ch}_3(E) - \Gamma\operatorname{ch}_1(E)) \geq 0.
\end{aligned}$$

Proof. By Theorems 4.1 and 5.4, a triple/double CY3 satisfies Assumption 6.2. Furthermore, we can take the 1-cycle Γ to be $\Gamma := \delta_X H^2 - \operatorname{td}_{X,2} = \gamma H^2$. □

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7. Construction of Bridgeland stability conditions

The goal of this section is to prove Theorem 1.3 in the introduction. First, let us recall the definition of Bridgeland stability condition.

Definition 7.1 ([Bri07]). Let $\mathcal{D}$ be a triangulated category. Fix a lattice Λ of finite rank and a group homomorphism $\mathrm{cl}\colon K(\mathcal{D}) \to \Lambda$.

A *stability condition* on $\mathcal{D}$ (with respect to (Λ, cl)) is a pair $(Z, \mathcal{A})$ consisting of a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ and the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ satisfying the following axioms.

1. We have $Z \circ \mathrm{cl}(\mathcal{A} \setminus \{0\}) \subset \mathbb{H} \cup \mathbb{R}_{<0}$, where $\mathbb{H}$ is the upper half plane.
2. Every nonzero object in the heart $\mathcal{A}$ has a Harder–Narasimhan filtration with respect to μ_Z-stability. Here, we define a Z-slope function μ_Z as

$$\mu_Z := -\frac{\Re Z}{\Im Z}\colon \mathcal{A} \to \mathbb{R} \cup \{+\infty\},$$

and define μ_Z-stability on the abelian category $\mathcal{A}$ in a usual way.
3. There exists a quadratic form q on Λ satisfying the following conditions.
   ◦ q is negative definite on the kernel of Z,
   ◦ For every μ_Z-semistable object $E \in \mathcal{A}$, we have $q(\mathrm{cl}(E)) \geq 0$.

The group homomorphism Z is called a *central charge*, and the axiom (3) is called the *support property*.

Let $\mathrm{Stab}_\Lambda(\mathcal{D})$ be a set of stability conditions on $\mathcal{D}$ with respect to (Λ, cl). Then the set $\mathrm{Stab}_\Lambda(\mathcal{D})$ has a structure of a complex manifold [Bri07]. Moreover, there is an action of the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$, where $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is the universal covering of the group

$$\mathrm{GL}^+(2,\mathbb{R}) := \{g \in \mathrm{GL}(2,\mathbb{R}) : \det(g) > 0\}.$$

Let us consider the case when $\mathcal{D} = D^b(X)$, where X is a double/triple cover CY3. In this case, we fix a lattice Λ to be the image of the morphism

$$\mathrm{cl} := \left(H^3 \mathrm{ch}_0, H^2 \mathrm{ch}_1, H \mathrm{ch}_2, \mathrm{ch}_3\right)\colon K(X) \to H^{2*}(X, \mathbb{Q}).$$

We simply denote as $\mathrm{Stab}(X) := \mathrm{Stab}_\Lambda(D^b(X))$. Following [BMS16, BMT14], we explain an explicit construction of stability conditions on $D^b(X)$. Let us recall several notions from [BMS16]. Fix real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$.

The heart corresponding to a stability condition is constructed as a tilt of $\mathrm{Coh}^\beta(X)$. Let us define a slope function $\nu'_{\beta,\alpha}$ on $\mathrm{Coh}^\beta(X)$ as

$$\nu'_{\beta,\alpha} := \frac{H \mathrm{ch}_2^\beta - \frac{1}{2}\alpha^2 H^3 \mathrm{ch}_0^\beta}{H^2 \mathrm{ch}_1^\beta}\colon \mathrm{Coh}^\beta(X) \to \mathbb{R} \cup \{+\infty\}.$$

Compared with the function $\nu_{\beta,\alpha}$ defined in Section 2, we have

$$\nu'_{\beta,\alpha} = \nu_{\beta, \frac{1}{2}(\beta^2+\alpha^2)} - \beta. \tag{19}$$

We define full subcategories $\mathcal{T}'_{\beta,\alpha}, \mathcal{F}'_{\beta,\alpha}$ of $\mathrm{Coh}^\beta(X)$ as

$$\mathcal{T}'_{\beta,\alpha} := \left\langle T \in \mathrm{Coh}^\beta(X) : T \text{ is } \nu'_{\beta,\alpha}\text{-semistable with } \nu'_{\beta,\alpha}(E) > 0\right\rangle,$$
$$\mathcal{F}'_{\beta,\alpha} := \left\langle F \in \mathrm{Coh}^\beta(X) : F \text{ is } \nu'_{\beta,\alpha}\text{-semistable with } \nu'_{\beta,\alpha}(E) \leq 0\right\rangle.$$

