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Symmetrisation and the Feigin–Frenkel centre

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Abstract

For complex simple Lie algebras of types B, C, and D, we provide new explicit formulas for the generators of the commutative subalgebra $\mathfrak{z}(\hat{\mathfrak{g}}) \subset \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ known as the *Feigin–Frenkel centre*. These formulas make use of the symmetrisation map as well as of some well-chosen symmetric invariants of $\mathfrak{g}$. There are some general results on the rôle of the symmetrisation map in the explicit description of the Feigin–Frenkel centre. Our method reduces questions about elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ to questions on the structure of the symmetric invariants in a type-free way. As an illustration, we deal with type G_2 by hand. One of our technical tools is the map $\mathsf{m}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$ introduced here. As the results show, a better understanding of this map will lead to a better understanding of $\mathfrak{z}(\hat{\mathfrak{g}})$.

Introduction

Let G be a complex reductive group. Set $\mathfrak{g} = \operatorname{Lie} G$. As is well known, the algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ of *symmetric* $\mathfrak{g}$-invariants and the centre $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ are polynomial algebras with $\operatorname{rk}\mathfrak{g}$ generators. Therefore there are several isomorphisms between them. Two of these isomorphisms can be distinguished, the one given by the *symmetrisation map*, which is a homomorphism of $\mathfrak{g}$-modules, and the Duflo isomorphism, which is a homomorphism of algebras. Both of them exist for any finite-dimensional complex Lie algebra.

The symmetrisation map is defined in the infinite-dimensional case as well. However, no analogue of the Duflo isomorphism for Lie algebras $\mathfrak{q}$ with $\dim \mathfrak{q} = \infty$ is known. Furthermore, one may need to complete $\mathcal{U}(\mathfrak{q})$ in order to replace $\mathcal{Z}(\mathfrak{q})$ with an interesting related object, see e.g. [Kac84]. In this paper, we are dealing with the most notable class of infinite-dimensional Lie algebras, namely affine Kac–Moody algebras $\hat{\mathfrak{g}}$, and the related centres at the critical level.

The Feigin–Frenkel centre $\mathfrak{z}(\hat{\mathfrak{g}})$ is a remarkable commutative subalgebra of the enveloping algebra $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$. The central elements of the completed enveloping algebra $\tilde{\mathcal{U}}_\kappa(\hat{\mathfrak{g}})$ at the critical level $\kappa = -\mathrm{h}^\vee$ can be obtained from the elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ by employing the vertex algebra structure [Fre07, § 4.3.2]. The structure of $\mathfrak{z}(\hat{\mathfrak{g}})$ is described by a theorem of Feigin and Frenkel [FF92], hence the name. This algebra provides a quantisation of the local Hitchin system [BD, § 2]. Elements $S \in \mathfrak{z}(\hat{\mathfrak{g}})$ give rise to higher Hamiltonians of the Gaudin model, which describes a completely integrable quantum spin chain [FFR94].

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The classical counterpart of $\mathfrak{z}(\hat{\mathfrak{g}})$ is the Poisson-commutative subalgebra of $\mathfrak{g}[t]$-invariants in $\mathcal{S}(\mathfrak{g}[t,t^{-1}])/(\mathfrak{g}[t]) \cong \mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])$, which is a polynomial ring with infinitely many generators according to a direct generalisation of a Raïs–Tauvel theorem [RT92]. Explicit formulas for the elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ appeared first in type A [CT06, CM09] following Talalaev's discovery [Tal06] of explicit higher Gaudin Hamiltonians. Then they were extended to all classical types in [Mol13]. The construction of [Mol13] relies on the Schur–Weyl duality involving the Brauer algebra. Type G_2 is covered by [MRR16]. The subject is beautifully summarised in [Mol18].

Unlike the finite-dimensional case, no natural isomorphism between the algebras $\mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{g}[t]}$ and $\mathfrak{z}(\hat{\mathfrak{g}})$ is known. Also, generally speaking, an element of $\mathfrak{z}(\hat{\mathfrak{g}})$ cannot be obtained by the symmetrisation ϖ from a homogeneous $\mathfrak{g}[t]$-invariant in $\mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])$. At the same time, some of the elements do come in this way, see Example 5.2, which is dealing with the Pfaffians of $\mathfrak{so}_{2n}$. In this paper, we show that for all classical Lie algebras, ϖ can produce generators of $\mathfrak{z}(\hat{\mathfrak{g}})$. The symmetrisation map is not a homomorphism of algebras. However, it is a homomorphism of $\mathfrak{g}[t^{-1}]$-modules and it behaves well with respect to taking various limits.

According to a striking result of Rybnikov [Ryb08], $\mathfrak{z}(\hat{\mathfrak{g}})$ is the centraliser in $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ of a single quadratic element $\mathcal{H}[-1]$, see § 1.1. This fact is crucial for our considerations.

Any $\mathcal{Y} \in \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ can be expressed as a sum

$$\varpi(Y_k) + \varpi(Y_{k-1}) + \cdots + Y_1 + Y_0 \quad \text{with } Y_j \in \mathcal{S}^j(t^{-1}\mathfrak{g}[t^{-1}]). \tag{0.1}$$

Here $Y_k = \mathrm{gr}(\mathcal{Y})$ if $Y_k \neq 0$. Note that $\sum_{0\leqslant j\leqslant k} \varpi(Y_j)$ is a $\mathfrak{g}$-invariant if and only if each Y_j is a $\mathfrak{g}$-invariant. In the following, we consider only elements with $Y_0 = 0$.

A *polarisation* of a $\mathfrak{g}$-invariant $F \in \mathcal{S}(\mathfrak{g})$ is a $\mathfrak{g}$-invariant in $\mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])$, see § 1.5 for the definition of a polarisation. However, $\mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{g}}$ is not generated by elements of this sort, see (3.7) for an example.

There are finite sets of elements $\{S_1,\ldots,S_\ell\} \subset \mathfrak{z}(\hat{\mathfrak{g}})$ with $\ell = \mathrm{rk}\,\mathfrak{g}$, called *complete sets of Segal–Sugawara vectors*, see § 1.6 for the definition, that are of vital importance for the understanding of $\mathfrak{z}(\hat{\mathfrak{g}})$. We prove that if $\mathfrak{g}$ is either a classical Lie algebra or an exceptional Lie algebra of type G_2, then there is a complete set $\{S_k\}$ of Segal–Sugawara vectors such that all the terms Y_j occurring in presentations (0.1) for S_k are polarisations of symmetric invariants of $\mathfrak{g}$. The map m, defined in § 1.4, plays a crucial rôle in the selection of suitable $\mathfrak{g}$-invariants. In particular, if $F[-1] \in \mathcal{S}^k(\mathfrak{g}t^{-1})$ is obtained from $F \in \mathcal{S}^k(\mathfrak{g})^{\mathfrak{g}}$ using the canonical isomorphism $\mathfrak{g}t^{-1} \cong \mathfrak{g}$, then $\varpi(F[-1]) \in \mathfrak{z}(\hat{\mathfrak{g}})$ if and only if $\mathsf{m}(F) = 0$, see Theorem 3.5 and the remark after it. More generally, if $H \in \mathcal{S}^k(\mathfrak{g})^{\mathfrak{g}}$ is such that

$$\mathsf{m}^d(H) = \mathsf{m}(\mathsf{m}^{d-1}(H)) \in \mathcal{S}(\mathfrak{g}) \quad \text{for all } 1 \leqslant d < k/2, \tag{0.2}$$

then there is a way to produce an element of $\mathfrak{z}(\hat{\mathfrak{g}})$ corresponding to H, see Theorem 3.11 and (2.4).

First, for $F = \xi_1 \ldots \xi_m \in \mathcal{S}^m(\mathfrak{g})$ and $\bar{a} = (a_1,\ldots,a_m) \in \mathbb{Z}^m_{<0}$, set

$$\varpi(F)[\bar{a}] = \frac{1}{m!} \sum_{\sigma\in \mathrm{S}_m} \xi_{\sigma(1)} t^{a_1} \ldots \xi_{\sigma(m)} t^{a_m} \in \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}]), \tag{0.3}$$

then extend this notation to all elements $F \in \mathcal{S}^m(\mathfrak{g})$ by linearity. According to Lemma 2.1, $\varpi(F)[\bar{a}] = \varpi(F[\bar{a}])$ for the $\bar{a}$-polarisation $F[\bar{a}] \in \mathcal{S}^m(t^{-1}\mathfrak{g}[t^{-1}])$ of F.

The expression $\varpi(\tau^r F[-1])\cdot 1$ encodes a sum of $(1/(m+r)!)c(r,\bar{a})\varpi(F)[\bar{a}]$, where the vectors $\bar{a} \in \mathbb{Z}^m_{<0}$ are such that $\sum_{j=1}^m a_j = -m-r$ and $c(r,\bar{a}) \in \mathbb{N}$ are certain combinatorially defined coefficients, which we do not compute explicitly. It is not clear whether any interesting combinatorial identity can be produced in this context.

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For each classical Lie algebra $\mathfrak{g}$, there is a set of generators $\{H_1,\dots,H_\ell\}\subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\mathsf{m}(H_k)\in\mathbb{C}H_j$ for some j depending on k, see §§ 2, 4, 7 and in particular Propositions 2.3, 4.3, 7.5. In types A and C, we are using the coefficients of the characteristic polynomial. In the orthogonal case, one has to work with $\det(I_n-q(F_{ij}))^{-1}$ instead. In type A_{n-1},

$$\mathsf{m}(\tilde\Delta_k)=\frac{(n-k+2)(n-k+1)}{k(k-1)}\tilde\Delta_{k-2};$$

in type C_n,

$$\mathsf{m}(\Delta_{2k})=\frac{(2n-2k+3)(2n-2k+2)}{2k(2k-1)}\Delta_{2k-2};$$

and finally for $\mathfrak{g}=\mathfrak{so}_n$, we have

$$\mathsf{m}(\Phi_{2k})=\frac{(n+2k-3)(n+2k-2)}{2k(2k-1)}\Phi_{2k-2}.$$

This leads to the following complete sets of Segal–Sugawara vectors:

$$\left\{\tilde S_{k-1}=\varpi(\tilde\Delta_k[-1])+\sum_{1\le r<(k-1)/2}\binom{n-k+2r}{2r}\varpi(\tau^{2r}\tilde\Delta_{k-2r}[-1])\cdot 1\mid 2\le k\le n\right\}\ \text{in type }\mathsf{A}_{n-1};$$

$$\left\{S_k=\varpi(\Delta_{2k}[-1])+\sum_{1\le r<k}\binom{2n-2k+2r+1}{2r}\varpi(\tau^{2r}\Delta_{2k-2r}[-1])\cdot 1\mid 1\le k\le n\right\}\ \text{in type }\mathsf{C}_n;$$

$$\left\{S_k=\varpi(\Phi_{2k}[-1])+\sum_{1\le r<k}\binom{n+2k-2}{2r}\varpi(\tau^{2r}\Phi_{2k-2r}[-1])\cdot 1\mid 1\le k<\ell\right\}\ \text{for }\mathfrak{so}_n\text{ with }n=2\ell-1$$

with the addition of $S_\ell=\varpi(\mathrm{Pf}[-1])$ for $\mathfrak{so}_n$ with $n=2\ell$.

The result in type A is not new. It follows via a careful rewriting from the formulas of [CT06, CM09]. We are not giving a new proof. Quite the contrary, we use the statement in type A in order to extend the formula to other types.

Our formulas for $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ describe the same elements as [Mol13], for the case of the Pfaffian-type Segal–Sugawara vector, see § 5; a more general result is recently obtained in [Mol21].

The advantage of our method is that it reduces questions about elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ to questions on the structure of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ in a type-free way. For example, it is possible to deal with type G_2 by hand unlike [MRR16], see (6.3). It is quite probable that other exceptional types can be handled on a computer. Conjecturally, each exceptional Lie algebra possesses a set $\{H_k\}$ of generating symmetric invariants such that each H_k satisfies (0.2).

One of the significant applications of the Feigin–Frenkel centre is related to *Vinberg's quantisation problem*. The symmetric algebra $\mathcal{S}(\mathfrak{g})$ carries a Poisson structure extended from the Lie bracket on $\mathfrak{g}$ by the Leibniz rule. To each $\mu\in\mathfrak{g}^*\cong\mathfrak{g}$, one associates the *Mishchenko–Fomenko subalgebra* $\mathcal{A}_\mu\subset\mathcal{S}(\mathfrak{g})$, which is an extremely interesting Poisson-commutative subalgebra [MF78]. In [Vin91], Vinberg proposed to find a commutative subalgebra $\mathcal{C}_\mu\subset\mathcal{U}(\mathfrak{g})$ such that $\mathrm{gr}(\mathcal{C}_\mu)=\langle\mathrm{gr}(Y)\mid Y\in\mathcal{C}_\mu\rangle_{\mathbb{C}}$ coincides with $\mathcal{A}_\mu$. Partial solutions to this problem are obtained in [NO96, Tar00]. The breakthrough came in [Ryb06], where a certain commutative subalgebra $\tilde{\mathcal{A}}_\mu\subset\mathcal{U}(\mathfrak{g})$ is constructed as an image of $\mathfrak{z}(\hat{\mathfrak{g}})$, cf. (8.1).

In [MY19, § 3.3], sets of generators $\{H_k\mid 1\leqslant k\leqslant \ell\}$ of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\tilde{\mathcal{A}}_\mu$ is generated by $\varpi(\partial_\mu^m H_k)$, cf. (8.3), are exhibited in types B, C, and D. For the symplectic Lie algebra, $H_k=\Delta_{2k}$, in the orthogonal case $H_k=\Phi_{2k}$ with the exception of $H_\ell=\mathrm{Pf}$ in type D_ℓ. Results of this paper provide a different proof for [MY19, Theorem 3.2]. We have pushed the symmetrisation map to the level of $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$.

In § 8.2, we briefly consider Gaudin algebras $\mathcal{G}$. If $\mathfrak{g}$ is a classical Lie algebra, then the *two-points* Gaudin subalgebra $\mathcal{G} \subset \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g})$ is generated by the symmetrisations of certain bi-homogeneous $\mathfrak{g}$-invariants in $\mathcal{S}(\mathfrak{g} \oplus \mathfrak{g})$, see Theorem 8.4.

1. Preliminaries and notation

Let $\mathfrak{g} = \operatorname{Lie} G$ be a non-Abelian complex reductive Lie algebra. The Feigin–Frenkel centre $\mathfrak{z}(\hat{\mathfrak{g}})$ is the centre of the universal affine vertex algebra associated with the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level [FF92, Fre07]. There is an injective homomorphism $\mathfrak{z}(\hat{\mathfrak{g}}) \hookrightarrow \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ and $\mathfrak{z}(\hat{\mathfrak{g}})$ can be viewed as a commutative subalgebra of $\mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ [Fre07, § 3.3]. Each element of $\mathfrak{z}(\hat{\mathfrak{g}})$ is annihilated by the adjoint action of $\mathfrak{g}$, cf. [Mol18, § 6.2.].

1.1 The Feigin–Frenkel centre as a centraliser

We set $\mathfrak{g}[b] := \mathfrak{g}t^b$ and $x[b] := xt^b$ for $x \in \mathfrak{g}$. Furthermore, $\hat{\mathfrak{g}}^- := t^{-1}\mathfrak{g}[t^{-1}]$. According to [Ryb08], $\mathfrak{z}(\hat{\mathfrak{g}})$ is the centraliser in $\mathcal{U}(\hat{\mathfrak{g}}^-)$ of the following quadratic element

$$\mathcal{H}[-1] = \sum_{a=1}^{\dim \mathfrak{g}} x_a[-1]x_a[-1],$$

where $\{x_1, \ldots, x_{\dim \mathfrak{g}}\}$ is any basis of $\mathfrak{g}$ that is orthonormal with respect to a fixed $\mathfrak{g}$-invariant non-degenerate *scalar product* (,). In this paper, a scalar product is a symmetric bilinear form.

1.2 The symmetrisation map

For any complex Lie algebra $\mathfrak{q}$, let $\varpi\colon \mathcal{S}^k(\mathfrak{q}) \to \mathfrak{q}^{\otimes k}$ be the canonical symmetrisation map. Following the usual convention, we let ϖ stand also for the symmetrisation map from $\mathcal{S}(\mathfrak{q})$ to $\mathcal{U}(\mathfrak{q})$. Let $\operatorname{gr}(X) \in \mathcal{S}(\mathfrak{q})$ be the *symbol* of $X \in \mathcal{U}(\mathfrak{q})$. Then $\operatorname{gr}(\varpi(Y)) = Y$ for $Y \in \mathcal{S}^k(\mathfrak{q})$ by the construction.

1.3 The antipode

Let us define the anti-involution ω on $\mathcal{U}(\hat{\mathfrak{g}}^-)$ to be the $\mathbb{C}$-linear map such that $\omega(y[k]) = -y[k]$ for each $y \in \mathfrak{g}$ and

$$\omega(y_1[k_1]y_2[k_2]\ldots y_m[k_m]) = (-y_m[k_m])\ldots(-y_2[k_2])(-y_1[k_1]).$$

Let also ω be the analogues anti-involution on $\mathcal{U}(\mathfrak{q})$ for any complex Lie algebra $\mathfrak{q}$.

Clearly, $\omega(\mathcal{H}[-1]) = \mathcal{H}[-1]$. Therefore ω acts on $\mathfrak{z}(\hat{\mathfrak{g}})$. For $Y_j \in \mathcal{S}^j(\hat{\mathfrak{g}}^-)$, we have $\omega(\varpi(Y_j)) = (-1)^j\varpi(Y_j)$. A non-zero element $\mathcal{Y} \in \mathcal{U}(\hat{\mathfrak{g}}^-)$ presented in the form (0.1) is an eigenvector of ω if and only if either all Y_j with even j or all Y_j with odd j are zero.

1.4 The map m

For $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{C}) = \operatorname{End}(\mathbb{C}^N)$ and $1 \leqslant r \leqslant k$, consider the linear map

$$\mathsf{m}_r\colon \mathfrak{gl}_N^{\otimes k} \to \mathfrak{gl}_N^{\otimes(k-r+1)} \text{ that sends } \xi_1 \otimes \cdots \otimes \xi_k \text{ to } \xi_1\xi_2\ldots\xi_r \otimes \xi_{r+1} \otimes \cdots \otimes \xi_k.$$

Note that clearly

$$\mathsf{m}_r \circ \mathsf{m}_s = \mathsf{m}_{r+s-1}. \tag{1.1}$$

Via the adjoint representation of $\mathfrak{g}$, the map m_r leads to a map $\mathfrak{g}^{\otimes k} \to \operatorname{End}(\mathfrak{g}) \otimes \mathfrak{g}^{\otimes(k-r)}$, which we denote by the same symbol. Explicitly, the map

$$\mathsf{m}_r\colon \mathfrak{g}^{\otimes k} \to \operatorname{End}(\mathfrak{g}) \otimes \mathfrak{g}^{\otimes(k-r)} \text{ sends } y_1 \otimes \cdots \otimes y_k \text{ to } \operatorname{ad}(y_1)\operatorname{ad}(y_2)\ldots\operatorname{ad}(y_r) \otimes y_{r+1} \otimes \cdots \otimes y_k.$$

Observe that

$$\operatorname{ad}(y_1)\operatorname{ad}(y_2)\ldots\operatorname{ad}(y_{2r+1}) + \operatorname{ad}(y_{2r+1})\ldots\operatorname{ad}(y_2)\operatorname{ad}(y_1) \in \mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g},$$

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where $\mathfrak{so}(\mathfrak{g}) = \{\xi \in \mathrm{End}(\mathfrak{g}) \mid (\xi(x), y) = -(x, \xi(y))\, \forall x, y \in \mathfrak{g}\}$; the isomorphism $\Lambda^2\mathfrak{g} \cong \mathfrak{so}(\mathfrak{g})$ is given by

$$(y_1 \wedge y_2)(x) = (y_2, x)y_1 - (y_1, x)y_2$$

for $y_1, y_2, x \in \mathfrak{g}$.

We embed $\mathcal{S}^k(\mathfrak{g})$ in $\mathfrak{g}^{\otimes k}$ via ϖ. Set $\mathsf{m} = \mathsf{m}_3$. Then $\mathsf{m}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. For example, if $Y = y_1y_2y_3 \in \mathcal{S}^3(\mathfrak{g})$, then

$$\begin{aligned}\mathsf{m}(Y) = \tfrac{1}{6}(&\mathrm{ad}(y_1)\mathrm{ad}(y_2)\mathrm{ad}(y_3) + \mathrm{ad}(y_3)\mathrm{ad}(y_2)\mathrm{ad}(y_1) + \mathrm{ad}(y_1)\mathrm{ad}(y_3)\mathrm{ad}(y_2)\\ &+ \mathrm{ad}(y_2)\mathrm{ad}(y_3)\mathrm{ad}(y_1) + \mathrm{ad}(y_2)\mathrm{ad}(y_1)\mathrm{ad}(y_3) + \mathrm{ad}(y_3)\mathrm{ad}(y_1)\mathrm{ad}(y_2)) \in \mathfrak{so}(\mathfrak{g}).\end{aligned}$$

Similarly one defines $\mathsf{m}_{2r+1}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{(k-2r-1)}(\mathfrak{g})$ for each odd $2r+1 \leqslant k$. Note that each m_{2r+1} is G-equivariant. It is convenient to put $\mathsf{m}(\mathcal{S}^k(\mathfrak{g})) = 0$ for $k \leqslant 2$.

Suppose that $\mathfrak{g}$ is simple. There is a G-stable decomposition $\Lambda^2\mathfrak{g} = \mathfrak{g} \oplus V$. This V will be called the *Cartan component* of $\Lambda^2\mathfrak{g}$. If $\mathfrak{g}$ is not of type A, then V is irreducible. For certain elements $H \in \mathcal{S}^k(\mathfrak{g})$, we have $\mathsf{m}(H) \in \mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. Note that the embedding $\mathfrak{g} \hookrightarrow \mathfrak{so}(\mathfrak{g})$ is canonical: it is given by the adjoint action of $\mathfrak{g}$. If $\mathsf{m}(H) \in \mathcal{S}^{k-2}(\mathfrak{g})$, then $\mathsf{m}_{2r+1}(H) = \mathsf{m}_{2r-1} \circ \mathsf{m}(H)$, because of (1.1). Since $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$, we have $\mathsf{m}(\mathcal{S}^3(\mathfrak{g})^{\mathfrak{g}}) = 0$.

1.5 Polarisations and fully symmetrised elements

For elements $y_1, \ldots, y_m \in \mathfrak{g}$ and a vector $\bar{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{<0}$, set $\Upsilon[\bar{a}] = \prod_{i=1}^m y_i[a_i] \in \mathcal{S}(\hat{\mathfrak{g}}^-)$. If we consider the product $Y = \prod_i y_i \in \mathcal{S}^m(\mathfrak{g})$, then there is no uniquely defined sequence of factors y_i. However, the *$\bar{a}$-polarisation* $Y[\bar{a}] := (1/m!) \sum_{\sigma \in \mathsf{S}_m} \Upsilon[\sigma(\bar{a})]$ of Y is well defined. We extend this notion to all elements of $\mathcal{S}^m(\mathfrak{g})$ by linearity. Linear combinations of the elements

$$\varpi(Y[\bar{a}]) \in \mathcal{U}(\hat{\mathfrak{g}}^-)$$

are said to be *fully symmetrised*. Note that $\varpi(H)$ is fully symmetrised if $H \in \mathcal{S}^m(\mathfrak{g}t^{-1})$. If $a_i = a$ for all i, then $\Upsilon[\bar{a}] = Y[\bar{a}]$ and we denote it simply by $Y[a]$.

The evaluation Ev_1 at $t = 1$ defines an isomorphism $\mathsf{Ev}_1\colon \mathcal{S}(\mathfrak{g}[a]) \to \mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$-modules. For $F \in \mathcal{S}(\mathfrak{g})$, let $F[a]$ stand for $\mathsf{Ev}_1^{-1}(F) \in \mathcal{S}(\mathfrak{g}[a])$. Then $\varpi(F)[a] := \varpi(F[a])$ is fully symmetrised.

1.6 Segal–Sugawara vectors

Set $\tau = -\partial_t$. According to [FF92], $\mathfrak{z}(\hat{\mathfrak{g}})$ is a polynomial algebra in infinitely many variables with a distinguished set of 'generators' $\{S_1, \ldots, S_\ell\}$ such that $\ell = \mathrm{rk}\,\mathfrak{g}$ and

$$\mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C}[\tau^m(S_k) \mid 1 \leqslant k \leqslant \ell, m \geqslant 0].$$

We have $\mathrm{gr}(S_k) = H_k[-1]$ with $H_k \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and $\mathbb{C}[H_1, \ldots, H_\ell] = \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. The set $\{S_k\}$ is said to be a *complete set of Segal–Sugawara vectors*. The symbols of $\tau^m(S_k)$ generate $\mathcal{S}(\hat{\mathfrak{g}}^-)^{\mathfrak{g}[t]}$ in accordance with [RT92].

Suppose that we have $\tilde{S}_k \in \mathfrak{z}(\hat{\mathfrak{g}})$ with $1 \leqslant k \leqslant \ell$ and $\mathrm{gr}(S_k) = \tilde{H}_k[-1]$, where $\tilde{H}_k \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, for each k. The structural properties of $\mathfrak{z}(\hat{\mathfrak{g}})$ imply that $\{\tilde{S}_k\}$ is a complete set of Segal–Sugawara vectors if and only if the set $\{\tilde{H}_k\}$ generates $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$.

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1.7 Symmetric invariants

For a finite-dimensional Lie algebra $\mathfrak{q}$, we have $\mathcal{S}(\mathfrak{q}) \cong \mathbb{C}[\mathfrak{q}^*]$. For any reductive Lie algebra, there is an isomorphism of $\mathfrak{g}$-modules $\mathfrak{g} \cong \mathfrak{g}^*$. For $\xi \in (\mathfrak{gl}_n)^*$, write

$$\det(qI_n - \xi) = q^n - \Delta_1(\xi)q^{n-1} + \cdots + (-1)^k\Delta_k(\xi)q^{n-k} + \cdots + (-1)^n\Delta_n(\xi). \qquad (1.2)$$

Then $\mathcal{S}(\mathfrak{gl}_n)^{\mathfrak{gl}_n} = \mathbb{C}[\Delta_1, \ldots, \Delta_n]$.

Let $\mathfrak{f} \subset \mathfrak{g}$ be a reductive subalgebra. Then there is an $\mathfrak{f}$-stable subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$, whereby also $\mathfrak{g}^* \cong \mathfrak{f}^* \oplus \mathfrak{m}^*$. Identifying $\mathfrak{f}$ with $\mathfrak{f}^*$, one defines the *restriction* $H|_{\mathfrak{f}}$ of $H \in \mathcal{S}(\mathfrak{g})$ to $\mathfrak{f}$. This is the image of H in $\mathcal{S}(\mathfrak{g})/\mathfrak{m}\mathcal{S}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{f})$.

In cases $n = 2\ell$, $\mathfrak{f} = \mathfrak{sp}_{2\ell}$ and $n = 2\ell + 1$, $\mathfrak{f} = \mathfrak{so}_n$, the restrictions $\Delta_{2k}|_{\mathfrak{f}}$ with $1 \leqslant k \leqslant \ell$ form a generating set in $\mathcal{S}(\mathfrak{f})^{\mathfrak{f}}$. In the case $\mathfrak{f} = \mathfrak{so}_n$ with $n = 2\ell$, the restriction of the determinant $\Delta_{2\ell}$ is the square of the Pfaffian and has to be replaced by the Pfaffian in the generating set.

Explicit formulas for basic symmetric invariants of the exceptional Lie algebras are less transparent.

Let $\mathfrak{g}$ be simple. The inclusions $\mathfrak{g} \subset \mathcal{S}(\mathfrak{g})$ are ruled by the symmetric invariants. The key point here is that $\mathcal{S}(\mathfrak{g})$ is a free module over $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ [Kos63]. If $\{H_1, \ldots, H_\ell\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a generating set consisting of homogeneous elements and $\deg H_i = d_i + 1$, then to each i corresponds a *primitive* copy of $\mathfrak{g}$ in $\mathcal{S}^{d_i}(\mathfrak{g})$. The non-primitive copies are obtained as linear combinations of the primitive ones with coefficients from $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$.

1.8 Miscellaneousness

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, we let ℓ stand for $\dim \mathfrak{h} = \operatorname{rk} \mathfrak{g}$ and $W = W(\mathfrak{g}, \mathfrak{h})$ stand for the Weyl group of $\mathfrak{g}$. The fundamental weights of a simple Lie algebra $\mathfrak{g}$ are π_k with $1 \leqslant k \leqslant \ell$ and $V(\lambda)$ stands for an irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight $\lambda = \sum_{k=1}^{\ell} c_k\pi_k$. Please keep in mind that the Vinberg–Onishchik numbering [VO88, Tables] of simple roots (and fundamental weights) is used. If $\alpha \in \mathfrak{h}^*$ is a positive root, then $\{e_\alpha, h_\alpha, f_\alpha\} \subset \mathfrak{g}$ is an $\mathfrak{sl}_2$-triple associated with α.

An automorphism $\sigma \in \operatorname{Aut}(\mathfrak{g})$ extends to $\mathfrak{g}[t^{-1}]$ by setting $\sigma(t^{-1}) = t^{-1}$. In this context, σ stands also for the corresponding automorphism of $\mathcal{S}(\mathfrak{g}[t^{-1}])$. If we take a σ-invariant product $(\ ,\ )$, then $\sigma(\mathcal{H}[-1]) = \mathcal{H}[-1]$. Therefore σ acts on $\mathfrak{z}(\hat{\mathfrak{g}})$.

If $\sigma \in \operatorname{Aut}(\mathfrak{g})$ is of finite order m, then it leads to a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{m-1}$. In the case $m = 2$, we have $\mathfrak{g}_1 = \{\xi \in \mathfrak{g} \mid \sigma(\xi) = -\xi\}$.

Throughout the paper:

- $\{x_i\}$ is an orthonormal basis of $\mathfrak{g}$;
- in the sums $\sum_i x_i$ or $\sum_{i,j} x_ix_j$, the ranges are from 1 to $\dim \mathfrak{g}$ for i and for j;
- $\bar{b} = (b_1, b_2) \in \mathbb{Z}^2_{<0}$ and $\mathcal{H}[\bar{b}]$ stands for $\sum_i x_i[b_1]x_i[b_2] \in \mathcal{U}(\hat{\mathfrak{g}}^-)$ and also for the symbol of this sum (in the sense of § 1.2);
- G_ξ stands for the stabiliser of ξ and it is always clear from the context, which G-action is considered, $\mathfrak{g}_\xi = \operatorname{Lie} G_\xi$;
- $\mathfrak{q}$ stands for an arbitrary unspecified complex Lie algebra;
- if $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ is a subalgebra, then $\operatorname{gr}(\mathcal{A}) := \langle \operatorname{gr}(x) \mid x \in \mathcal{A}\rangle_{\mathbb{C}} \subset \mathcal{S}(\mathfrak{q})$.

2. Explicit formulas in type A

In type A, there are several explicit formulas for the Segal–Sugawara vectors [CT06, CM09], see also [Mol18, § 7.1]. One of them actually uses symmetrisation. One can form the matrix

$$E[-1] + \tau = (E_{ij}[-1]) + \tau I_n$$

with $E_{ij} \in \mathfrak{gl}_n$ and calculate its column determinant and symmetrised determinant. Due to the fact that this matrix is *Manin*, see [Mol18, Definition 3.1.1, p. 48, Lemma 7.1.2], the results are the same. The symmetrised version is more suitable for our purpose. The elements S_j are coefficients of τ^{n-j} in

$$\det_{\mathrm{sym}}(E[-1]+\tau) = \varpi(\Delta_n[-1]) + \varpi(\tau\Delta_{n-1}[-1]) + \cdots + \varpi(\tau^{n-2}\Delta_2[-1]) + \varpi(\tau^{n-1}\Delta_1[-1]) + \tau^n.$$

Assume the conventions that

$$\tau x[a] - x[a]\tau = [\tau, x[a]] = \tau(x[a]) = -ax[a-1]$$

and $\tau{\cdot}1 = 0$. This leads for example to $\tau x[-1]{\cdot}1 = x[-2]$. Note that ϖ acts on the summands of $\tau^{n-k}\Delta_k[-1]$ as on products of n factors. It permutes τ with elements of $\mathfrak{gl}_n[-1]$.

Let θ be a Weyl involution of $\mathfrak{g}$, i.e. there is a θ-stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\theta|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. As is well known, $\theta(\Delta_k) = (-1)^k\Delta_k$ if $\mathfrak{g} = \mathfrak{gl}_n$. In particular, $\theta(\mathcal{H}[-1]) = \mathcal{H}[-1]$ and θ acts on $\mathfrak{z}(\hat{\mathfrak{g}})$. Hence one can always modify the Segal–Sugawara vectors in such a way that they become eigenvectors of θ. The resulting simplified forms are

$$\begin{aligned} S_n = {} & \varpi(\Delta_n[-1]) + \varpi(\tau^2\Delta_{n-2}[-1]){\cdot}1 + \cdots + \varpi(\tau^{2r}\Delta_{n-2r}[-1]){\cdot}1 + \cdots \\ & + \varpi(\tau^{n-u}\Delta_u[-1]){\cdot}1 \quad \text{with } u = n - 2\left[\frac{n-1}{2}\right], \end{aligned} \tag{2.1}$$

$$S_k = \varpi(\Delta_k[-1]) + \sum_{1\leqslant r<k/2} \binom{n-k+2r}{2r} \varpi(\tau^{2r}\Delta_{k-2r}[-1]){\cdot}1. \tag{2.2}$$

We will see that there is a direct connection with the symmetrisation and that one could have used ω instead of θ in order to simplify the formulas. The following two lemmas are valid for all Lie algebras.

Lemma 2.1. *Take $Y = y_1 \ldots y_m \in \mathcal{S}^m(\mathfrak{g})$ and $\bar{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{<0}$. Then in $\mathcal{U}(\hat{\mathfrak{g}}^-)$, we have*

$$\mathbb{Y}[\bar{a}] := \sum_{\sigma\in \mathtt{S}_m} y_{\sigma(1)}[a_1]\ldots y_{\sigma(m)}[a_m] = \varpi\Bigg(\sum_{\sigma\in\mathtt{S}_m} y_1[a_{\sigma(1)}]\ldots y_m[a_{\sigma(m)}]\Bigg) = m!\varpi(Y[\bar{a}])$$

in the notation of § 1.5.

Proof. It suffices to show that $\mathbb{Y}[\bar{a}]$ is invariant under all $\mathtt{t}_i = (i\,i+1) \in \mathtt{S}_m$ with $1 \leqslant i < m$. For each $\sigma \in \mathtt{S}_m$, both monomials

$$y_{\sigma(1)}[a_1]\ldots y_{\sigma(i)}[a_i]y_{\sigma(i+1)}[a_{i+1}]\ldots y_{\sigma(m)}[a_m] \quad \text{and} \quad y_{\sigma(1)}[a_1]\ldots y_{\sigma(i+1)}[a_i]y_{\sigma(i)}[a_{i+1}]\ldots y_{\sigma(m)}[a_m]$$

appear in $\mathbb{Y}[\bar{a}]$ with the same coefficient 1. Let $s(\sigma, i)$ stand for their sum. Then

$$s(\sigma,i) - \mathtt{t}_i(s(\sigma,i)) = \cdots [y_{\sigma(i)}[a_i], y_{\sigma(i+1)}[a_{i+1}]] \cdots + \cdots [y_{\sigma(i+1)}[a_i], y_{\sigma(i)}[a_{i+1}]] \cdots = 0,$$

because $[y_{\sigma(i)}[a_i], y_{\sigma(i+1)}[a_{i+1}]] = [y_{\sigma(i)}, y_{\sigma(i+1)}][a_i + a_{i+1}] = -[y_{\sigma(i+1)}[a_i], y_{\sigma(i)}[a_{i+1}]]$. Since $\mathbb{Y}[\bar{a}] = \frac{1}{2}\sum_\sigma s(\sigma, i)$ for each i, we are done. □

Lemma 2.2. *Take $F \in \mathcal{S}^m(\mathfrak{g})$ and $r \geqslant 1$. Then $\varpi(\tau^r F[-1]){\cdot}1$ is fully symmetrised and therefore is an eigenvector of ω corresponding to the eigenvalue $(-1)^m$.*

Proof. Notice that $\varpi(\tau^r(F+F')[-1]){\cdot}1 = \varpi(\tau^r F[-1]){\cdot}1 + \varpi(\tau^r F'[-1]){\cdot}1$ for any $F' \in \mathcal{S}^m(\mathfrak{g})$. Hence we may assume that $F = y_1 \ldots y_m$ with $y_j \in \mathfrak{g}$. By the construction, $\varpi(\tau^r F[-1]){\cdot}1$ is the

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sum of terms

$$\frac{1}{(m+r)!}c(r,\bar a)\sum_{\sigma\in\mathsf{S}_m} y_{\sigma(1)}[a_1]\dots y_{\sigma(m)}[a_m] \quad \text{with } c(r,\bar a)\in\mathbb{N},$$

taken over all vectors $\bar a=(a_1,\dots,a_m)\in\mathbb{Z}^m_{<0}$ such that $\sum a_j=-(m+r)$. The scalars $c(r,\bar a)$ depend on $(m,r,\bar a)$ in an elementary combinatorial way. Each summand here is a fully symmetrised element by Lemma 2.1. Hence the desired conclusion follows. □

Let $z=(1/n)I_n$ be a central element of $\mathfrak{g}=\mathfrak{gl}_n$ and let $\tilde\Delta_k\in\mathcal{S}^k(\mathfrak{sl}_n)$ denote the restriction of Δ_k to $\mathfrak{sl}_n$. Then

$$\Delta_k=\tilde\Delta_k+(n-k+1)z\tilde\Delta_{k-1}+\binom{n-k+2}{2}z^2\tilde\Delta_{k-2}+\dots+\binom{n-2}{k-2}z^{k-2}\tilde\Delta_2+\binom{n}{k}z^k. \quad (2.3)$$

Fix $\mathfrak{h}=\langle E_{ii}\mid 1\leqslant i\leqslant n\rangle_{\mathbb{C}}$. Let $\varepsilon_i\in\mathfrak{h}^*$ be a linear function such that $\varepsilon_i(E_{jj})=\delta_{i,j}$. For $E_{ii}\in\mathfrak{g}$, set $\tilde E_{ii}=E_{ii}-z$.

PROPOSITION 2.3. *In type* A, *we have*

$$\mathsf{m}_{2r+1}(\tilde\Delta_k)=\frac{(2r)!(k-2r)!}{k!}\binom{n-k+2r}{2r}\tilde\Delta_{k-2r}$$

if $k-2r>1$ *and* $\mathsf{m}(\tilde\Delta_3)=\mathsf{m}(\Delta_3)=0$.

Proof. Notice that the centre of $\mathfrak{g}$ plays a very specific rôle in m, since $\mathrm{ad}(z)=0$. In particular, $\mathsf{m}(\mathcal{S}^3(\mathfrak{gl}_n))=\mathsf{m}(\mathcal{S}^3(\mathfrak{sl}_n))\subset\Lambda^2\mathfrak{sl}_n$. Furthermore,

$$\mathsf{m}(\Delta_k)\in\mathsf{m}(\tilde\Delta_k)+\Lambda^2\mathfrak{sl}_n\otimes z\mathcal{S}^{k-4}(\mathfrak{g}),$$

where one can use the multiplication in either $\mathrm{End}(\mathfrak{gl}_n)$ or $\mathrm{End}(\mathfrak{sl}_n)$ for the definition of m. Therefore we can work either with $\mathfrak{sl}_n$ or with $\mathfrak{gl}_n$, whichever is more convenient.

Suppose that $Y=E_{ij}E_{ls}E_{up}$ is a factor of a monomial of Δ_k. Then

$$i\notin\{l,u\},\ j\notin\{s,p\},\ l\neq u, \text{ and } s\neq p.$$

The $\mathfrak{h}$-weight of Y cannot be equal to either $2\varepsilon_1-\varepsilon_n-\varepsilon_{n-1}$ or $\varepsilon_1+\varepsilon_2-2\varepsilon_n$. These are the highest weights of the Cartan component of $\Lambda^2\mathfrak{sl}_n$. Hence $\mathsf{m}(\Delta_k)\in(\mathfrak{sl}_n\otimes\mathcal{S}^{k-3}(\mathfrak{g}))^{\mathfrak{g}}$. The image in question is a polynomial function on $(\mathfrak{sl}_n\oplus\mathfrak{g})^*\cong\mathfrak{sl}_n\oplus\mathfrak{g}$ of degree 1 in $\mathfrak{sl}_n$ and degree $k-3$ in $\mathfrak{g}$. Note that $\mathsf{m}(\Delta_3)$ is a $\mathfrak{gl}_n$-invariant in $\mathfrak{sl}_n$ and is thereby zero. Suppose that $n\geqslant k>3$.