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Here, $\nu'_{\beta,\alpha}$-stability is same as $\nu_{\beta,\frac{1}{2}(\beta^2+\alpha^2)}$-stability, and $\langle - \rangle$ denotes the extension closure in the abelian category $\mathrm{Coh}^{\beta}(X)$. We now define the *double-tilted heart* as

$$\mathcal{A}^{\beta,\alpha} := \left\langle \mathcal{F}'_{\beta,\alpha}[1], \mathcal{T}'_{\beta,\alpha} \right\rangle \subset D^b(X).$$

We also define a central charge function $Z^{a,b}_{\beta,\alpha} \colon \Lambda \to \mathbb{C}$ as

$$Z^{a,b}_{\alpha,\beta} := -\operatorname{ch}_3^{\beta} + bH\operatorname{ch}_2^{\beta} + aH^2\operatorname{ch}_1^{\beta} + i\left(H\operatorname{ch}_2^{\beta} - \frac{1}{2}\alpha^2 H^3\operatorname{ch}_0^{\beta}\right)$$

for real numbers $a, b \in \mathbb{R}$.

Finally, for a real number $\gamma > 0$, define U_γ to be a set of vectors $(\alpha, \beta, a, b) \in \mathbb{R}^4$ satisfying

$$\alpha > 0, \quad \alpha^2 + \left(\beta - \lfloor\beta\rfloor - \frac{1}{2}\right)^2 > \frac{1}{4}, \quad a > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha + \gamma. \tag{20}$$

Theorem 7.2 (cf. [BMS16, Proposition 8.10]). *Let X be a triple (resp. double) cover CY3, and put $\gamma := 2/9$ (resp. 1/3). Then there exists an injective continuous map*

$$U_\gamma \hookrightarrow \mathrm{Stab}(X), \quad (\alpha, \beta, a, b) \mapsto \left(Z^{a,b}_{\beta,\alpha}, \mathcal{A}_{\beta,\alpha}\right).$$

Furthermore, the orbit $\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cdot U_\gamma$ forms an open subset in the space $\mathrm{Stab}(X)$ of stability conditions.

We divide the proof of the above theorem into several steps. The arguments below are essentially the same as that in [BMS16, Section 8].

Proposition 7.3 (cf. [BMS16, Theorem 8.6]). *For every element $(\alpha, \beta, a, b) \in U_\gamma$ with $\alpha, \beta \in \mathbb{Q}$, the pair $\left(Z^{a,b}_{\beta,\alpha}, \mathcal{A}_{\beta,\alpha}\right)$ satisfies axioms (1) and (2) in Definition 7.1.*

Proof. First, we check the axiom (1) in Definition 7.1 for the pair $\left(Z^{a,b}_{\beta,\alpha}, \mathcal{A}_{\beta,\alpha}\right)$. As in the proof of [BMS16, Theorem 8.6], it is enough to show the inequality $Z^{a,b}_{\beta,\alpha}(F[1]) < 0$ for every $\nu'_{\beta,\alpha}$-semistable object F with $\nu'_{\beta,\alpha}(F) = 0$. By the inequality $\alpha^2 + \left(\beta - \lfloor\beta\rfloor - \frac{1}{2}\right)^2 > \frac{1}{4}$ in equation (20), we can apply Theorem 1.2 to the object F. Noting the equation (19), we get

$$\operatorname{ch}_3^{\beta}(F) \leq \left(\gamma + \frac{1}{6}\alpha^2\right) H^2 \operatorname{ch}_1^{\beta}(F). \tag{21}$$

Furthermore, together with the assumption $H\operatorname{ch}_2^{\beta}(F) = \frac{1}{2}\alpha^2 H^3 \operatorname{ch}_0^{\beta}$, the classical BG inequality (cf. [BMS16, Theorem 3.5]) $\overline{\Delta}_H(F) \geq 0$ gives the inequality

$$\left(H\operatorname{ch}_2^{\beta}(F)\right)^2 \leq \frac{1}{4}\alpha^2\left(H^2\operatorname{ch}_1^{\beta}(F)\right)^2. \tag{22}$$