Fortunately, $G(\mathfrak{sl}_n\oplus\mathfrak{h})$ is a dense subset of $\mathfrak{sl}_n\oplus\mathfrak{g}$. We calculate the restriction

$$\boldsymbol{f}=\mathsf{m}(\Delta_k)|_{\mathfrak{sl}_n\oplus\mathfrak{h}}$$

of $\mathsf{m}(\Delta_k)$ to $\mathfrak{sl}_n\oplus\mathfrak{h}$. Write $\boldsymbol{f}=\sum_{\nu=1}^{L}\xi_\nu\otimes\boldsymbol{H}_\nu$ with $\xi_\nu\in\mathfrak{sl}_n$ and pairwise different monomials $\boldsymbol{H}_\nu\in\mathcal{S}^{k-3}(\mathfrak{h})$ in $\{E_{ii}\}$. Since $\mathsf{m}(\Delta_k)$ is an element of $\mathfrak{h}$-weight zero, $\xi_\nu\in\mathfrak{h}$ for each ν. Thus one can say that $\boldsymbol{f}$ is an invariant of the Weyl group $W(\mathfrak{g},\mathfrak{h})\cong\mathsf{S}_n$. Without loss of generality assume that $\boldsymbol{H}_1=y_4\dots y_k$ with $y_s=E_{ss}$ for all $s\geqslant 4$. In order to understand $\boldsymbol{f}$, it suffices to calculate ξ_1. Let F be the polynomial obtained from Δ_3 by setting $E_{ij}=0$ for all (i,j) such that i or j belongs to $\{4,\dots,k\}$. Then $\xi_1=(3!(k-3)!/k!)\mathsf{m}(F)$.

Now take Y as above with $\{i,l,u\}=\{j,s,p\}=\{1,2,3\}$. Then:

- $\mathsf{m}(Y)(E_{14})=0$ if $i=j$ or $l=s$ or $u=p$;
- $\mathsf{m}(Y)(E_{14})=\frac{1}{6}E_{14}$ if $Y=E_{13}E_{32}E_{21}$;
- $\mathsf{m}(Y)(E_{14})=\frac{1}{6}E_{14}$ if $Y=E_{12}E_{23}E_{31}$.

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Besides, $\mathsf{m}(Y)(E_{vw}) = 0$ if $v, w \geqslant 4$. In the self-explanatory notation, $\eta = \mathsf{m}(\Delta_3^{(1,2,3)})$ is an invariant of $(\mathfrak{gl}_3 \oplus \mathfrak{gl}_{n-3})$ and η acts on $\mathfrak{gl}_3 = \langle E_{vw} \mid 1 \leqslant v, w \leqslant 3\rangle_{\mathbb{C}}$ as zero. Since $\Delta_3^{(1,2,3)}$ is a linear combination of $Y = E_{ij}E_{ls}E_{up}$ with $\{i,l,u\} = \{j,s,p\} = \{1,2,3\}$, the element η acts on $\mathfrak{g}$ as $\frac{1}{3}(E_{11} + E_{22} + E_{33})$. This implies that $\eta = \frac{1}{3}(\tilde{E}_{11} + \tilde{E}_{22} + \tilde{E}_{33})$. By the construction of F, we now have $\mathsf{m}(F) = \binom{n-k+2}{2} \sum_{l \notin \{4,\dots,k\}} \frac{1}{3}\tilde{E}_{ll}$ and hence

$$\xi_1 \otimes \boldsymbol{H}_1 = \frac{3!(k-3)!}{k!}\frac{1}{3}\binom{n-k+2}{2}\Bigg(\sum_{l \notin \{4,\dots,k\}} \tilde{E}_{ll}\Bigg) \otimes E_{44}\dots E_{kk}.$$

From this one deduces that up to the scalar

$$\frac{k-2}{3}\frac{3!(k-3)!}{k!}\binom{n-k+2}{2},$$

the restriction of $\mathsf{m}(\tilde{\Delta}_k)$ to $\mathfrak{sl}_n \oplus \mathfrak{h}$ coincides with the restriction $\tilde{\Delta}_{k-2}|_{\mathfrak{sl}_n \oplus \mathfrak{h}}$, where we regard $\tilde{\Delta}_{k-2}$ as an element of $\mathfrak{sl}_n \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. In particular, $\mathsf{m}(\tilde{\Delta}_k)$ is a symmetric invariant. More explicitly,

$$\mathsf{m}(\tilde{\Delta}_k) = \frac{(k-2)}{3}\frac{3!(k-3)!}{k!}\binom{n-k+2}{2}\tilde{\Delta}_{k-2} = \frac{2(k-2)!}{k!}\binom{n-k+2}{2}\tilde{\Delta}_{k-2}.$$

Iterating the map m, we obtain the result. □

Remark. Strictly speaking, $\mathsf{m}(\Delta_k)$ is not an element of $\mathcal{S}^{k-2}(\mathfrak{gl}_n)$. This is the reason for working with $\mathfrak{sl}_n$.

Now we can exhibit formulas for Segal–Sugawara vectors of t-degree k that are independent of n, i.e. these formulas are valid for all $n \geqslant k$. First of all notice that in view of (2.3), Formula (2.2) produces an element of $\mathfrak{z}(\widehat{\mathfrak{sl}_n})$ if we replace each Δ_{k-2r} with $\tilde{\Delta}_{k-2r}$. (This statement can be deduced from (2.1) as well.) Making use of Proposition 2.3, one obtains that for $H = \tilde{\Delta}_k$,

$$\tilde{S}_{k-1} = \varpi(H[-1]) + \sum_{1 \leqslant r < (k-1)/2} \binom{k}{2r}\varpi(\tau^{2r}\mathsf{m}_{2r+1}(H)[-1])\cdot 1 \tag{2.4}$$

is a Segal–Sugawara vector.

3. Commutators and Poisson brackets

In this section, we prove that Formula (2.4) is universal, i.e. that it is valid in all types, providing $\mathsf{m}_{2r+1}(H)$ is a symmetric invariant for each $r \geqslant 1$.

For $F \in \mathcal{S}^m(\mathfrak{g})$, set $\mathbb{X}_{F[-1]} := [\mathcal{H}[-1], \varpi(F)[-1]]$. Note that

$$\omega(\mathcal{H}[-1]\varpi(F)[-1]) = (-1)^{m+2}\varpi(F)[-1]\mathcal{H}[-1].$$

Hence ω multiplies $\mathbb{X}_{F[-1]}$ with $(-1)^{m+1}$. This implies that the symbol of $\mathbb{X}_{F[-1]}$ has degree $m + 1 - 2d$ with $d \geqslant 0$. Let $\mathcal{H}[-1]$ stand also for $\sum_i x_i[-1]x_i[-1] \in \mathcal{S}^2(\mathfrak{g}[-1])$.

The symmetric algebra $\mathcal{S}(\mathfrak{q})$ of a Lie algebra $\mathfrak{q}$ is equipped with the standard Poisson bracket $\{\ ,\ \}$ such that $\{x, y\} = [x, y]$ for $x, y \in \mathfrak{q}$. Using the standard filtration on $\mathcal{U}(\mathfrak{q})$, one can state that

$$\{\mathrm{gr}(X), \mathrm{gr}(Y)\} = [X, Y] + \mathcal{U}_{l+m-2}(\mathfrak{q}) \quad \text{if } X \in \mathcal{U}_l(\mathfrak{q})\backslash\mathcal{U}_{l-1}(\mathfrak{q}), Y \in \mathcal{U}_m(\mathfrak{q})\backslash\mathcal{U}_{m-1}(\mathfrak{q}).$$

The fact that $\{\mathcal{H}[-1], F[-1]\} = 0$ for $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$ follows from [FF92]. For convenience, we present a short proof here.

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LEMMA 3.1. *Take two arbitrary $\mathfrak{g}$-invariants F, F' in $\mathcal{S}(\mathfrak{g})$. Then $\{F[-1], F'[-1]\} = 0$.*

Proof. The Poisson bracket of two polynomial functions can be calculated by

$$\{f_1, f_2\}(\gamma) = [d_\gamma f_1, d_\gamma f_2](\gamma) \quad \text{for } \gamma \in (\hat{\mathfrak{g}}^-)^*. \tag{3.1}$$

In the case of $F[-1]$ and $F'[-1]$, the differentials $d_\gamma F[-1], d_\gamma F'[-1]$ at γ depend only on the (-1)-part of γ. More explicitly, if $\gamma(x[-1]) = \tilde{\gamma}(x)$ with $\tilde{\gamma} \in \mathfrak{g}^*$, then $d_\gamma F[-1] = (d_{\tilde{\gamma}} F)[-1]$ and the same identity hods for F'. We have $d_{\tilde{\gamma}} F, d_{\tilde{\gamma}} F' \in (\mathfrak{g}_\gamma)^{\mathfrak{g}_\gamma}$, since F and F' are $\mathfrak{g}$-invariants. Hence $[d_{\tilde{\gamma}} F, d_{\tilde{\gamma}} F'] = 0$ and also $[d_\gamma F[-1], d_\gamma F'[-1]] = 0$ for any $\gamma \in (\hat{\mathfrak{g}}^-)^*$. □

If $[\mathfrak{g}, \mathfrak{g}]$ is not simple, then the following assumption on the choice of the scalar product on $\mathfrak{g}$ is made in order to simplify the calculations.

(♦) There is a constant $C \in \mathbb{C}$ such that $\sum_{i=1}^{\dim \mathfrak{g}} \mathrm{ad}(x_i)^2(\xi) = C\xi$ for each $\xi \in [\mathfrak{g}, \mathfrak{g}]$.

The constant C depends on the scalar product in question.

From now on, assume that $\mathfrak{g}$ is semisimple. As the next step we examine the difference

$$X_{F[-1]} := \mathbb{X}_{F[-1]} - \varpi(\{\mathcal{H}[-1], F[-1]\})$$

and more general expressions, where the commutator is taken with $\mathcal{H}[\bar{b}]$. Our goal is to present $\mathbb{X}_{F[-1]}$ in the form (0.1). For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{U}(\hat{\mathfrak{g}}^-)$, the symbol $\mathrm{gr}([\mathcal{F}_1, \mathcal{F}_2])$ is equal to the Poisson bracket $\{\mathrm{gr}(\mathcal{F}_1), \mathrm{gr}(\mathcal{F}_2)\}$ if this Poisson bracket is non-zero.

3.1 Commutators, the first approximation

Fix $m \geqslant 1$. Then set $\check{j} = m - j$ for $1 \leqslant j < m$.

LEMMA 3.2. *For $Y = \hat{y}_1 \ldots \hat{y}_m \in \mathcal{S}(\hat{\mathfrak{g}}^-)$ with $\hat{y}_j = y_j[a_j]$, set*

$$X_Y = X_{Y,\bar{b}} = [\mathcal{H}[b_1, b_2], \varpi(Y)] - \varpi(\{\mathcal{H}[b_1, b_2], Y\}).$$

Then

$$\begin{aligned} X_Y = &\sum_{l=1}^{m} \sum_{\sigma \in \tilde{\mathsf{S}}_{m-1}} \sum_{j<p} \sum_{i,u} (\mathsf{m}(y_{\sigma(p)} \otimes y_l \otimes y_{\sigma(j)})(x_i), x_u) \\ &\times (c_{2,3}(j,p) \hat{y}_{\sigma(1)} \ldots \hat{y}_{\sigma(j-1)} x_i[b_1 + a_{\sigma(j)}] \hat{y}_{\sigma(j+1)} \ldots \hat{y}_{\sigma(p-1)} x_u[b_2 + a_l + a_{\sigma(p)}] \hat{y}_{\sigma(p+1)} \ldots \hat{y}_{\sigma(m-1)} \\ &+ c_{2,3}(j,p) \hat{y}_{\sigma(1)} \ldots \hat{y}_{\sigma(j-1)} x_i[b_2 + a_{\sigma(j)}] \hat{y}_{\sigma(j+1)} \ldots \hat{y}_{\sigma(p-1)} x_u[b_1 + a_l + a_{\sigma(p)}] \hat{y}_{\sigma(p+1)} \ldots \hat{y}_{\sigma(m-1)} \\ &+ (-1) c_{3,2}(j,p) \hat{y}_{\sigma(1)} \ldots \hat{y}_{\sigma(j-1)} x_i[b_1 + a_{\sigma(j)} + a_l] \hat{y}_{\sigma(j+1)} \ldots \hat{y}_{\sigma(p-1)} x_u[b_2 + a_{\sigma(p)}] \hat{y}_{\sigma(p+1)} \ldots \hat{y}_{\sigma(m-1)} \\ &+ (-1) c_{3,2}(j,p) \hat{y}_{\sigma(1)} \ldots \hat{y}_{\sigma(j-1)} x_i[b_2 + a_{\sigma(j)} + a_l] \hat{y}_{\sigma(j+1)} \ldots \hat{y}_{\sigma(p-1)} x_u[b_1 + a_{\sigma(p)}] \hat{y}_{\sigma(p+1)} \ldots \hat{y}_{\sigma(m-1)}), \end{aligned}$$

where $\tilde{\mathsf{S}}_{m-1}$ stands for the set of bijective maps from $\{1, \ldots, m-1\}$ to $\{1, \ldots, m\} \backslash \{l\}$ and we have $1 \leqslant j < p \leqslant m-1$. The constants $c_{2,3}(j,p)$, $c_{3,2}(j,p) \in \mathbb{Q}$ do not depend on Y, they depend only on m. Besides,

$$c_{2,3}(j,p) = c_{3,2}(\check{p}, \check{j}),$$

$c_{2,3}(j,p) \leqslant 0$ for all $j < p$, and $c_{2,3}(j,p) < 0$ if in addition $\check{j} \geqslant p$.

Proof. Set $\hat{Y} = \hat{y}_1 \ldots \hat{y}_m \in \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$. Let $\hat{x}_i^{(1)}$ stand for $x_i[b_1]$ and $\hat{x}_i^{(2)}$ for $x_i[b_2]$. Then

$$[\mathcal{H}[b_1, b_2], \hat{Y}] = \sum_{j=1,i=1}^{j=m,i=\dim \mathfrak{g}} (\hat{y}_1 \ldots \hat{y}_{j-1} \hat{x}_i^{(1)} [\hat{x}_i^{(2)}, \hat{y}_j] \hat{y}_{j+1} \ldots \hat{y}_m + \hat{y}_1 \ldots \hat{y}_{j-1} [\hat{x}_i^{(1)}, \hat{y}_j] \hat{x}_i^{(2)} \hat{y}_{j+1} \ldots \hat{y}_m).$$

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Furthermore,

$$[\mathcal{H}[b_1,b_2],\varpi(Y)] = \frac{1}{m!}\sum_{\sigma\in S_m}[\mathcal{H}[b_1,b_2],\sigma(\hat{Y})]. \tag{3.2}$$

Each summand of $[\mathcal{H}[b_1,b_2],\sigma(\hat{Y})]$ we regard as a formal non-commutative product. The symmetrisation of $P_Y = \{\mathcal{H}[b_1,b_2],Y\}$ resembles (3.2), but with a rather significant difference: the factor $\hat{x}_i^{(\upsilon)}$, which is not involved in $[\hat{x}_i^{(\nu)},\hat{y}_{\sigma(j)}]$, does not have to stay next to $[\hat{x}_i^{(\nu)},\hat{y}_{\sigma(j)}]$ (here we have $\{\nu,\upsilon\}=\{1,2\}$). The idea behind the computation of X_Y is to modify each term of $\varpi(P_Y)$ in such a way that the *wayward* factor $\hat{x}_i^{(\upsilon)}$ gets back to its place as in $[\mathcal{H}[b_1,b_2],\varpi(Y)]$. In this process, other commutators $\pm[\hat{x}_i^{(\upsilon)},\hat{y}_l]$ will appear. It is convenient to consider the differences

$$X_{\sigma(\hat{Y})} = [\mathcal{H}[b_1,b_2],\sigma(\hat{Y})] - \frac{1}{m+1}\sum_{j=1}^{m}\sum_{\upsilon=1}^{2}\overbrace{\hat{y}_{\sigma(1)}\cdots\hat{y}_{\sigma(j-1)}[\hat{x}_i^{(\nu)},\hat{y}_{\sigma(j)}]\hat{y}_{\sigma(j+1)}\cdots\hat{y}_{\sigma(m-1)}\hat{y}_{\sigma(m)}}^{\hat{x}_i^{(\upsilon)}\text{ at different places}},$$

where for each fixed j and υ, we add $m+1$ different formal products with $\hat{x}_i^{(\upsilon)}$ standing in $m+1$ different places. Then $X_Y = (1/m!)\sum_{\sigma\in S_m} X_{\sigma(\hat{Y})}$.

While modifying $\varpi(P_Y)$, one obtains products of the form

$$X(\sigma,i,\upsilon,\nu;j,p) = \hat{y}_{\sigma(1)}\ldots\hat{y}_{\sigma(j-1)}[\hat{x}_i^{(\upsilon)},\hat{y}_{\sigma(j)}]\hat{y}_{\sigma(j+1)}\ldots\hat{y}_{\sigma(p-1)}[\hat{x}_i^{(\nu)},\hat{y}_{\sigma(p)}]\hat{y}_{\sigma(p+1)}\ldots\hat{y}_{\sigma(m)}$$

with some coefficients; one also has to commute $\hat{x}_i^{(\upsilon)}$ with $[\hat{x}_i^{(\nu)},\hat{y}_j]$. The total sum of the products that correspond to $X_{\hat{Y}}$ and contain a double commutator $[\hat{x}_i^{(\upsilon)},[\hat{x}_i^{(\nu)},\hat{y}_j]]$ as a factor is

$$X_{\mathrm{dcom}}(\mathrm{id},j,i) := \frac{1}{m+1}\hat{y}_1\ldots\hat{y}_{j-1}(j[\hat{x}_i^{(2)},[\hat{x}_i^{(1)},\hat{y}_j]] + (m-j+1)[[\hat{x}_i^{(2)},\hat{y}_j],\hat{x}_i^{(1)}])\hat{y}_{j+1}\ldots\hat{y}_m.$$

Observe that

$$[\hat{x}_i^{(2)},[\hat{x}_i^{(1)},\hat{y}_j]] + [[\hat{x}_i^{(2)},\hat{y}_j],\hat{x}_i^{(1)}] = [[\hat{x}_i^{(2)},\hat{x}_i^{(1)}],\hat{y}_j] = 0.$$

Applying σ to all $\hat{y}_p$ with $1\leqslant p\leqslant m$, we obtain $X_{\mathrm{dcom}}(\sigma,j,i)$ from $X_{\mathrm{dcom}}(\mathrm{id},j,i)$. Set $X_{\mathrm{dcom}} = \sum_{j=1}^{m}\sum_{\sigma\in S_m}\sum_i X_{\mathrm{dcom}}(\sigma,j,i)$. We have $\mathrm{gr}(X_{\mathrm{dcom}})=0$, since

$$\mathrm{gr}(X_{\mathrm{dcom}}(\mathrm{id},j,i)) + \mathrm{gr}(X_{\mathrm{dcom}}(\vartheta,m-j+1,i)) = 0$$

for the transposition $\vartheta = (j(m-j+1))$.

Now consider expressions $[[\hat{x}_i^{(\upsilon)},[\hat{x}_i^{(\nu)},\hat{y}_j]],\hat{y}_l]$. In view of (♦),

$$\sum_i \mathrm{ad}(\hat{x}_i^{(1)})\mathrm{ad}(\hat{x}_i^{(2)})(y_j[a_j]) = \sum_i \mathrm{ad}(\hat{x}_i^{(2)})\mathrm{ad}(\hat{x}_i^{(1)})(y_j[a_j]) = Cy_j[a_j+b],$$

where $b=b_1+b_2$. This leads to

$$\sum_i[[\hat{x}_i^{(\upsilon)},[\hat{x}_i^{(\nu)},\hat{y}_j]],\hat{y}_l] = [Cy_j[a_j+b],\hat{y}_l] = C[y_j,y_l][a_j+a_l+b]. \tag{3.3}$$

Thus $\sum_i X_{\mathrm{dcom}}(\mathrm{id},j,i) - \varpi(\mathrm{gr}(\sum_i X_{\mathrm{dcom}}(\mathrm{id},j,i))) = C\sum_{l\neq j} c_{(j,l)} X_{\mathrm{dcom}}^{[j,l]}(\mathrm{id})$, where

$$X_{\mathrm{dcom}}^{[j,l]}(\mathrm{id}) = \begin{cases}\hat{y}_1\ldots\hat{y}_{l-1}[y_j,y_l][a_j+a_l+b]\hat{y}_{l+1}\ldots\hat{y}_{j-1}\hat{y}_{j+1}\ldots\hat{y}_m & \text{if } l<j\\ \hat{y}_1\ldots\hat{y}_{j-1}\hat{y}_{j+1}\ldots\hat{y}_{l-1}[y_j,y_l][a_j+a_l+b]\hat{y}_{l+1}\ldots\hat{y}_m & \text{if } l>j;\end{cases}$$

furthermore $c_{(j,l)}\in\mathbb{Q}$. Set $\sigma=(j\,l)$. The difference

$$\sum_i X_{\mathrm{dcom}}(\sigma,j,i) - \varpi\left(\mathrm{gr}\left(\sum_i X_{\mathrm{dcom}}(\sigma,j,i)\right)\right)$$

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has a summand

$$Cc_{(j,l)}X^{[j,l]}_{\rm dcom}(\sigma) = -Cc_{(j,l)}X^{[j,l]}_{\rm dcom}({\rm id}).$$

This proves that $\sum_{\sigma\in\mathtt{S}_m,j,i} X_{\rm dcom}(\sigma,j,i)=0$, i.e. the expressions containing double commutators $[\hat x_i^{(\upsilon)},[\hat x_i^{(\nu)},\hat y_j]]$ as factors have no contribution to X_Y.

In the modification of $(m+1)X_{\hat Y}$, a term

$$X({\rm id},i,\upsilon,\nu;j,l) = \hat y_1\ldots\hat y_{j-1}[\hat x_i^{(\upsilon)},\hat y_j]\hat y_{j+1}\ldots\hat y_{l-1}[\hat x_i^{(\nu)},\hat y_l]\hat y_{l+1}\ldots\hat y_m$$

appears j times with the coefficient 1 (these are the instances, where $\hat x_i^{(\upsilon)}$ is the wayward factor); it also appears $(m-l+1)$ times with the coefficient (-1) from the instances, where $\hat x_i^{(\nu)}$ is the wayward factor. Thereby

$$X_{\sigma(\hat Y)} = \frac{1}{m+1}\sum_{j<l}\sum_{\upsilon}\sum_{i}(j+l-m-1)X(\sigma,i,\upsilon,\nu;j,l).$$

Set $j'=m-j+1$. Then $j+j'=m+1$. Assume that $l\ne j'$ and $l>j$. For any $\sigma\in\mathtt{S}_m$, the symbol of $X(\sigma,i,\upsilon,\nu;j,l)$ is the same as the symbol of $X(\tilde\sigma,i,\upsilon,\nu;l',j')$, where

$$\tilde\sigma(\hat Y) = \hat y_{\sigma(1)}\ldots\hat y_{\sigma(l'-1)}\hat y_{\sigma(j)}\hat y_{\sigma(l'+1)}\ldots\hat y_{\sigma(j'-1)}\hat y_{\sigma(l)}\hat y_{\sigma(j'+1)}\ldots\hat y_{\sigma(m)};$$

furthermore, these two expressions appear in X_Y with opposite coefficients. This shows that $X_Y\in\mathcal U_{m-1}(\hat{\mathfrak g}^-)$. In order to get a better understanding of X_Y, we modify the terms $X(\sigma,i,\upsilon,\nu;j,l)$, which we consider as formal products.

Each $X(\sigma,i,\upsilon,\nu;j,l)$ has factors of two sorts: elements $\hat y_p$ (depicted as points in the diagram below) and two commutators $[\hat x_i^{(\upsilon)},\hat y_{\sigma(j)}]$, $[\hat x_i^{(\upsilon)},\hat y_{\sigma(l)}]$, which are depicted as stars. We move the commutator that is closer to the middle point of the product until the expression obtains a central symmetry. In the case $l\leqslant m/2$, this looks as follows:

$$\cdots\star\cdots\cdots\star\cdots\,|\,\cdots\cdots\cdots\cdots\cdots\cdots \overset{\text{modification}}{\rightsquigarrow} \cdots\star\cdots\cdots\cdots\cdots\,|\,\cdots\cdots\cdots\cdots\star\cdots\,.$$

See also Example 3.3 below. The commutator $[\hat x_i^{(\upsilon)},\hat y_{\sigma(l)}]$ is moving if and only if $l<j'$. After the modification, the products of m factors annihilate each other and X_Y is now a $\mathbb Q$-linear combination of products of $m-1$ factors, where in each term, $m-3$ factors are elements $\hat y_w$, one is a commutator $[\hat x_i^{(\upsilon)},\hat y_j]$, and another one is a commutator $[[\hat x_i^{(\nu)},\hat y_l],\hat y_p]$. A possible example in the case $\sigma={\rm id}$, is

$$\hat y_1\ldots\hat y_{j-1}[\hat x_i^{(\upsilon)},\hat y_j]\hat y_{j+1}\ldots\hat y_{l-1}\hat y_{l+1}\ldots\hat y_{p-1}[[\hat x_i^{(\nu)},\hat y_l],\hat y_p]\hat y_{p+1}\ldots\hat y_m.$$

It appears only if $l<j'$. First we deal with these expressions ‘qualitative’ and after that describe the coefficients.

Observe that for $y\in\mathfrak g$ and $a\in\mathbb Z_{<0}$, we have $y[a]=\sum_i(x_i,y)x_i[a]$. Assume for simplicity that $\{j,p,l\}=\{1,2,3\}$, disregard for the moment the other factors, and ignore the t-degrees of the elements. Consider the sum

$$\begin{aligned}\sum_i[x_i,y_1][y_3,[y_2,x_i]] &= \sum_{i,j,u}([x_i,y_1],x_j)x_j([y_3,[y_2,x_i]],x_u)x_u\\ &= \sum_{i,j,u}(x_i,[y_1,x_j])x_j(x_i,{\rm ad}(y_2){\rm ad}(y_3)(x_u))x_u\end{aligned}$$

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$$= \sum_{i,j,u} ((\mathrm{ad}(y_2)\mathrm{ad}(y_3)(x_u), x_i)x_i, [y_1, x_j])x_j x_u$$
$$= \sum_{j,u} (\mathrm{ad}(y_2)\mathrm{ad}(y_3)(x_u), [y_1, x_j])x_j x_u$$
$$= \sum_{j,u} (\mathrm{ad}(y_3)\mathrm{ad}(y_2)\mathrm{ad}(y_1)(x_j), x_u)x_j x_u. \tag{3.4}$$

Note that $\mathrm{ad}(y_3)\mathrm{ad}(y_2)\mathrm{ad}(y_1)(x_j) = \mathsf{m}(y_3 \otimes y_2 \otimes y_1)(x_j)$. If we recall the t-degrees, then the product $x_j x_u$ has to be replaced with $x_j[b_\upsilon + a_1]x_u[b_\nu + a_2 + a_3]$ in (3.4). The other factors $\hat{y}_w$ do not interfere with the transformations in (3.4).

In the process of changing the sequence of factors of $X(\mathrm{id}, i, \upsilon, \nu; j, l)$ with $j < l < j'$, the term $\cdots [\hat{x}_i^{(\upsilon)}, \hat{y}_j] \cdots [[\hat{x}_i^{(\nu)}, \hat{y}_l], \hat{y}_p] \cdots$ appears with the negative coefficient $(j + l - m - 1)$ as long as $l < p \leqslant j'$. This shows that indeed the constants $c_{2,3}(j,p)$ do not depend on Y, they depend only on m. Moreover, $c_{2,3}(j,p) = 0$ if $p > \check{j}$ and $c_{2,3}(j,p) < 0$ if $p \leqslant \check{j}$.

The symmetry $c_{2,3}(j,p) = c_{3,2}(\check{p}, \check{j})$ is justified by the fact that $\omega(X_Y) = (-1)^{m-1}X_Y$. A more direct way to see this, is to notice that if a factor $[\hat{x}_i^{(\nu)}, \hat{y}_l]$ moves from a place v to j' in some term, then $j < v < j'$ and there is a term with the apposite coefficient, where $[\hat{x}_i^{(\nu)}, \hat{y}_l]$ moves from v' to j. The first type of moves produces

$$(\text{coeff.})(\mathsf{m}(y_{\sigma(p)} \otimes y_l \otimes y_{\sigma(j)})(x_i), x_u) \ldots x_i[b_\upsilon + a_{\sigma(j)}] \ldots x_u[b_\nu + a_l + a_{\sigma(p)}] \ldots$$

and the second

$$(\text{the same coeff.})(x_u, \mathsf{m}(y_{\sigma(p')} \otimes y_l \otimes y_{\sigma(j')})(x_i)) \ldots x_u[b_\nu + a_l + a_{\sigma(p')}] \ldots x_i[b_\upsilon + a_{\sigma(j')}] \ldots.$$

We have $(x_i, \mathsf{m}(y_{\sigma(j)} \otimes y_l \otimes y_{\sigma(p)})(x_u)) = -(\mathsf{m}(y_{\sigma(p)} \otimes y_l \otimes y_{\sigma(j)})(x_i), x_u)$ and the scalar product $(\ ,\ )$ is symmetric. These facts confirm the symmetry of the constants and justifies the minus signs in front of $c_{3,2}(j,p)$ in the answer. □

Example 3.3. Consider the case $m = 6$. One obtains that

$$\begin{aligned} X_Y = \frac{1}{7!} \sum_{\sigma \in \mathsf{S}_6, i} & (4X(\sigma,i,1,2;5,6) - 4X(\sigma,i,1,2;1,2) + 3X(\sigma,i,1,2;4,6) - 3X(\sigma,i,1,2;1,3) \\ & + 2X(\sigma,i,1,2;3,6) - 2X(\sigma,i,1,2;1,4) + 2X(\sigma,i,1,2;4,5) - 2X(\sigma,i,1,2;2,3) \\ & + X(\sigma,i,1,2;2,6) - X(\sigma,i,1,2;1,5) + X(\sigma,i,1,2;3,5) - X(\sigma,i,1,2;2,4)) \\ & + (\text{the similar expression for } (\upsilon,\nu) = (2,1)). \end{aligned}$$

Take $\sigma = \mathrm{id}$. Performing the modification

$$X(\mathrm{id}, i, 1, 2; 5, 6) = \hat{y}_1\hat{y}_2\hat{y}_3\hat{y}_4[\hat{x}_i^{(1)}, \hat{y}_5][\hat{x}_i^{(2)}, \hat{y}_6] \overset{\text{modification}}{\rightsquigarrow} [\hat{x}_i^{(1)}, \hat{y}_5]\hat{y}_1\hat{y}_2\hat{y}_3\hat{y}_4[\hat{x}_i^{(2)}, \hat{y}_6],$$

we move the factor $[\hat{x}_i^{(1)}, \hat{y}_5]$ to the first place producing commutators with $\hat{y}_4$, $\hat{y}_3$, $\hat{y}_2$, $\hat{y}_1$ on the way. In the same manner,

$$X(\mathrm{id}, i, 1, 2; 1, 2) = [\hat{x}_i^{(1)}, \hat{y}_1][\hat{x}_i^{(2)}, \hat{y}_2]\hat{y}_3\hat{y}_4\hat{y}_5\hat{y}_6 \overset{\text{modification}}{\rightsquigarrow} [\hat{x}_i^{(1)}, \hat{y}_1]\hat{y}_3\hat{y}_4\hat{y}_5\hat{y}_6[\hat{x}_i^{(2)}, \hat{y}_2]$$

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and for $\tilde\sigma=(1\,5\,3)(2\,6\,4)$,

$$X(\tilde\sigma,i,1,2;1,2)=[\hat x_i^{(1)},\hat y_5][\hat x_i^{(2)},\hat y_6]\hat y_1\hat y_2\hat y_3\hat y_4 \overset{\text{modification}}{\rightsquigarrow} [\hat x_i^{(1)},\hat y_5]\hat y_1\hat y_2\hat y_3\hat y_4[\hat x_i^{(2)},\hat y_6].$$

Performing the modification

$$X(\mathrm{id},i,1,2;4,5)=\hat y_1\hat y_2\hat y_3[\hat x_i^{(1)},\hat y_4][\hat x_i^{(2)},\hat y_5]\hat y_6 \overset{\text{modification}}{\rightsquigarrow} \hat y_1[\hat x_i^{(1)},\hat y_4]\hat y_2\hat y_3[\hat x_i^{(2)},\hat y_5]\hat y_6,$$

we move the factor $[\hat x_i^{(1)},\hat y_4]$ to the second place producing commutators with $\hat y_3$ and $\hat y_2$. The non-zero constants $c^-(j,p):=-c_{2,3}(j,p)$ are

$$c^-(1,2)=\frac{4}{7!},\quad c^-(1,3)=\frac{7}{7!},\quad c^-(1,4)=\frac{9}{7!},\quad c^-(1,5)=\frac{10}{7!},\quad c^-(2,3)=\frac{2}{7!},\quad c^-(2,4)=\frac{3}{7!}.$$

Instead of the usual symmetrisation map, one can consider a *weighted* 'symmetrisation' or rather shuffle, where each permutation is added with a scalar coefficient assigned by a certain function Ψ. We will need only a very particular case of this construction. Let $\Psi\colon \mathsf{S}_{k+2}\to\mathbb{Q}$ be a *weight function*, satisfying the following assumptions:

(A) $\Psi(\sigma)$ depends only on $j=\sigma(k+1)$ and $p=\sigma(k+2)$, i.e. $\Psi(\sigma)=\Psi(j,p)$;
(B) $\Psi(j,p)=\Psi(j',p')$ if $j'=k+3-j$.

Then set

$$\varpi_{\rm wt}(y_1\dots y_k\otimes y_{k+1}\otimes y_{k+2})=\sum_{\sigma\in\mathsf{S}_{k+2}}\Psi(\sigma)\,y_{\sigma(1)}\otimes\dots\otimes y_{\sigma(k+2)}$$

for $y_j\in\mathfrak q$. Let also $\varpi_{\rm wt}$ stand for the corresponding map from $\mathcal{S}^k(\mathfrak q)\otimes\mathfrak q\otimes\mathfrak q$ to $\mathcal{U}(\mathfrak q)$. Condition (B) guaranties that $\omega(\varpi_{\rm wt}(F))=(-1)^k\varpi_{\rm wt}(F)$ for each $F\in\mathcal{S}^k(\mathfrak q)\otimes\mathfrak q\otimes\mathfrak q$. In the case $\Psi(\sigma)=1/(k+2)!$, the map $\varpi_{\rm wt}$ coincides with ϖ. Keep in mind that each appearing $\varpi_{\rm wt}$ may have its own weight function.

Suppose that $Y\in\mathcal{S}^m(\mathfrak g)$, $\bar a\in\mathbb{Z}^m_{<0}$, and we want to merge them in order to obtain an element of $\mathcal{S}^m(\hat{\mathfrak g}^-)$. The only canonical way to do so is to replace $\bar a$ with the orbit $\mathsf{S}_m\bar a$, add over this orbit, and divide by $|\mathsf{S}_m\bar a|$ as we have done in § 1.5. The result is the $\bar a$-polarisation $Y[\bar a]$ of Y. Set

$$X_{Y[\bar a]}=[\mathcal{H}[b_1,b_2],\varpi(Y[\bar a])]-\varpi(\{\mathcal{H}[b_1,b_2],Y[\bar a]\}).$$

For different numbers $u,v,l\in\{1,\dots,m\}$, let $\bar a^{u,v,l}\in\mathbb{Z}^{m-3}_{<0}$ be the vector obtained from $\bar a$ by removing a_u, a_v, and a_l. Let $\langle u,v,l\rangle$ be a triple such that $l<v$ and $u\ne l,v$. Write $\mathsf{m}(Y)=\sum_{w=1}^L\xi_w\otimes R_w$ with $\xi_w\in\Lambda^2\mathfrak g$, $R_w\in\mathcal{S}^{m-3}(\mathfrak g)$.

Proposition 3.4. *The element $X_{Y[\bar a]}$ is equal to*

$$\sum_w\sum_{\langle u,v,l\rangle}\sum_v\sum_{i,j}(\xi_w(x_i),x_j)\varpi_{\rm wt}(R_w[\bar a^{u,v,l}]\otimes x_i[b_\upsilon+a_u]\otimes x_j[b_\nu+a_l+a_v]),$$

where $\Psi(j,p)=2c_{2,3}(j,p)$ if $j<p$ and $\Psi(j,p)=2c_{3,2}(p,j)$ if $j>p$ for the weight function Ψ.

Proof. Using the linearity, we may assume that $Y=y_1\dots y_m$. The symmetry in t-degrees allows one to add the expressions appearing in the formulation of Lemma 3.2 over the triples $(y_e[a_{\sigma(p)}],y_f[a_l],e_g[a_{\sigma(j)}])$ with $\{e,f,g\}=\{\sigma(p),l,\sigma(j)\}$ while keeping $x_i[b_\upsilon+a_{\sigma(j)}]$, $x_u[b_\nu+a_l+a_{\sigma(p)}]$ and $x_i[b_\upsilon+a_{\sigma(j)}+a_l]$, $x_u[b_\nu+a_{\sigma(p)}]$ at their places. In this way the coefficient $\mathsf{m}(y_{\sigma(p)}\otimes y_l\otimes y_{\sigma(j)})$ is replaced with $\mathsf{m}(y_{\sigma(p)}y_ly_{\sigma(j)})$ and thereby ξ_w with

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$1 \leqslant w \leqslant L$ come into play. It remains to count the scalars and describe the weight function.

Suppose that $j < p$. Then

$$\frac{2}{m!} 3! c_{2,3}(j,p) = \frac{3!}{m!} \Psi(j,p)$$

and thereby $\Psi(j,p) = 2c_{2,3}(j,p)$. Analogously, $\Psi(p,j) = 2c_{3,2}(j,p)$. □

Theorem 3.5. *For $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$ with $m \geqslant 4$, the symmetrisation $\varpi(F)[-1]$ is an element of the Feigin–Frenkel centre if and only if $\mathsf{m}(F) = 0$.*

Proof. According to [Ryb08], $\varpi(F)[-1] \in \mathfrak{z}(\hat{\mathfrak{g}})$ if and only if it commutes with $\mathcal{H}[-1]$. In view of Lemma 3.1, this is the case if and only if $X_{F[-1]} = 0$. Lemma 3.2 describes this element. It states that $c_{2,3}(j,p), c_{3,2}(j,p) \leqslant 0$ and $c_{2,3}(j,p) < 0$ if $p \leqslant \check{j}$ as well as $c_{3,2}(j,p) < 0$ if $\check{p} \leqslant j$. Since $\varpi(F)[-1]$ is fully symmetrised, we can use Proposition 3.4. It immediately implies that if $\mathsf{m}(F) = 0$, then $X_{F[-1]} = 0$.

Suppose that $\mathsf{m}(F) \neq 0$. Write $\mathsf{m}(F) = \sum_{w=1}^{L} \xi_w \otimes R_w$ with $\xi_w \in \Lambda^2\mathfrak{g}$ and linearly independent $R_w \in \mathcal{S}^{m-3}(\mathfrak{g})$. If $\xi \in \Lambda^2\mathfrak{g}$ is non-zero, then there are i, j such that $(\xi(x_i), x_j) \neq 0$.

Set $\boldsymbol{c} = 2\sum_{j<p}(c_{2,3}(j,p) + c_{3,2}(j,p))$. According to Lemma 3.2, $\boldsymbol{c} < 0$. Hence

$$\frac{m!}{(m-3)!} \boldsymbol{c} \sum_{w=1}^{w=L} \sum_{i,j} (\xi_w(x_i), x_j) x_i[-2] x_j[-3] R_w[-1]$$

is a non-zero element of $\mathcal{S}(\hat{\mathfrak{g}}^-)$. In view of the same lemma, this expression is equal to $\mathrm{gr}(X_{F[-1]})$. Thus $X_{F[-1]} \neq 0$. This completes the proof. □

Remark. If $\mathfrak{g}$ is simple, then $\mathfrak{g}^{\mathfrak{g}}$ is equal to zero and $\mathcal{S}^2(\mathfrak{g})^{\mathfrak{g}}$ is spanned by $\mathcal{H} = \sum_i x_i^2$. By our convention, $\mathsf{m}(\mathcal{S}^m(\mathfrak{g})) = 0$ if $m \leqslant 2$. Furthermore, $\mathsf{m}(\mathcal{S}^3(\mathfrak{g})^{\mathfrak{g}}) \subset (\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$. Therefore Theorem 3.5 holds for $m \leqslant 3$ as well.