By the inequalities (21) and (22), we obtain

$$\begin{aligned} Z^{a,b}_{\beta,\alpha}(F[1]) &= \Re Z^{a,b}_{\beta,\alpha}(F[1]) \\ &\leq \left(\gamma + \frac{1}{6}\alpha^2\right) H^2 \operatorname{ch}_1^{\beta}(F) + \frac{1}{2}|b|\alpha H^2 \operatorname{ch}_1^{\beta}(F) - aH^2\operatorname{ch}_1^{\beta}(F) < 0. \end{aligned}$$

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Now the axiom (2) is also satisfied since we assume $\alpha, \beta \in \mathbb{Q}$ (see the proof of [BMS16, Theorem 8.6] for the detail). □

Next, we discuss the support property. Let us put

$$\overline{\nabla}_H^{\alpha,\beta,\gamma}(E) := 3\gamma\alpha^2\left(H^3\operatorname{ch}_0^\beta(E)\right)^2 + 2\left(H\operatorname{ch}_2^\beta(E)\right)\left(2H\operatorname{ch}_2^\beta(E) - 3\gamma H^3\operatorname{ch}_0^\beta(E)\right) \\ - 6\left(H^2\operatorname{ch}_1^\beta(E)\right)\left(\operatorname{ch}_3^\beta(E) - \gamma H^2\operatorname{ch}_1^\beta(E)\right)$$

for an object $E \in D^b(X)$.

Proposition 7.4 (cf. [BMS16, Lemmas 8.5, 8.8]). *Fix an element $(\alpha, \beta, a, b) \in U_\gamma$. Then there exists an interval $I_{\alpha,\gamma}^{a,b} \subset \mathbb{R}$ such that for every $K \in I_{\alpha,\gamma}^{a,b}$, the quadratic form $Q_K^{\alpha,\beta,\gamma} := K\overline{\Delta}_H + \overline{\nabla}_H^{\alpha,\beta,\gamma}$ is negative definite on the kernel of the central charge function $Z_{\beta,\alpha}^{a,b}$.*

Furthermore, if we assume $(\alpha, \beta) \in \mathbb{Q}$, then every $Z_{\beta,\alpha}^{a,b}$-semistable object E satisfies the inequality $Q_K^{\alpha,\beta,\gamma}(E) \geq 0$.

Proof. Let us prove the first assertion. The vectors $\left(1, 0, \frac{1}{2}\alpha^2, \frac{1}{2}b\alpha\right)$ and $(0, 1, 0, a)$ forms the basis the kernel of $Z_{\beta,\alpha}^{a,b}$. With respect to this basis, the quadratic form $K\overline{\Delta}_H + \overline{\nabla}_H^{\alpha,\beta,\gamma}$ is represented by the matrix

$$\begin{pmatrix} -\alpha^2 K + \alpha^4 & -\frac{3}{2}b\alpha^2 \\ -\frac{3}{2}b\alpha^2 & K - 6(a-\gamma) \end{pmatrix}. \tag{23}$$

When $b = 0$, the matrix (23) is negative definite if and only if $K \in \left(\alpha^2, 6(a-\gamma)\right) =: I_{\alpha,\gamma}^{a,b=0}$. This interval is nonempty by equation (20).

When $b \neq 0$, we also need to require the determinant of the matrix (23) to be positive, i.e.,

$$\alpha^2\left(-K^2 + \left(6(a-\gamma) + \alpha^2\right)K - 6(a-\gamma)\alpha^2 - \frac{9}{4}b^2\alpha^2\right) > 0. \tag{24}$$

The solution space $K \in I_{\alpha,\gamma}^{a,b}$ of the inequality (24) forms a nonempty open interval since we have, by equation (20),

$$a - \gamma > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha.$$

We can prove the second assertion as in [BMS16, Lemma 8.8], using the BG type inequality obtained in Theorem 1.2. □

Finally we are able to prove Theorem 7.2.

Proof of Theorem 7.2. By Propositions 7.3 and 7.4, the pair $\left(Z_{\beta,\alpha}^{a,b}, \mathcal{A}^{\beta,\alpha}\right)$ is a stability condition on $D^b(X)$ for every element $(\alpha, \beta, a, b) \in U_\gamma$ with $\alpha, \beta \in \mathbb{Q}$. We can deform them to the real parameters (α, β) by the support property in Proposition 7.4. See [BMS16, Proposition 8.10] for the precise proof. □

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