We will be using weighted shuffles ϖ_{wt} of Poisson *half-brackets*. If $Y = \hat{y}_1 \dots \hat{y}_m \in \mathcal{S}(\hat{\mathfrak{g}}^-)$, then

$$\varpi_{\mathrm{wt}}(Y, b_1, b_2) := \sum_{j=1,i=1}^{j=m,i=\dim\mathfrak{g}} \varpi_{\mathrm{wt}}(Y/\hat{y}_j \otimes x_i[b_1] \otimes [x_i[b_2], \hat{y}_j]). \tag{3.5}$$

Strictly speaking, here ϖ_{wt} is a linear map from $\mathcal{S}^m(\hat{\mathfrak{g}}^-)$ in $\mathcal{U}(\hat{\mathfrak{g}}^-)$ depending on (b_1, b_2) and the choice of a weight function Ψ. The absence of wt in the lower index indicates that we are taking the usual symmetrisation.

3.2 Iterated shuffling

Another general fact about Lie algebras $\mathfrak{q}$ will be needed. Suppose that $Y = y_1 \dots y_m \in \mathcal{S}^m(\mathfrak{q})$ and $x \in \mathfrak{q}$. Write $Y = (1/m)\sum_{1\leqslant j\leqslant m} y_j \otimes Y^{(j)}$ with $Y^{(j)} = Y/y_j$. Then

$$\sum_{1\leqslant j\leqslant m} [x, y_j] Y^{(j)} = \{x, Y\}. \tag{3.6}$$

Proposition 3.6. *Let $\mathcal{F}[\check{a}] = \varpi(F[\bar{a}]) \in \mathcal{U}(\hat{\mathfrak{g}}^-)$ be a fully symmetrised element corresponding to a polynomial $F \in \mathcal{S}^m(\mathfrak{g})$ and a vector $\bar{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m_{<0}$. Suppose that $\mathsf{m}_{2r+1}(F) \in \mathcal{S}^{m-2r}(\mathfrak{g})$*

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for all $m/2 > r \geqslant 1$. Then:

(i) $X_{F[\bar a]} = [\mathcal{H}[\bar b], \mathcal{F}[\breve a]] - \varpi(\{\mathcal{H}[\bar b], F[\bar a]\})$ *is a sum of weighted symmetrisations*

$$\varpi_{\rm wt}(\mathsf{m}(F)[\bar a^{l,j}], b_1 + a_l, b_2 + a_j), \quad \varpi_{\rm wt}(\mathsf{m}(F)[\bar a^{l,j}], b_2 + a_l, b_1 + a_j),$$

where $l \neq j$ and $\bar a^{l,j}$ is obtained from $\bar a$ by removing a_l and a_j;

(ii) *for every weight function Ψ, there is a constant $c \in \mathbb{Q}$, which is independent of F, such that $\mathcal{P}_{F[\bar a]} = \varpi_{\rm wt}(F[\bar a], b_1, b_2) - c\varpi(F[\bar a], b_1, b_2)$ is a sum of $\varpi_{\rm wt}(\mathsf{m}(F)[\bar a^{(1)}], b_\upsilon + \gamma_\upsilon, b_\nu + \gamma_\nu)$ with different weight functions, whereby $\bar a^{(1)}$ is a subvector of $\bar a$ with $m-2$ entries and $\bar\gamma \in \mathbb{Z}^2_{<0}$ is constructed from the complement $\bar a \backslash \bar a^{(1)}$ of $\bar a^{(1)}$;*

(iii) $\mathbb{X}_{F[a]} = [\mathcal{H}[b_1, b_2], \mathcal{F}[\breve a]]$ *is a sum of*

$$C(\bar a^{(r)}, \bar\gamma)\varpi(\mathsf{m}_{2r+1}(F)[\bar a^{(r)}], b_\upsilon + \gamma_\upsilon, b_\nu + \gamma_\nu),$$

where $0 \leqslant r < m/2$, $\bar a^{(r)}$ is a subvector of $\bar a$ with $m - 2r$ entries, $\bar\gamma \in \mathbb{Z}^2_{<0}$ is constructed from $\bar a \backslash \bar a^{(r)}$, and the coefficients $C(\bar a^{(r)}, \bar\gamma) \in \mathbb{Q}$ are independent of F.

Proof. Since we are working with a fully symmetrised element, Proposition 3.4 applies. In the same notation, write $\mathsf{m}(F) = \sum_{w=1}^{L} \xi_w \otimes R_w$. By our assumptions, $\mathsf{m}(F) \in \mathcal{S}^{m-2}(\mathfrak{g})$. In particular, $\xi_w \in \mathfrak{g}$ for each w. Observe that

$$\sum_{i,j} (\xi_w(x_i), x_j) x_i[b_\upsilon] x_j[b_\nu] = \sum_{i,j} x_i[b_\upsilon]([\xi_w, x_i], x_j) x_j[b_\nu] = \sum_i x_i[b_\nu][\xi_w, x_i[b_\upsilon]].$$

Thereby part (i) follows from Proposition 3.4 in view of (3.6).

(ii) Note that $\omega(\mathcal{P}_{F[\bar a]}) = (-1)^{m+1}\mathcal{P}_{F[\bar a]}$, because of the assumption (B) imposed on all weight functions. By the construction, the image of $\varpi_{\rm wt}(F[\bar a], b_1, b_2)$ in $\mathcal{S}^{m+1}(\hat{\mathfrak{g}}^-)$ is equal to $c\sum_i \{x_i[b_2], F[\bar a]\} x_i[b_1]$ for some $c \in \mathbb{Q}$. This constant c depends only on the weight function Ψ. For this c, we have $\deg \mathrm{gr}(\mathcal{P}_{F[\bar a]}) \leqslant m$.

The element $\mathcal{P}_{F[\bar a]}$ is a linear combination of products, where each product contains $m+1$ linear factors. Let us symmetrise the summands of $\mathcal{P}_{F[\bar a]}$ by changing the sequence of factors in them. Note that there is no need to commute factors $\hat y_j = y_j[a_{\sigma(j)}]$ and $\hat y_l = y_l[a_{\sigma(l)}]$, since $\mathcal{P}_{F[\bar a]}$ is symmetric in the $\hat y_p$. There is no sense in commuting $\hat x_i^{(1)}$ and $\hat x_i^{(2)}$ either. After this symmetrisation all products of $m+1$ factors annihilate each other and $\mathcal{P}_{F[\bar a]}$ becomes a linear combination of products containing m factors. Now we symmetrise these new products. Because of the antipode symmetry, they disappear after the symmetrisation and now $\mathcal{P}_{F[\bar a]}$ is a linear combination of products containing $m-1$ factors. Furthermore, each non-zero summand must contain certain factors according to one of the types listed below:

(1) $[\hat x_i^{(\nu)}, y_j]$ and $[[\hat x_i^{(\upsilon)}, \hat y_l], \hat y_p]$;
(2) $[\hat y_p, [\hat y_l, [\hat y_j, \hat x_i^{(\upsilon)}]]] = \mathrm{ad}(y_p)\mathrm{ad}(y_l)\mathrm{ad}(y_j)(x_i[b_\upsilon + a_{\sigma(p)} + a_{\sigma(l)} + a_{\sigma(j)}])$ and $\hat x_i^{(\nu)}$;
(3) $[[\hat y_p, \hat x_i^{(\upsilon)}], [\hat y_l, \hat x_i^{(\nu)}]] = [[y_p, x_i], [y_l, x_i]][a_{\sigma(p)} + a_{\sigma(l)} + b_1 + b_2]$;
(4) $[\hat y_p, [\hat x_i^{(\upsilon)}, [\hat x_i^{(\nu)}, \hat y_j]]]$;
(5) $[\hat x_i^{(\upsilon)}, [\hat y_p, [\hat y_j, \hat x_i^{(\nu)}]]] = [[\hat x_i^{(\upsilon)}, \hat y_p], [\hat y_j, \hat x_i^{(\nu)}]] - [\hat y_p, [\hat x_i^{(\upsilon)}, [\hat x_i^{(\nu)}, \hat y_j]]]$.

The terms of type (4) disappear if we add over all i and permute p and j, because of the properties of $\mathcal{H}[\bar b]$, cf. (3.3). The terms of type (3) disappear if we permute l and p. Therefore the terms of type (5) disappear as well.

One can deal with the terms of types (1) and (2) in the same way as in Lemma 3.2 and Proposition 3.4. They lead to $\gamma = (a_j, a_l)$ and $\gamma = (a_j + a_l, 0)$ as well as $\gamma = (0, a_j + a_l)$.

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Note that the commutators of type (2) are easier to understand, since there is no need to permute the t-degrees, and at the same time they give rise to half-brackets.

(iii) We are presenting $\mathbb{X}_{F[\bar{a}]}$ in the form (0.1) and can state at once that it has terms of degrees $m+1-2d$ only. Note that $[\mathcal{H}[\bar{b}], \mathcal{F}[\check{\bar{a}}]]$ can be viewed as a weighted symmetrisation $\varpi_{\rm wt}(F[\bar{a}], b_1, b_2)$ if we choose $\Psi(j,j+1)=\Psi(j+1,j)=1$ and $\Psi(j,l)=0$ in the case $|l-j|>1$. The term of degree $m+1$ is the Poisson bracket $\{\mathcal{H}[\bar{b}], F[\bar{a}]\}$. Here $r=0$ and $\bar{\gamma}=0$. In degree $m-1$, we obtain images in $\mathcal{S}^{m-1}(\hat{\mathfrak{g}}^-)$ of the weighted symmetrisations described in part (i). Further terms, which are of degrees $m-3, m-5, m-7$, and so on, are described by the iterated application of part (ii). At all steps, we obtain combinatorially defined rational coefficients, which are independent of F. □

Example 3.7. Suppose that $F \in \mathcal{S}^4(\mathfrak{g})^{\mathfrak{g}}$ and that $\mathfrak{g}$ is simple. Here we have $\mathsf{m}(F) \in (\Lambda^2\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ and $\dim(\Lambda^2\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}} = 1$. This subspace is spanned by $\mathcal{H} = \sum x_i^2$. Hence $\mathsf{m}(F) = \mathcal{H}$ up to a scalar. As we will proof in § 3.3, $S = \varpi(F[-1]) + 6\varpi(\tau^2\mathsf{m}(F)[-1])\cdot 1$ is a Segal–Sugawara vector. Making use of the fact that $\tau^2(\mathcal{H}[-1]) \in \mathfrak{z}(\hat{\mathfrak{g}})$, one can write S as a sum $\varpi(F[-1]) + B\mathcal{H}[-2]$ for some scalar $B \in \mathbb{C}$.

The only possible vector $\bar{\gamma}$ that can appear in Proposition 3.6(iii) is $(-1,-1)$. Therefore the commutator $[\mathcal{H}[-1], \varpi(F[-1])]$ is equal to $B\varpi(\{\mathcal{H}[-2], \mathcal{H}[-1]\}) = B[\mathcal{H}[-2], \mathcal{H}[-1]]$. In the orthonormal basis $\{x_i\}$, we have

$$\{\mathcal{H}[-2], \mathcal{H}[-1]\} = 4\sum_{i,j,s}([x_i, x_j], x_s)x_s[-3]x_i[-2]x_j[-1]. \tag{3.7}$$

3.3 Poisson (half-)brackets

Suppose that $\hat{Y} = \hat{y}_1 \dots \hat{y}_m \in \mathcal{S}^m(\hat{\mathfrak{g}}^-)$ and $\hat{y}_j = y_j[a_j]$. Then $P_{\hat{Y}} := \{\mathcal{H}[\bar{b}], \hat{Y}\} = P_{\hat{Y}}(b_1, b_2) + P_{\hat{Y}}(b_2, b_1)$, where

$$P_{\hat{Y}}(b_\upsilon, b_\nu) = \sum_{j=1,i=1}^{j=m,i=\dim\mathfrak{g}} [x_i[b_\nu], \hat{y}_j]x_i[b_\upsilon]\hat{Y}/\hat{y}_j = \sum_{j,i,u}([x_u, x_i], y_j)x_u[b_\nu + a_j]x_i[b_\upsilon]\hat{Y}/\hat{y}_j.$$

Note that in the case $b_\nu + a_j = b_\upsilon$, each summand $([x_u, x_i], y_j)x_u[b_\nu + a_j]x_i[b_\upsilon]\hat{Y}/\hat{y}_j$ is annihilated by $([x_i, x_u], y_j)x_i[b_\nu + a_j]x_u[b_\upsilon]\hat{Y}/\hat{y}_j$. Hence

$$P_{\hat{Y}}(b_1, b_2) = \sum_{j:\, a_j \neq b_1 - b_2} \sum_{i,u}([x_i, x_u], y_j)x_i[b_2 + a_j]x_u[b_1]\hat{Y}/\hat{y}_j. \tag{3.8}$$

The product $(\ ,\ )$ extends to a non-degenerate $\mathfrak{g}$-invariant scalar product on $\mathcal{S}(\hat{\mathfrak{g}}^-)$. We will assume that $(\mathfrak{g}[a], \mathfrak{g}[d]) = 0$ for $a \neq d$, that $(x[a], y[a]) = (x, y)$ for $x, y \in \mathfrak{g}$, and that

$$(\xi_1 \dots \xi_{\boldsymbol{k}}, \eta_1 \dots \eta_{\boldsymbol{m}}) = \delta_{\boldsymbol{k},\boldsymbol{m}} \sum_{\sigma \in \mathsf{S}_{\boldsymbol{k}}} (\xi_1, \eta_{\sigma(1)}) \dots (\xi_{\boldsymbol{k}}, \eta_{\sigma(\boldsymbol{k})})$$

if $\xi_j, \eta_j \in \mathcal{S}(\hat{\mathfrak{g}}^-)$, $\boldsymbol{m} \geqslant \boldsymbol{k}$. Let $\mathcal{B}$ be a monomial basis of $\mathcal{S}(\hat{\mathfrak{g}}^-)$ consisting of the elements $\hat{v}_1 \dots \hat{v}_{\boldsymbol{k}}$, where $\hat{v}_j = v_j[d_j]$ and $v_j \in \{x_i\}$. Then $\mathcal{B}$ is an orthogonal, but not an orthonormal basis. For instance, if $\Xi = x_1^{\gamma_1} \dots x_{\boldsymbol{k}}^{\gamma_{\boldsymbol{k}}}$ with $\boldsymbol{k} \leqslant \dim\mathfrak{g}$, then $(\Xi, \Xi) = \gamma_1! \dots \gamma_{\boldsymbol{k}}!$.

Set $M := m+1$, $\mathcal{B}(M) := \mathcal{B} \cap \mathcal{S}^M(\hat{\mathfrak{g}}^-)$, and write

$$P_{\hat{Y}}(b_1, b_2) = \sum_{\mathbb{V} \in \mathcal{B}(M)} A(\mathbb{V})\mathbb{V} \quad \text{with } A(\mathbb{V}) \in \mathbb{C},$$

expressing each $\hat{Y}/\hat{y}_j$ in the basis $\mathcal{B}$.

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LEMMA 3.8. *If $A(\mathbb{V}) \neq 0$ and $\mathbb{V} = \hat{v}_1 \ldots \hat{v}_M$, then $\boldsymbol{p} := \{p \mid d_p = b_1\} \neq \varnothing$. Furthermore,*

$$A(\mathbb{V}) = \sum_{p\in\boldsymbol{p}, l\notin\boldsymbol{p}} (\mathbb{V},\mathbb{V})^{-1}\left(\hat{Y}, \frac{\mathbb{V}}{\hat{v}_l\hat{v}_p}[v_l, v_p][d_l - b_2]\right). \tag{3.9}$$

Proof. The first statement is clear, cf. (3.8). It remains to calculate the coefficient of $\mathbb{V}$ in $P_{\hat{Y}}(b_1, b_2)$. Pick a pair (p, l) with $p \in \boldsymbol{p}$ and $l \notin \boldsymbol{p}$. If we take into account only those summands of $P_{\hat{Y}}(b_1, b_2)$, where $\mathbb{V}^{p,l} = \mathbb{V}/(\hat{v}_p\hat{v}_l)$ is a summand of $\hat{Y}/\hat{y}_j$ for some j, the factor $\hat{v}_l$ is $x_u[a_j + b_2]$, and $\hat{v}_p$ appears as $x_i[b_1]$, then the coefficient is

$$\sum_{j=1}^{m}(\hat{y}_j, [v_l, v_p][d_l - b_2])(\hat{Y}/\hat{y}_j, \mathbb{V}^{p,l})(\mathbb{V}^{p,l}, \mathbb{V}^{p,l})^{-1} = (\mathbb{V}^{p,l}, \mathbb{V}^{p,l})^{-1}(\hat{Y}, \mathbb{V}^{p,l}[v_l, v_p][d_l - b_2]).$$

If one adds these expressions over the pairs (p, l), then certain instances may be counted more than once. If $\hat{v}_p = \hat{v}_{p'}$ for some $p' \neq p$ or $\hat{v}_l = \hat{v}_{l'}$ for some $l' \neq l$, then (p, l') or (p', l) has to be omitted from the summation. In other words, it is necessary to divide the contribution of (p, l) by the multiplicities γ_p and γ_l of $v_p[d_p]$, $v_l[d_l]$ in $\mathbb{V}$. Since $(\mathbb{V},\mathbb{V}) = \gamma_p\gamma_l(\mathbb{V}^{p,l}, \mathbb{V}^{p,l})$, the result follows. □

The Poisson bracket $P_{\hat{Y}}$ is not multi-homogeneous with respect to $\hat{\mathfrak{g}}^- = \bigoplus_{d\leqslant -1}\mathfrak{g}[d]$. If $b_1 \neq b_2$, then in general the 'halves' of $P_{\hat{Y}}$ have different multi-degrees and neither of them has to be multi-homogeneous. We need to split $P_{\hat{Y}}(b_1, b_2)$ into smaller pieces. For $\bar{a} \in \mathbb{Z}^m_{<0}$, set $\mathcal{S}^{\bar{a}}(\hat{\mathfrak{g}}^-) = \prod_{j=1}^m \mathfrak{g}[a_j] \subset \mathcal{S}(\hat{\mathfrak{g}}^-)$.

Let $\bar{\alpha} = \{\alpha_1^{r_1}, \ldots, \alpha_s^{r_s}\}$ be a multi-set such that $\alpha_i \neq \alpha_j$ for $i \neq j$, $\alpha_j \in \mathbb{Z}_{<0}$ for all $1 \leqslant j \leqslant s$, $\sum_{j=1}^s r_j = M$, and $r_j > 0$ for all j. Set $\mathcal{S}^{\bar{\alpha}}(\hat{\mathfrak{g}}^-) := \prod_{j=1}^s \mathcal{S}^{r_j}(\mathfrak{g}[\alpha_j])$, $\mathcal{B}(\bar{\alpha}) := \mathcal{B} \cap \mathcal{S}^{\bar{\alpha}}(\hat{\mathfrak{g}}^-)$. Fix different $i, j \in \{1, \ldots, s\}$. Assume that a monomial $\mathbb{V} = \hat{v}_1 \ldots \hat{v}_M \in \mathcal{B}(\bar{\alpha})$ with $\hat{v}_l = v_l[d_l]$ is written in such a way that $d_l = \alpha_i$ for $1 \leqslant l \leqslant r_i$ and $d_l = \alpha_j$ for $r_i < l \leqslant r_i + r_j$. Finally suppose that $F \in \mathcal{S}^m(\mathfrak{g})$. In this notation, set

$$\mathbb{W}[F, \bar{\alpha}, (i, j)] := \sum_{\mathbb{V}\in\mathcal{B}(\bar{\alpha})} A(\mathbb{V})\mathbb{V} \quad \text{with}$$

$$A(\hat{v}_1 \ldots \hat{v}_M) = (\mathbb{V},\mathbb{V})^{-1} \sum_{\substack{1\leqslant l\leqslant r_i,\\ r_i<p\leqslant r_i+r_j}} \left(F, [v_l, v_p]\prod_{u\neq p,l} v_u\right). \tag{3.10}$$

Clearly, $\mathbb{W}[F, \bar{\alpha}, (j, i)] = -\mathbb{W}[F, \bar{\alpha}, (i, j)]$.

PROPOSITION 3.9. *Let $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$ be fixed. Then the elements $\mathbb{W}[\bar{\alpha}, (i, j)] = \mathbb{W}[F, \bar{\alpha}, (i, j)]$ satisfy the following 'universal' relations:*

$$\sum_{j:j\neq i} \mathbb{W}[\bar{\alpha}, (i, j)] = 0 \quad \textit{for each } i \leqslant s.$$

These relations are independent of F.

Proof. Follow the notation of (3.10). Note that for each $1 \leqslant l \leqslant r_i$,

$$\left(F, \sum_{w:\, w\neq l}[v_l, v_w]\prod_{u:\, u\neq l,w} v_u\right) = \left(\{F, v_l\}, \prod_{w:\, w\neq l} v_w\right) = 0.$$

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Of course, here we are adding also over the pairs (l,w) with $\hat{v}_w \in \mathfrak{g}[\alpha_i]$ if $r_i > 1$. However, $[v_l, v_w] = -[v_w, v_l]$ and hence the coefficient of $\mathbb{V}$ in $\sum_{j\neq i} \mathbb{W}[\bar{\alpha},(i,j)]$ is equal to

$$(\mathbb{V},\mathbb{V})^{-1} \sum_{1\leqslant l\leqslant r_i} \left(F, \sum_{w\colon w\neq l} [v_l, v_w] \prod_{u\colon u\neq l,w} v_u\right) = \sum_{1\leqslant l\leqslant r_i} 0 = 0.$$

This completes the proof. $\square$

Proposition 3.10. *For $Y[\bar{a}]$ with $Y = y_1 \ldots y_m \in \mathcal{S}^m(\mathfrak{g})$ and $\bar{a} \in \mathbb{Z}^m_{<0}$ (see § 1.5 for the notation), the rescaled Poisson half-bracket*

$$\boldsymbol{P}[\bar{a}] := |\mathsf{S}_m \bar{a}| P_{Y[\bar{a}]}(b_1, b_2) = |\mathsf{S}_m \bar{a}| \sum_u \{x_u[b_2], Y[\bar{a}]\} x_u[b_1]$$

is equal to the sum of $\mathbb{W}[Y,\bar{\alpha},(i,j)]$ with $i<j$ over all multi-sets $\bar{\alpha}$ as above such that the multi-set $\{a_1,\ldots,a_m\}$ of entries of $\bar{a}$ can be obtained from $\bar{\alpha}$ by removing one $\alpha_j = b_1$ and replacing one α_i with $\alpha_i - b_2$.

Proof. For each multi-homogeneous component of $\boldsymbol{P}[\bar{a}]$, the multi-set of t-degrees is obtained from the entries of $\bar{a}$ by appending b_1 and replacing one a_l with $a_l + b_2$. Moreover, here $b_1 \neq a_l + b_2$, cf. (3.8). This explains the restrictions on $\bar{\alpha}$.

For $\bar{\alpha}$ and (i,j) satisfying the assumptions of the proposition, we have to compare the coefficients $A(\mathbb{V})$ of $\mathbb{V} \in \mathcal{B}(\bar{\alpha})$ given by (3.10) and (3.9). The key point here is the observation that $(Y[\bar{a}],\mathbb{V}) = |\mathsf{S}_m\bar{a}|^{-1}(Y, v_1\ldots v_m)$ for any $\mathbb{V} = \hat{v}_1 \ldots \hat{v}_m \in \mathcal{B} \cap \mathcal{S}^{\bar{a}}(\hat{\mathfrak{g}}^-)$.

In a more relevant setup, suppose that a summand $(y_{\sigma(1)}, [v_l, v_p]) \prod_{w\neq 1, u\neq l,p} (y_{\sigma(w)}, v_u)$ of the scalar product on the right hand side of (3.10) is non-zero for some $\sigma \in \mathsf{S}_m$ and some l,p. Then there is exactly one choice, prescribed by $(d_1,\ldots,d_M)$, of the t-degrees for a monomial of $Y[\bar{a}]$ such that the corresponding summand

$$(y_{\sigma(1)}[\alpha_i - b_2], [v_l, v_p][d_l - b_2]) \prod_{w\neq 1, u\neq l,p} (y_{\sigma(w)}[d_u], \hat{v}_u)$$

of the scalar product on the right hand side of (3.9) is non-zero as well. By our assumptions on the scalar product, these summands are equal. $\square$

Theorem 3.11. *Suppose that $\mathsf{m}_{2r+1}(H)$ with $H \in \mathcal{S}^k(\mathfrak{g})^{\mathfrak{g}}$ is a symmetric invariant for each $r \geqslant 1$. Then (2.4) provides a Segal–Sugawara vector S associated with H.*

Proof. Since $\mathsf{m}_{2l+1}(H) \in \mathcal{S}(\mathfrak{g})$ for any l, we can say that $\mathsf{m}_{2r+1}(H) = \mathsf{m}^r(H)$, cf. (1.1). By Lemma 2.2, each $\varpi(\tau^{2r}\mathsf{m}^r(H)[-1])\cdot 1$ is a fully symmetrised element. It can be written as a sum of $\tilde{c}(r,\bar{a})\varpi(\mathsf{m}^r(H)[\bar{a}])$, where $\bar{a} \in \mathbb{Z}^{k-2r}_{<0}$ and the coefficients $\tilde{c}(r,\bar{a}) \in \mathbb{Q}$ depend only on k, r, and $\bar{a}$. The coefficients of (2.4) depend only on k and r. Combining this observation with Propositions 3.6(iii) and 3.10, we obtain that

$$[\mathcal{H}[-1], S] = \sum C(r,\bar{\alpha},i,j)\varpi(\mathbb{W}[\mathsf{m}_{2r+1}(H), \alpha_1^{r_1},\ldots,\alpha_d^{r_s},(i,j)]),$$

where again the coefficients $C(r,\bar{\alpha},i,j) \in \mathbb{Q}$ do not depend on H. For a given degree k, one obtains a bunch of $(r,\bar{\alpha})$, which depends only on k, and each appearing coefficient depends on k, r, $\bar{\alpha}$, and (i,j). In type A, for each $k \geqslant 2$, we find the invariant $\tilde{\Delta}_k$ such that the corresponding commutator $[\mathcal{H}[-1], \tilde{S}_{k-1}]$ vanishes, cf. (2.4).

For each $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$, the elements $\mathbb{W}[F,\bar{\alpha},(i,j)]$ are linearly dependent. They satisfy the 'universal' relations, see Proposition 3.9. At the same time, for $m = k-2r$, the coefficients $C(r,\bar{\alpha},i,j)$ provide a relation among $\mathbb{W}[\tilde{\Delta}_m,\bar{\alpha},(i,j)]$. Our goal is to prove that this relation holds for $\mathbb{W}[\mathsf{m}^r(H),\bar{\alpha},(i,j)]$ as well. To this end, it suffices to show that the terms

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$\mathbb{W}[\tilde{\Delta}_m, \bar{\alpha}, (i,j)] = \mathbb{W}[(i,j)]$ with fixed m and fixed $\bar{\alpha}$ do not satisfy any other linear relation, not spanned by the 'universal' ones.

We consider the complete simple graph with s vertices $1,\dots,s$ and identify pairs (i,j) with the corresponding (oriented) edges. Now one can say that a linear relation among the polynomials $\mathbb{W}[(i,j)]$ is given by its coefficients on the edges. Note that if $s=2$, then $\mathbb{W}[F,\bar{\alpha},(1,2)] = 0$ for each $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$, cf. Example 3.12(i) below. Therefore assume that $s \geqslant 3$.

Suppose there is a relation and that the coefficient of $\mathbb{W}[(i,j)]$ is non-zero. We work in the basis

$$\{E_{uv}[d], (E_{11} + \cdots + E_{ww} - wE_{(w+1)(w+1)})[d] \mid u \neq v, d < 0\}.$$

Choose $\hat{y}_1 = E_{12}[\alpha_i]$, $\hat{y}_2 = E_{21}[\alpha_j]$ and let all other factors $\hat{y}_l$ with $2 < l \leqslant M$ be elements of $t^{-1}\mathfrak{h}[t^{-1}]$. Assume that $\hat{y}_3 = (E_{11} - E_{22})[\alpha_p]$ with $p \neq i,j$ and that $(E_{11} - E_{22}, y_l) = 0$ for all $l > 3$. Then the monomial $\hat{y}_1 \dots \hat{y}_M$ appears with a non-zero coefficient only in $\mathbb{W}[(i,j)]$, $\mathbb{W}[(i,p)]$, and $\mathbb{W}[(p,j)]$. This means that in the triangle (i,j,p) at least one of the edges (i,p) and (j,p) has a non-zero coefficient as well.

We erase all edges with zero coefficients on them. Now the task is to modify the relation or, equivalently, the graph, by adding scalar multiplies of the universal relations in such a way that all edges disappear.

If a vertex l is connected with j, remove the edge (j,l) using the universal relation 'at l'. In this way j becomes isolated. This means that there is no edge left. $\square$

Example 3.12. Keep the assumption $F \in \mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$.

(i) Suppose that $s=2$. Then $\mathbb{W}(F,\bar{\alpha},(1,2)) = 0$ according to the universal relation. This provides a different proof of Lemma 3.1.

Also in the case of $\bar{a} = (-3, (-1)^{m-1})$, we have $\{\mathcal{H}[-1], F[\bar{a}]\} \in \mathfrak{g}[-3]\mathfrak{g}[-2]\mathcal{S}^{m-1}(\mathfrak{g}[-1])$.

(ii) Now suppose that $s=3$. Then $\mathbb{W}(F,\bar{\alpha},(1,2)) = -\mathbb{W}(F,\bar{\alpha},(1,3)) = \mathbb{W}(F,\bar{\alpha},(2,3))$.

4. Type C

There is a very suitable matrix realisation, where $\mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$ is the linear span of the elements F_{ij} with $i,j \in \{1,\dots,2n\}$ such that

$$F_{ij} = E_{ij} - \epsilon_i\,\epsilon_j\,E_{j'i'}, \tag{4.1}$$

with $i' = 2n-i+1$ and $\epsilon_i = 1$ for $i \leqslant n$, $\epsilon_i = -1$ for $i > n$. Of course, $F_{ij} = \pm F_{j'i'}$. Set

$$\mathfrak{h} = \langle F_{jj} \mid 1 \leqslant j \leqslant n\rangle_{\mathbb{C}}.$$

The symmetric decomposition $\mathfrak{gl}_{2n} = \mathfrak{sp}_{2n} \oplus \mathfrak{p}$ leads to explicit formulas for symmetric invariants of $\mathfrak{sp}_{2n}$. One writes $E_{ij} = \frac{1}{2}F_{ij} + \frac{1}{2}(E_{ij} + \epsilon_i\,\epsilon_j\,E_{j'i'})$, expands the coefficients Δ_k of (1.2) accordingly, and then sets $(E_{ij} + \epsilon_i\,\epsilon_j\,E_{j'i'}) = 0$. Up to the multiplication with 2^k, this is equivalent to replacing each E_{ij} with F_{ij} in the formulas for $\Delta_k \in \mathcal{S}(\mathfrak{gl}_{2n})$. As is well known, the restriction of Δ_{2k+1} to $\mathfrak{sp}_{2n}$ is equal to zero for each k.

Until the end of this section, Δ_{2k} stands for the symmetric invariant of $\mathfrak{g} = \mathfrak{sp}_{2n}$ that is equal to the sum of the principal $(2k \times 2k)$-minors of the matrix (F_{ij}).

Lemma 4.1. *For each $k \geqslant 2$, we have $\mathsf{m}(\Delta_{2k}) \in (\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g}))^{\mathfrak{g}}$.*

Proof. Note that $\Lambda^2\mathfrak{g} = V(2\pi_1 + \pi_2) \oplus \mathfrak{g}$. In the standard notation [VO88, Ref. Chapter, § 2], we have $2\pi_1 + \pi_2 = 3\varepsilon_1 + \varepsilon_2$. Assume that $y_1y_2y_3$ is a factor of a summand of Δ_{2k} of weight $3\varepsilon_1 + \varepsilon_2$

and $y_s \in \{F_{ij}\}$ for each s. Then:

- either $F_{1(2n)} \in \{y_1, y_2, y_3\}$ and some $y_j \neq F_{1(2n)}$ lies in the first row or the last column;
- or all three elements y_s must lie in the union of the first row and the last column.

Each of the two possibilities contradicts the definition of Δ_{2k}. Thus, $\mathsf{m}(\Delta_k) \in (\mathfrak{g} \otimes \mathcal{S}^{2k-3}(\mathfrak{g}))^{\mathfrak{g}}$. □

Lemma 4.2. *We have:*

(i) $\mathsf{m}(\Delta_6) = ((n-2)(2n-3)/15)\Delta_4$; *and*
(ii) $\mathsf{m}(\partial_{F_{11}}\Delta_4) = \frac{-1}{3}(2n-1)(2n-2)F_{11}$.

Proof. According to Lemma 4.1, $\mathsf{m}(\Delta_6) \in (\mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}}$. Observe that $\mathcal{S}^3(\mathfrak{g})$ contains exactly two linearly independent copies of $\mathfrak{g}$: one is equal to $\{\xi\Delta_2 \mid \xi \in \mathfrak{sp}_{2n}\}$, the other is primitive. Therefore

$$(\mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}} = \mathcal{S}^4(\mathfrak{g})^{\mathfrak{g}} = \langle \Delta_4, \Delta_2^2 \rangle_{\mathbb{C}}.$$

By the construction, Δ_2^2 contains the summand F_{11}^4. Since F_{11}^3 cannot be a factor of a summand of Δ_6, we conclude that $\mathsf{m}(\Delta_6)$ is proportional to Δ_4.

Let $y \otimes F_{11}^2F_{22}$ be a summand of $\mathsf{m}(\Delta_6)$. Then $y \in \mathbb{C}F_{22}$. Also $y = (-3!3!/6!)\mathsf{m}(\partial_{F_{22}}\Delta_4^{[1]})$, where $\Delta_4^{[1]} \in \mathcal{S}^4(\mathfrak{sp}_{2n-2})$ stands for Δ_4 of $\mathfrak{sp}_{2n-2} \subset \mathfrak{g}_{F_{11}}$. Next we compute $\eta = \mathsf{m}(\partial_{F_{22}}\Delta_4^{[1]})(F_{12})$. This will settle part (ii). So far we have shown that

$$\varpi(\partial_{F_{11}}\Delta_4) \in \mathcal{U}(\mathfrak{g}) \text{ acts as } cF_{11} \text{ on } \mathfrak{g} \text{ and on } \mathbb{C}^{2n} \tag{4.2}$$

and part (ii) describes this constant c, which is to be computed.

Recall that $F_{12} = -F_{(2n-1)2n}$. The terms of $\Delta_4^{[1]}$ have neither 1 nor $2n$ in the indices. Therefore a non-zero action on F_{12} comes only from the following summands of $\Delta_4^{[1]}$:

$$F_{22}F_{(2n-1)j}F_{js}F_{s(2n-1)}, \quad -F_{22}F_{(2n-1)s}F_{s's'}F_{s(2n-1)}, \tag{4.3}$$

$$F_{(2n-1)(2n-1)}F_{j2}F_{sj}F_{2s}, \quad -F_{(2n-1)(2n-1)}F_{s2}F_{s's'}F_{2s}. \tag{4.4}$$

One easily computes that

$$\mathsf{m}(F_{(2n-1)j}F_{js}F_{s(2n-1)})(F_{12}) = \begin{cases} \frac{1}{6}F_{12} & \text{if } j \neq s', \\ \frac{1}{3}F_{12} & \text{if } j = s',\ \text{because } F_{s's} = 2E_{s's}, \end{cases}$$

and that $\mathsf{m}(F_{(2n-1)s}F_{s's'}F_{s(2n-1)})(F_{12}) = -\frac{1}{6}F_{12}$. There are $2n-4$ choices for s in line (4.3). If s is fixed, then there are $2n-5$ possibilities for j, since $j \neq s$, but the choice $j = s'$ has to be counted twice. Applying the symmetry $F_{uv} = \pm F_{v'u'}$, we see that the terms in line (4.4) are the same as in (4.3). Now

$$\eta = \frac{1}{3}((2n-4)^2 + (2n-4))F_{12} = \frac{(2n-4)(2n-3)}{3}F_{12}.$$

Hence $y = ((n-2)(2n-3)/30)F_{22}$. Since $((n-2)(2n-3)/30)F_{11} \otimes F_{11}F_{22}^2$ is also a summand of $\mathsf{m}(\Delta_6)$, we obtain $\mathsf{m}(\Delta_6) = ((n-2)(2n-3)/15)\Delta_4$. □

Proposition 4.3. *We have*

$$\mathsf{m}_{2r+1}(\Delta_{2k}) = \frac{(2k-2r)!(2r)!}{(2k)!}\binom{2n-2k+2r+1}{2r}\Delta_{2k-2r}.$$

Proof. First we have to show that $\mathsf{m}(\Delta_{2k}) \in \mathcal{S}^{2k-2}(\mathfrak{g})$. By Lemma 4.1, $\mathsf{m}(\Delta_{2k})$ is a G-invariant polynomial function on $\mathfrak{g} \oplus \mathfrak{g}$. We use again the fact that $G(\mathfrak{g} \oplus \mathfrak{h})$ is dense in $\mathfrak{g} \oplus \mathfrak{g}$.

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Examine first the summands of Δ_{2k} that lie in $\mathcal{S}^3(\mathfrak{g})\mathcal{S}^{2k-3}(\mathfrak{h})$. Such a summand has the form $y_1y_2y_3(F_{i_1i_1}\dots F_{i_si_s})^2F_{j_1j_1}\dots F_{j_uj_u}$. Here $j_a\ne j_b, j_b'$ if $a\ne b$ and the product $y_1y_2y_3$ is an element of weight zero lying in $\mathcal{S}^3(\mathfrak{f})$, where $\mathfrak{f}$ is a subalgebra of $\mathfrak{g}$ isomorphic either to $\mathfrak{sp}_6$ or $\mathfrak{sp}_4$. Furthermore, the numbers i_b, i_b' with $1\leqslant b\leqslant s$ do not appear among the indices of the elements of $\mathfrak{f}$ and at most three different numbers j_b with $1\leqslant b\leqslant u$ can appear among the indices of the elements of $\mathfrak{f}$.

If $u>3$, we can change at least one $F_{j_bj_b}$ to $F_{j_b'j_b'}=-F_{j_bj_b}$ without altering the other factors and produce a different summand of Δ_{2k}. These two expressions annihilate each other. Therefore $u=3$ or $u=1$. Suppose $u=3$ and that there is no way to annihilate the term via $F_{j_bj_b}\mapsto F_{j_b'j_b'}$. Then $\mathfrak{f}\cong\mathfrak{sp}_6$ and $y_1y_2y_3F_{j_1j_1}F_{j_2j_2}F_{j_3j_3}$ is a summand of the determinant $\Delta_6^{(\mathfrak{f})}\in\mathcal{S}^6(\mathfrak{f})$. Since $\mathsf{m}(\Delta_6)$ is proportional to Δ_4 by Lemma 4.2 and $F_{j_1j_1}F_{j_2j_2}F_{j_3j_3}$ cannot appear in Δ_4, we conclude that terms $y\otimes(F_{i_1i_1}\dots F_{i_si_s})^2F_{j_1j_1}\dots F_{j_uj_u}$ with $u>1$ do not appear in $\mathsf{m}(\Delta_{2k})$.

Fix an element $\boldsymbol{H}=F_{11}F_{22}^2\dots F_{ll}^2$ with $l=k-1$. We compute the coefficient of $\boldsymbol{H}$ in $\mathsf{m}(\Delta_{2k})$. This coefficient is equal to $(-1)^k(3!(2k-3)!/(2k)!)\mathsf{m}(\partial_{F_{11}}\Delta_4^{(l)})$, where $\Delta_4^{(l)}$ is Δ_4 of the $\mathfrak{sp}_{2n-2k+4}$-subalgebra generated by F_{ij} with $i,j\notin\{2,2',\dots,l,l'\}$. According to Lemma 4.2, $\mathsf{m}(\partial_{F_{11}}\Delta_4^{(l)})=\frac{-1}{3}(2n-2k+3)(2n-2k+2)F_{11}$. Making use of the action of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ on $\mathfrak{h}\oplus\mathfrak{h}$, we conclude that $\mathsf{m}(\Delta_{2k})$ is proportional to Δ_{2k-2}, more explicitly

$$\mathsf{m}(\Delta_{2k})=\frac{2(k-1)(2n-2k+3)(2n-2k+2)(2k-3)!}{(2k)!}\Delta_{2k-2}$$

and with some simplifications

$$\mathsf{m}(\Delta_{2k})=\frac{(2n-2k+3)(2n-2k+2)}{2k(2k-1)}\Delta_{2k-2}=\binom{2k}{2}^{-1}\binom{2n-2k+3}{2}\Delta_{2k-2}.$$

Iterating the map m, one obtains the result. □

Theorem 4.4. *For $\mathfrak{g}=\mathfrak{sp}_{2n}$ and $1\leqslant k\leqslant n$,*

$$S_k=\varpi(\Delta_{2k}[-1])+\sum_{1\leqslant r<k}\binom{2n-2k+2r+1}{2r}\varpi(\tau^{2r}\Delta_{2k-2r}[-1])\cdot 1$$

is a Segal–Sugawara vector.

5. Several exceptional examples

There are instances, where our methods work very well.

Example 5.1. Suppose that $\mathfrak{g}=\mathfrak{so}_8$. Then $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})=\mathsf{S}_3$, where $\mathrm{Inn}(\mathfrak{g})$ is the group of inner automorphisms. There are two Segal–Sugawara vectors, say S_2 and S_3, such that their symbols are $\mathfrak{g}$-invariants of degree 4 in $\mathcal{S}(\mathfrak{g}[-1])$. Assume that S_2 and S_3 are fixed vectors of ω. Then each of them is a sum $\varpi(Y_4)+\varpi(Y_2)$, cf. (0.1). Each element in $\mathcal{S}^2(\hat{\mathfrak{g}}^-)^{\mathfrak{g}}$ is proportional to $\mathcal{H}[\bar b]$ for some $\bar b$. Hence it is also an invariant of S_3. Without loss of generality we may assume that the symbols of S_2 and S_3 are Pfaffians $\mathrm{Pf}_2[-1]$, $\mathrm{Pf}_3[-1]$ related to different matrix realisations of $\mathfrak{so}_8$. Then for each of them there is an involution $\sigma\in\mathsf{S}_3$ such that $\sigma(Y_4)=-Y_4$. Replacing S_j with $S_j-\sigma(S_j)$, we see that $\tilde S_2=\varpi(\mathrm{Pf}_2[-1])$ and $\tilde S_3=\varpi(\mathrm{Pf}_3[-1])$ are also Segal–Sugawara vectors. In view of Theorem 3.5, this implies that $\mathsf{m}(\mathrm{Pf}_2)=\mathsf{m}(\mathrm{Pf}_3)=0$.

Automorphisms of $\mathfrak{g}$ make themselves extremely useful. We will see the full power of this devise in § 7, which deals with the orthogonal case. At the moment notice the following thing, any $\sigma \in \mathrm{Aut}(\mathfrak{g})$ acts on $\mathcal{S}(\mathfrak{g})$ in the natural way and induces a map $\sigma_{(m)} : \mathcal{S}^m(\mathfrak{g}) \to \mathcal{S}^m(\mathfrak{g})$. Let $\mathfrak{v}_m \otimes \mathfrak{g}$ be the isotypic component of $\mathcal{S}^m(\mathfrak{g})$ corresponding to $\mathfrak{g}$. Then σ acts on $\mathfrak{v}_m$ and for this action, we have $\sigma(v) \otimes \sigma(x) = \sigma_{(m)}(v \otimes x)$, where $v \in \mathfrak{v}_m$, $x \in \mathfrak{g}$.

An interesting story is related to Pfaffians in higher ranks.

Example 5.2 (The Pfaffians). Take $\mathfrak{g} = \mathfrak{so}_{2n}$. If $n < 4$, these algebras appear in type A. In the case $n = 4$, the Pfaffian-like Segal–Sugawara vectors are examined in Example 5.1. Suppose that $n > 4$ and that $\mathfrak{so}_{2n} \subset \mathfrak{gl}_{2n}$ consists of the skew-symmetric with respect to the antidiagonal matrices. The highest weight of the Cartan component of $\Lambda^2\mathfrak{g}$ is $\pi_1 + \pi_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Assume that $\pi_1 + \pi_3$ appears as the weight of a factor $y_1 y_2 y_3$ for a summand $y_1 \ldots y_n$ of the Pfaffian Pf. Then up to the change of indices, we must have

$$y_1 = (E_{1i} - E_{i'1'}), \quad y_2 = (E_{1j} - E_{j'1'}),$$

where $i' = 2n + 1 - i$. If this is really the case, then the determinant $\Delta_{2n} \in \mathcal{S}^{2n}(\mathfrak{gl}_{2n})$ has a summand $E_{1i}E_{1j} \ldots$, a contradiction.

Thus $\mathsf{m}(\mathrm{Pf}) \in (\mathfrak{g} \otimes \mathcal{S}^{n-3}(\mathfrak{g}))^{\mathfrak{g}}$. If n is odd, then there is no copy of $\mathfrak{g}$ in $\mathcal{S}^{n-3}(\mathfrak{g})$ and we conclude at once that the image of the Pfaffian under m is zero.

Suppose that n is even. Then we can rely on the fact that $G(\mathfrak{g} \oplus \mathfrak{h})$ is dense in $\mathfrak{g} \oplus \mathfrak{g}$. Fix a factor $\boldsymbol{H} \in \mathcal{S}^{n-3}(\mathfrak{h})$ of a summand of Pf. Without loss of generality assume that $\boldsymbol{H} = \prod_{s>3}(E_{ss} - E_{s's'})$. Let $\mathrm{Pf}^{(3)}$ be the Pfaffian of the subalgebra spanned by

$$E_{ij} - E_{j'i'},\ E_{ij'} - E_{ji'},\ E_{i'j} - E_{j'i} \quad \text{with } i, j \leqslant 3.$$

Since this subalgebra is isomorphic to $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ write also $\tilde{\Delta}_3^{(4)}$ for $\mathrm{Pf}^{(3)}$. By the construction, $(3!(n-3)!/n!)\mathsf{m}(\mathrm{Pf}^{(3)}) \otimes \boldsymbol{H}$ is a summand of $\mathsf{m}(\mathrm{Pf})$. For a Weyl involution θ of $\mathfrak{sl}_4$, we have $\theta(\tilde{\Delta}_3^{(4)}) = -\tilde{\Delta}_3^{(4)}$. Therefore $\varpi(\tilde{\Delta}_3^{(4)})$ acts as zero on any irreducible self-dual $\mathfrak{sl}_4$-module, in particular, on $\mathfrak{sl}_4$ and on $\Lambda^2\mathbb{C}^4 = \mathbb{C}^6$. Now we can conclude that $\mathsf{m}(\mathrm{Pf}^{(3)}) = 0$ and hence $\mathsf{m}(\mathrm{Pf}) = 0$. Thus $\varpi(\mathrm{Pf})[-1]$ is a Segal–Sugawara vector for each n.

Keep the assumption that n is even. Another way to see that $\mathsf{m}(\mathrm{Pf}) = 0$ is to use an outer involution $\sigma \in \mathrm{Aut}(\mathfrak{so}_{2n})$ such that $\sigma(\mathrm{Pf}) = -\mathrm{Pf}$. Here $\sigma(v) = -v$ for $v \in \mathfrak{v}_{n-1}$ such that $v \otimes \mathfrak{g}$ is the primitive copy of $\mathfrak{g}$ that gives rise to Pf and also $\sigma(\mathsf{m}(\mathrm{Pf})) = -\mathsf{m}(\mathrm{Pf})$. At the same time, σ acts as id on $\mathfrak{v}_{n-3}$. Therefore σ acts as id on $(\mathfrak{g} \otimes \mathcal{S}^{n-3}(\mathfrak{g}))^{\mathfrak{g}}$. Since $\mathsf{m}(\mathrm{Pf}) \in (\mathfrak{g} \otimes \mathcal{S}^{n-3}(\mathfrak{g}))^{\mathfrak{g}}$, it must be zero.

Explicit formulas for the Pfaffian-type Segal–Sugawara vector $\mathrm{Pf}F[-1] \in \mathcal{U}(\mathfrak{so}_{2n}[t^{-1}])$ are given in [Mol13, Roz14]. In the basis $\{F^{\circ}_{ij} = E_{ij} - E_{ji} \mid 1 \leqslant i < j \leqslant 2n\}$ for $\mathfrak{so}_{2n}$, the vector $\mathrm{Pf}F[-1]$ is written as a sum of monomials with pairwise commuting factors, see [Mol13] and [Mol18, Equation (8.11)]. Hence it coincides with the symmetrisation of its symbol, in our notation, $\mathrm{Pf}F[-1] = \varpi(\mathrm{Pf})[-1]$. Example 5.2 provides a different proof for [Mol18, Proposistion 8.4].

Another easy to understand instance is provided by the invariant of degree 5 in type E_6.

Example 5.3. Suppose that $\mathfrak{g}$ is a simple Lie algebra of type E_6. Let $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be a homogeneous invariant of degree 5. Then $\mathsf{m}(H) \in (\Lambda^2\mathfrak{g} \otimes \mathcal{S}^2(\mathfrak{g}))^{\mathfrak{g}}$. Here $\Lambda^2\mathfrak{g} = V(\pi_3) \oplus \mathfrak{g}$ and $\mathcal{S}^2(\mathfrak{g}) = V(2\pi_6) \oplus V(\pi_1 + \pi_5) \oplus \mathbb{C}$. Therefore $\mathsf{m}(H) = 0$.

Recall that we are considering only semisimple $\mathfrak{g}$ now and that $(\ ,\ )$ is fixed in such a way that $\mathcal{H} \in \mathcal{U}(\mathfrak{g})$ acts on $\mathfrak{g}$ as $C\mathrm{id}_{\mathfrak{g}}$ for some $C \in \mathbb{C}$.

LEMMA 5.4. *There is $c_1 \in \mathbb{C}$ depending on the scalar product $(\ ,\ )$ such that $\sum_i x_i[\xi, x_i] = c_1\xi$ in $\mathcal{U}(\mathfrak{g})$ for any $\xi \in \mathfrak{g}$. Furthermore, $\mathcal{H}(\xi) = \sum_i \mathrm{ad}(x_i)^2(\xi) = -2c_1\xi$, i.e. $C = -2c_1$.*

Proof. For each $\xi \in \mathfrak{g}$, set $\psi_1(\xi) = \sum_i x_i[\xi, x_i] = \sum_i x_i\xi x_i - \mathcal{H}\xi$. Since $[\mathcal{H}, \xi] = 0$, we have then $\omega(\psi_1(\xi)) = -\psi_1(\xi)$. Note that

$$\psi_1(\xi) = \sum_i [x_i, [\xi, x_i]] + \sum_i [\xi, x_i]x_i = -\mathcal{H}(\xi) + \omega(\psi_1(\xi)) = -\mathcal{H}(\xi) - \psi_1(\xi).$$

Thereby $2\psi_1(\xi) = -\mathcal{H}(\xi) = -C\xi$. □

LEMMA 5.5. *Let $\mathfrak{g}$ be a simple Lie algebra of rank at least 2. Then $\mathsf{m}(\mathcal{H}^3) \notin \mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g})$.*

Proof. Choose an orthogonal basis of $\mathfrak{h}$ such that at least one element in it is equal to h_α for a simple root α. If the root system of $\mathfrak{g}$ is not simply laced, suppose that α is a long root. Suppose further that either $\alpha = \alpha_1$ or $\alpha = \alpha_\ell$. Changing the scalar product if necessary, we may assume that $h_\alpha \in \{x_i\}$. Consider the summand $\xi \otimes h_\alpha^3$ of $\mathsf{m}(\mathcal{H}^3)$. We have

$$\xi = \frac{3!3!}{6!}(6\mathsf{m}(h_\alpha\mathcal{H}) + 8\mathsf{m}(h_\alpha^3)).$$

Set $\xi_0 = \mathsf{m}(h_\alpha\mathcal{H})$. Note that $h_\alpha\mathcal{H} \in \mathcal{U}(\mathfrak{g})$ acts on $\mathfrak{g}$ as a scalar multiple of $\mathrm{ad}(h_\alpha)$. In view of Lemma 5.4, the sum $\sum_i x_i h_\alpha x_i$ acts on $\mathfrak{g}$ as another multiple of $\mathrm{ad}(h_\alpha)$. Hence $\xi_0 \in \mathfrak{g}$.

It remains to show that $\eta = \mathrm{ad}(h_\alpha)^3$ is not an element of $\mathfrak{g} \subset \mathfrak{so}(\mathfrak{g})$. Let α' be the unique simple root not orthogonal to α. Observe that $\eta(e_\alpha) = 8e_\alpha$ and $\eta(e_{\alpha'}) = -e_{\alpha'}$. Set $\gamma = \alpha + \alpha'$. Then $e_\gamma \neq 0$ and $\eta(e_\gamma) = e_\gamma$. Since $1 \neq 8 - 1$, we conclude that indeed $\eta \notin \mathfrak{g}$. □

PROPOSITION 5.6. *Let $\mathfrak{g}$ be an exceptional simple Lie algebra. Suppose that $H \in \mathcal{S}^6(\mathfrak{g})^{\mathfrak{g}}$. Then there is $\boldsymbol{b} \in \mathbb{C}$ such that $\mathsf{m}(H - \boldsymbol{b}\mathcal{H}^3) \in \mathbb{C}\mathcal{H}^2 \subset \mathcal{S}^4(\mathfrak{g})$.*

Proof. Let V be the Cartan component of $\Lambda^2\mathfrak{g}$. A straightforward calculation shows that V appears in $\mathcal{S}^3(\mathfrak{g})$ with multiplicity one as in the following table.

Type	The highest weight of V	$\mathcal{S}^3(\mathfrak{g})$
E_6	π_3	$V(3\pi_6) \oplus V(\pi_1 + \pi_5 + \pi_6) \oplus V(\pi_3) \oplus V(\pi_1 + \pi_5) \oplus \mathfrak{g}$
E_7	π_5	$V(3\pi_5) \oplus V(\pi_2 + \pi_6) \oplus V(\pi_5) \oplus V(2\pi_1) \oplus \mathfrak{g}$
E_8	π_2	$V(3\pi_1) \oplus V(\pi_1 + \pi_7) \oplus V(\pi_2) \oplus \mathfrak{g}$
F_4	π_3	$V(3\pi_4) \oplus V(2\pi_1 + \pi_4) \oplus V(\pi_3) \oplus V(\pi_2) \oplus \mathfrak{g}$
G_2	$3\pi_1$	$V(3\pi_1) \oplus V(2\pi_1 + \pi_2) \oplus V(3\pi_2) \oplus V(\pi_1) \oplus \mathfrak{g}$

We have $\mathsf{m}(H) \in (V \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}} \oplus (\mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}}$. The first summand here is one-dimensional. Since $\mathsf{m}(\mathcal{H}^3) \notin \mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g})$ by Lemma 5.5, there is $\boldsymbol{b} \in \mathbb{C}$ such that $\mathsf{m}(\tilde{H}) \in \mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g})$ for $\tilde{H} = H - \boldsymbol{b}\mathcal{H}^3$.

The degrees of basic symmetric invariants $\{H_k \mid 1 \leqslant k \leqslant l\}$ indicate that $\mathcal{S}^3(\mathfrak{g})$ contains exactly one copy of $\mathfrak{g}$. (This is also apparent in the table above.) Hence $(\mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}} = \mathcal{S}^4(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}\mathcal{H}^2$. □

COROLLARY 5.7. *Keep the assumption that $\mathfrak{g}$ is exceptional. Then $\tilde{H} = H - \boldsymbol{b}\mathcal{H}^3$ of Proposition 5.6 satisfies (0.2) and there are $R(1), R(2) \in \mathbb{C}$ such that*

$$S_2 = \varpi(\tilde{H})[-1] + R(1)\varpi(\tau^2\mathcal{H}^2[-1])\cdot 1 + R(2)\varpi(\tau^4\mathcal{H}[-1])\cdot 1 \qquad (5.1)$$

is an element of $\mathfrak{z}(\hat{\mathfrak{g}})$.

Proof. The first statement follows from Proposition 5.6 and Example 3.7. More explicitly, $\mathsf{m}(\tilde{H}) \in \mathbb{C}\mathcal{H}^2$, since there is no other symmetric invariant of degree four. Now the existence of $R(1)$ and $R(2)$ follows from Theorem 3.11. □

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6. Type $\mathbf{G_2}$

Let $\mathfrak{g}$ be a simple Lie algebra of type G_2. Then $\ell = 2$. The algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ has two generators, $\mathcal{H}$ and $\Delta_6 \in \mathcal{S}^6(\mathfrak{g})$. In this section, we compute the constant $\boldsymbol{b}$ of Proposition 5.6 for $H = \Delta_6$ and $R(1), R(2)$ of (5.1). All our computations are done by hand. A computer-aided explicit formula for a Segal–Sugawara vector of t-degree 6 is obtained in [MRR16].

First we choose a matrix realisation of $\mathfrak{g} \subset \mathfrak{so}_7$. The embedding $\iota\colon \mathfrak{sl}_3 \to \mathfrak{so}_7$ is fixed by

$$\iota(E_{ij}) = E_{ij} - E_{(7-j)(7-i)}$$

for $i \neq j$. We choose a basis of $\mathfrak{h} \subset \mathfrak{sl}_3$ as $\{h_1, h_2\}$ with $h_1 = \operatorname{diag}(1,-1,0)$, $h_2 = \operatorname{diag}(1,1,-2)$ and extend it to a basis of $\mathfrak{sl}_3$ by adding e_i, f_i with $1 \leqslant i \leqslant 3$ in the semi-standard notation, e.g. $e_3 = E_{13}$, $f_1 = E_{21}$, $f_3 = E_{31}$. Let $\varepsilon_i \in \mathfrak{h}^*$ with $1 \leqslant i \leqslant 3$ be the same as in § 2. Now it remains to describe the complement of $\mathfrak{sl}_3 = (\mathfrak{so}_6 \cap \mathfrak{g})$, which is isomorphic to $\mathbb{C}^3 \oplus (\mathbb{C}^3)^*$.

Matrix (6.1) presents an element of $\mathbb{C}^3 \oplus (\mathbb{C}^3)^* \subset \mathfrak{so}_7$:

$$\left(\begin{array}{cccccc|c} & & & -\beta & \gamma & 0 & \sqrt{2}\mathtt{a} \\ & & & \alpha & 0 & -\gamma & \sqrt{2}\mathtt{b} \\ & & & 0 & -\alpha & \beta & \sqrt{2}\mathtt{c} \\ \mathtt{b} & -\mathtt{a} & 0 & & & & \sqrt{2}\gamma \\ -\mathtt{c} & 0 & \mathtt{a} & & & & \sqrt{2}\beta \\ 0 & \mathtt{c} & -\mathtt{b} & & & & \sqrt{2}\alpha \\ \hline -\sqrt{2}\alpha & -\sqrt{2}\beta & -\sqrt{2}\gamma & -\sqrt{2}\mathtt{c} & -\sqrt{2}\mathtt{b} & -\sqrt{2}\mathtt{a} & 0 \end{array}\right). \tag{6.1}$$

With a certain abuse of notation, we denote the elements of the corresponding basis by the same symbols, for instance,

$$\mathtt{a} = \sqrt{2}E_{17} - E_{42} + E_{53} - \sqrt{2}E_{76}$$

as a vector of $\mathbb{C}^3$.

The scalar product $(\ ,\ )$ is such that $\mathcal{H} = \Delta_2$ with

$$\Delta_2 = 2e_1f_1 + 2e_2f_2 + 2e_3f_3 + \tfrac{1}{2}h_1^2 + \tfrac{1}{6}h_2^2 - \tfrac{2}{3}(\mathtt{a}\alpha + \mathtt{b}\beta + \mathtt{c}\gamma).$$

The basic invariant of degree 6, Δ_6, is chosen as the restriction to $\mathfrak{g}$ of the coefficient $\Delta_6^{(7)}$ of degree 6 in (1.2) written for $\mathfrak{gl}_7$. In this case, the restriction of Δ_6 to $\mathfrak{sl}_3$ is equal to $-\tilde{\Delta}_3^2$, where $\tilde{\Delta}_3$ is the determinant of $\mathfrak{sl}_3$. For future use, we record

$$[\mathtt{a},\alpha] = \operatorname{diag}(-2,1,1) \in \mathfrak{sl}_3, \quad [\mathtt{b},\beta] = \operatorname{diag}(1,-2,1) \in \mathfrak{sl}_3, \quad [\mathtt{c},\gamma] = h_2,$$
$$[\alpha,\mathtt{c}] = 3f_3, \quad [\beta,\mathtt{c}] = 3f_2, \quad [\mathtt{a},\mathtt{b}] = -2\gamma, \quad [\gamma,\beta] = 2\mathtt{a}, \quad [\mathtt{b},\mathtt{c}] = -2\alpha, \quad [\beta,\mathtt{a}] = 3e_1.$$

The decomposition $\mathfrak{g} = (\mathbb{C}^3)^* \oplus \mathfrak{g} \oplus \mathbb{C}^3$ is a $\mathbb{Z}/3\mathbb{Z}$-grading induced by an (inner) automorphism σ of $\mathfrak{g}$. Note that our basis for $\mathfrak{g}$ consists of eigenvectors of σ.

Recall that S_2 is given by (5.1) and that we are computing the constants occurring there. There is an easy part of the calculation. It concerns the projection of $\mathsf{m}(\Delta_2^3)$ on $(V \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}}$. As we already know, the highest weight of V is $3\pi_1$. Next choose a monomial of weight $3\pi_1$, for instance, $e_3^2 f_1$.

Lemma 6.1. *Let $\xi \otimes e_3^2 f_1$ be a summand of $\mathsf{m}(\Delta_2^3)$. Then $\xi(e_3) = \frac{6}{5} f_2$.*

Proof. Observe that in Δ_2^3, the factor $e_3^2 f_1$ appears only in the summand $24 e_3^2 f_3^2 e_1 f_1$. By the construction, we have

$$\xi = \frac{24 \times 3! \times 3!}{6!}\mathsf{m}(f_3^2 e_1) = \frac{6}{5}\mathsf{m}(f_3^2 e_1).$$

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Note that $[e_1, e_3] = 0$. Hence

$$\begin{aligned}\tfrac{5}{6}\xi(e_3) &= \tfrac{1}{6}(2\mathrm{ad}(e_1)\mathrm{ad}(f_3)^2 + 2\mathrm{ad}(f_3)\mathrm{ad}(e_1)\mathrm{ad}(f_3))(e_3)\\ &= \tfrac{1}{6}(-2\mathrm{ad}(f_2)\mathrm{ad}(f_3) - 4\mathrm{ad}(f_3)\mathrm{ad}(f_2))(e_3) = -[f_2, [f_3, e_3]] = f_2\end{aligned}$$

and the result follows. □

In the above computation, we did not see the projection of $\mathsf{m}(\Delta_2^3)$ on $(\mathfrak{g} \otimes \mathcal{S}^3(\mathfrak{g}))^{\mathfrak{g}}$, which is equally important. Set $h_3 = [e_3, f_3]$. Note that $\{f_3, h_3, e_3\}$ is an $\mathfrak{sl}_2$-triple associated with the highest root of $\mathfrak{g}$. In the following lemma, $\mathfrak{sl}_2$ means $\langle f_3, h_3, e_3\rangle_{\mathbb{C}}$.

Lemma 6.2. *Let $\eta \otimes e_3^2 f_3$ be a summand of $\mathsf{m}(\Delta_2^3)$. Then η acts as $\frac{48}{5}\mathrm{ad}(f_3)$ on $\langle e_3, f_3, h_3\rangle_{\mathbb{C}}$ and as $\frac{42}{5}\mathrm{ad}(f_3)$ on the 11-dimensional $\mathfrak{sl}_2$-stable complement of this subspace.*

Proof. In this case, one has to pay a special attention to the summand $8e_3^3 f_3^3$ of Δ_2^3. In the product $(e_3 f_3)(e_3 f_3)(e_3 f_3)$, there are six choices of (e_3, e_3, f_3) such that f_3 and one of the elements e_3 belong to one and the same copy of Δ_2; there are also three other choices. These first six choices are absorbed in $(3!3!/6!)24\mathsf{m}(f_3\Delta_2)$. Note that $(3! \times 3! \times 24)/6! = \frac{6}{5}$. The contribution to η of the three other choices is $(3!3!/6!)24\mathsf{m}(e_3 f_3^2)$. Hence $\eta = \frac{6}{5}(\mathsf{m}(f_3\Delta_2) + \mathsf{m}(e_3 f_3^2))$.

The element $\varpi(\Delta_2) \in \mathcal{U}(\mathfrak{g})$ acts on $\mathfrak{g}$ as a scalar. That scalar is 8 in our case. According to Lemma 5.4, the sum $\sum_i x_i f_3 x_i \in \mathcal{U}(\mathfrak{g})$ is equal to $\mathcal{H} f_3 - 4f_3$. Thus, $\mathsf{m}(f_3\Delta_2) = (8 - \frac{4}{3})\mathrm{ad}(f_3)$ and $\frac{6}{5}\mathsf{m}(f_3\Delta_2) = 8\mathrm{ad}(f_3)$.

Now consider $\eta_0 = (e_3 f_3^2 + f_3^2 e_3 + f_3 e_3 f_3) \in \mathcal{U}(\mathfrak{sl}_2)$. Clearly, η_0 acts as zero on a trivial $\mathfrak{sl}_2$-module; for the defining representation on $\mathbb{C}^2 = \langle v_1, v_2\rangle_{\mathbb{C}}$ with $e_3 v_1 = 0$, one obtains $\eta_0(v_1) = v_2$ and $\eta_0(v_2) = 0$. This suffices to state that $\frac{6}{5}\mathsf{m}(e_3 f_3^2)$ acts as $\frac{2}{5}\mathrm{ad}(f_3)$ on the $\mathfrak{sl}_2$-stable complement of $\langle f_3, h_3, e_3\rangle_{\mathbb{C}}$. Finally, $\eta_0(f_3) = 0$ by the obvious reasons, $\eta_0(e_3) = -2h_3 - 2h_3 = 4\mathrm{ad}(f_3)(e_3)$ and $\eta_0(h_3) = 4\mathrm{ad}(f_3)(h_3)$ as well, since η_0 acts on $\mathfrak{g}$ as an element of $\mathfrak{so}(\mathfrak{g})$. All computations are done now and the proof is finished. □

Let $\mathsf{pr}\colon \mathfrak{so}_7 \to \mathfrak{g}$ be the orthogonal projection. In order to work with Δ_6, one needs to compute the images under pr of $F_{ij} = E_{ij} - E_{(8-j)(8-i)} \in \mathfrak{so}_7$. For the elements of $\mathfrak{gl}_3 \subset \mathfrak{so}_6$, this is easy, the task reduces to F_{ii} with $1 \leqslant i \leqslant 3$, where we have

$$\mathsf{pr}(F_{11}) = \tfrac{1}{6}(3h_1 + h_2), \quad \mathsf{pr}(F_{22}) = \tfrac{1}{6}(-3h_1 + h_2), \quad \mathsf{pr}(F_{33}) = \tfrac{-1}{3}h_2.$$

The elements of $F_{ij} \in \mathfrak{so}_6$ with $1 \leqslant i \leqslant 3$, $4 \leqslant j \leqslant 6$ project with the coefficient $\frac{1}{3}$ on the corresponding letters in (6.1), e.g, $\mathsf{pr}(F_{14}) = \frac{-1}{3}\beta$, $\mathsf{pr}(F_{15}) = \frac{1}{3}\gamma$, and so on. The elements F_{i7} project with the coefficient $\frac{\sqrt{2}}{3}$ on the corresponding letters, e.g, $\mathsf{pr}(F_{17}) = \frac{\sqrt{2}}{3}\mathsf{a}$. Finally, the elements F_{7i} project with the coefficient $\frac{-\sqrt{2}}{3}$ on the corresponding letters, e.g, $\mathsf{pr}(F_{71}) = \frac{-\sqrt{2}}{3}\alpha$. An explicit formula for Δ_6 can be obtained by replacing first E_{ij} with F_{ij} in $\Delta_6^{(7)} \in \mathcal{S}^6(\mathfrak{gl}_7)$ and then replacing F_{ij} with $\mathsf{pr}(F_{ij})$. We write down some of the terms of Δ_6:

$$\begin{aligned}\Delta_6 = -\tilde{\Delta}_3^2 &- \tfrac{4}{27}\mathsf{c}^3 e_3^2 f_1 - \tfrac{4}{9}\mathsf{c}\beta f_3 e_3^2 f_1 + \tfrac{2}{3}\mathsf{c}\alpha f_2 e_3^2 f_1 + \tfrac{1}{9}\mathsf{c}\alpha h_1 f_3 e_3^2 - \tfrac{1}{27}\mathsf{c}\alpha h_2 f_3 e_3^2\\ &- \tfrac{4}{9}\mathsf{b}\alpha f_2 f_3 e_3^2 + \cdots.\end{aligned}$$

With this knowledge we can attack the computation of $\mathsf{m}(\Delta_6)$. The first challenge is to understand the term $\tilde{\xi} \otimes e_3^2 f_1$.

Lemma 6.3. *For $\tilde{\xi}$ as above, we have $\tilde{\xi}(e_3) = \frac{5}{18}f_2$.*

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Proof. Once again, we rely on a direct computation. The terms of $\tilde\Delta_3$ containing e_3 as a factor are $e_3f_1f_2$ and $e_3f_3(\frac{1}{2}h_1-\frac{1}{6}h_2)$. Thereby the contribution of $-\tilde\Delta_3^2$ to $\tilde\xi$ is

$$\tfrac{1}{20}\mathsf{m}\big(-2f_1f_2^2-2f_2f_3\big(\tfrac{1}{2}h_1-\tfrac{1}{6}h_2\big)\big)$$

and this element of $\mathrm{End}(\mathfrak{g})$ maps e_3 to $\frac{3}{20}f_2$.

Since $\sigma(\Delta_6)=\Delta_6$, the summands of Δ_6 that contain $e_3^2f_1$ as a factor are of tri-degrees $(3,3,0)$ or $(1,4,1)$ with respect to the $\mathbb{Z}/3\mathbb{Z}$-grading $\mathfrak{g}=(\mathbb{C}^3)^*\oplus\mathfrak{sl}_3\oplus\mathbb{C}^3$. By the weight considerations, the first possibility occurs only for the monomial $\mathsf{c}^3e_3^2f_1$. Record that

$$\mathrm{ad}(\mathsf{c})^3(e_3)=\mathrm{ad}(\mathsf{c})^2(-\mathsf{a})=[\mathsf{c},2\beta]=-6f_2.$$

The coefficient of $\mathsf{c}^3e_3^2f_1$ in Δ_6 is equal to $\frac{-4}{27}$. The monomials of the tri-degree $(1,4,1)$ are $\mathsf{c}\beta f_3e_3^2f_1$ and $\mathsf{c}\alpha f_2e_3^2f_1$. Their coefficients are $\frac{-4}{9}$ and $\frac{2}{3}$, respectively.

Next

$$\mathsf{m}(\mathsf{c}\beta f_3)(e_3)=\tfrac{1}{6}(\mathrm{ad}([\beta,\mathsf{c}])\mathrm{ad}(f_3)+2\mathrm{ad}(f_3)\mathrm{ad}([\beta,\mathsf{c}]))(e_3)=\tfrac{3}{2}\mathrm{ad}(f_2)\mathrm{ad}(f_3)(e_3)=-\tfrac{3}{2}f_2$$

and

$$\mathsf{m}(\mathsf{c}\alpha f_2)(e_3)=\tfrac{1}{2}(\mathrm{ad}(f_2)\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha)+\mathrm{ad}(f_2)\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c}))(e_3)=\tfrac{1}{2}[\mathrm{diag}(1,-2,1),f_2]=\tfrac{3}{2}f_2.$$

Summing up, we have

$$\tilde\xi(e_3)=\tfrac{1}{20}\big(3+\tfrac{8}{9}+\tfrac{2}{3}+1\big)f_2=\tfrac{1}{20}\big(5+\tfrac{5}{9}\big)f_2=\tfrac{1}{4}\big(1+\tfrac{1}{9}\big)f_2=\tfrac{5}{18}f_2,$$

and we are done. □

COROLLARY 6.4. *We have $\boldsymbol{b}=\frac{25}{108}$ and the invariant $\tilde H$ of Proposition 5.6 is equal to $\Delta_6-\frac{25}{108}\Delta_2^3$.*

Proof. By the definition of $\boldsymbol{b}$, we must have $(\tilde\xi-\boldsymbol{b}\xi)(e_3)=0$. From Lemmas 6.1 and 6.3 we obtain that $\boldsymbol{b}=\frac{5}{18}\times\frac{5}{6}=\frac{25}{108}$. □

Next we deal with $\tilde\eta$ for the summand $\tilde\eta\otimes e_3^2f_3$ of $\mathsf{m}(\Delta_6)$.

LEMMA 6.5. *For $\tilde\eta$ as above, we have*

$$\tilde\eta(\mathsf{a})=\tfrac{1}{20}\big(\tfrac{-2}{9}-\tfrac{28}{9}-\tfrac{4}{3}+\tfrac{2}{9}\big)\mathrm{ad}(f_3)(\mathsf{a})=\tfrac{-2}{9}\mathrm{ad}(f_3)(\mathsf{a})$$

and

$$\tilde\eta(h_3)=\tfrac{1}{20}\big(\tfrac{1}{3}+\tfrac{8}{27}+\tfrac{4}{9}+\tfrac{1}{27}\big)\mathrm{ad}(f_3)(h_3)=\tfrac{10}{9\times 20}\mathrm{ad}(f_3)(h_3)=\tfrac{1}{18}\mathrm{ad}(f_3)(h_3).$$

Proof. We go through the relevant summands of Δ_6. In $-\tilde\Delta_3^2$, these are $\frac{-1}{36}e_3^2f_3^2(3h_1-h_2)^2$ and $\frac{-1}{3}e_3^2f_1f_2f_3(3h_1-h_2)$. The corresponding contributions to $\tilde\eta$ are

$$\tfrac{-1}{18}\mathsf{m}(f_3(3h_1-h_2)^2)\quad\text{and}\quad\tfrac{-1}{3}\mathsf{m}(f_1f_2(3h_1-h_2))$$

multiplied by $\frac{1}{20}$. We are going to keep the factor $\frac{1}{20}$ in the background. Note that $\mathsf{m}(f_3(3h_1-h_2)^2)$ acts on a as $4\mathrm{ad}(f_3)$, and hence we add $\frac{-2}{9}$. Since $2-4+2=0$, the second of the above elements acts on a as zero. If we consider the action on h_3 instead, then the contribution of the first term is zero and $\mathsf{m}(f_1f_2(3h_1-h_2))$ acts as $\mathrm{ad}([f_1,f_2])=-\mathrm{ad}(f_3)$.

On account of σ, the other relevant terms have tri-degrees $(3,3,0)$, $(0,3,3)$, $(1,4,1)$, where the former two possibilities occur for $\frac{4}{27}\mathsf{c}^2\mathsf{b}e_3^2f_3$ and $\frac{4}{27}\alpha^2\beta e_3^2f_3$. Here

$$\begin{aligned}\mathsf{m}(\mathsf{c}^2\mathsf{b})(h_3)&=\tfrac{1}{3}(\mathrm{ad}(\mathsf{b})\mathrm{ad}(\mathsf{c})\mathrm{ad}(\mathsf{c})+\mathrm{ad}(\mathsf{c})\mathrm{ad}(\mathsf{b})\mathrm{ad}(\mathsf{c}))(h_3)\\&=\tfrac{1}{3}(-2\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c})-4\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha))(h_3)=(-2\mathrm{ad}(f_3)-2\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha))(h_3).\end{aligned}$$

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Since $[\alpha, h_3] = \alpha$ and $[\mathsf{c}, \alpha] = -3f_3$, the contribution in question is $\frac{4}{27}\mathrm{ad}(f_3)$. Similarly,

$$\begin{aligned}\mathsf{m}(\alpha^2\beta)(h_3) &= \tfrac{1}{3}(\mathrm{ad}(\beta)\mathrm{ad}(\alpha)\mathrm{ad}(\alpha) + \mathrm{ad}(\alpha)\mathrm{ad}(\beta)\mathrm{ad}(\alpha))(h_3)\\ &= \tfrac{1}{3}(2\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha) + 4\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c}))(h_3) = (2\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c}) - 2\mathrm{ad}(f_3))(h_3).\end{aligned}$$

Since $[\mathsf{c}, h_3] = \mathsf{c}$, $[\alpha, \mathsf{c}] = 3f_3$, we obtain again $\frac{4}{27}\mathrm{ad}(f_3)$. A slightly different story happens at a, namely,

$$\begin{aligned}\mathsf{m}(\mathsf{c}^2\mathsf{b})(\mathsf{a}) &= (-2\mathrm{ad}(f_3) - 2\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha) + \mathrm{ad}(\mathsf{c})\mathrm{ad}(\mathsf{c})\mathrm{ad}(\mathsf{b}))(\mathsf{a})\\ &= (-2\mathrm{ad}(f_3) - 2\mathrm{ad}(f_3) + 4\mathrm{ad}(f_3))(\mathsf{a}) = (-4+4)\mathrm{ad}(f_3)(\mathsf{a}) = 0,\\ \mathsf{m}(\alpha^2\beta)(\mathsf{a}) &= (2\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c}) - 2\mathrm{ad}(f_3) + \mathrm{ad}(\alpha)\mathrm{ad}(\alpha)\mathrm{ad}(\beta))(\mathsf{a})\\ &= (8-2-6)\mathrm{ad}(f_3)(\mathsf{a}) = 0.\end{aligned}$$

Now consider the terms of the tri-degree $(1,4,1)$. Let $\varepsilon_i - \varepsilon_j$ be the weight of the fourth element from $\mathfrak{sl}_3$. Assume first that $i \ne j$. Then $\varepsilon_3 - \varepsilon_1 = (\varepsilon_i - \varepsilon_j) + \varepsilon_s - \varepsilon_l$ for some s and l. One of the possibilities is $i = 3, j = 1$, and $s = l$. The other two come from the decomposition $\varepsilon_1 - \varepsilon_3 = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 - \varepsilon_3)$.

In the case $s = l$, the relevant term is $\frac{4}{9}e_3^2 f_3^2 \mathsf{b}\beta$ and its contribution to $\tilde{\eta}$ is $\frac{8}{9}\mathsf{m}(f_3\mathsf{b}\beta)$. Since both b and β commute with f_3 and h_3, we see that $\mathsf{m}(f_3\mathsf{b}\beta)(h_3) = 0$. Furthermore,

$$\mathsf{m}(f_3\mathsf{b}\beta)(\mathsf{a}) = \mathrm{ad}(\mathsf{b})\mathrm{ad}(\beta)(\mathsf{c}) - \tfrac{1}{2}\mathsf{c} = -3\mathsf{c} - \tfrac{1}{2}\mathsf{c} = \tfrac{-7}{2}\mathrm{ad}(f_3)(\mathsf{a}).$$

In this way the summand $\frac{-28}{9}$ appears in the first formula of the lemma.

In the case $s \ne l$, the relevant terms are $\frac{-4}{9}\mathsf{b}\alpha f_2 f_3 e_3^2$ and $\frac{-4}{9}\mathsf{c}\beta f_1 f_3 e_3^2$. On h_3, each of the elements $\mathsf{m}(\mathsf{b}\alpha f_2)$, $\mathsf{m}(\mathsf{c}\beta f_1)$ acts as $\frac{-1}{2}\mathrm{ad}(f_3)$. Thus, $\frac{4}{9}$ appears in the second formula. Further,

$$\begin{aligned}\mathsf{m}(\mathsf{b}\alpha f_2)(\mathsf{a}) &= \tfrac{1}{6}(\mathrm{ad}(f_2)\mathrm{ad}(\mathsf{b})\mathrm{ad}(\alpha) + 2\mathrm{ad}(\alpha)\mathrm{ad}(f_2)\mathrm{ad}(\mathsf{b}))(\mathsf{a})\\ &= \tfrac{1}{6}(\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha) + 2\mathrm{ad}(\alpha)\mathrm{ad}(\mathsf{c}))(\mathsf{a}) = \tfrac{1}{2}\mathrm{ad}(f_3)(\mathsf{a}) + \mathrm{ad}(f_3)(\mathsf{a}) = \tfrac{3}{2}\mathrm{ad}(f_3)(\mathsf{a}).\end{aligned}$$

In the same fashion $\mathsf{m}(\mathsf{c}\beta f_1)(\mathsf{a}) = \frac{3}{2}\mathrm{ad}(f_3)(\mathsf{a})$. This justifies $\frac{-4}{3}$ in the first formula.

The final term, which is $\frac{1}{27}\mathsf{c}\alpha(3h_1 - h_2)f_3e_3^2$, fulfils the case $i = j$, $s = 3$, $l = 1$. Here we have $\mathsf{m}((3h_1 - h_2)\mathsf{c}\alpha)(\mathsf{a}) = 6\mathrm{ad}(f_3)(\mathsf{a})$, hence the last summand in the first formula is $\frac{2}{9}$. Similarly,

$$\mathsf{m}((3h_1 - h_2)\mathsf{c}\alpha)(h_3) = \tfrac{1}{6}(2\mathrm{ad}((\alpha)\mathrm{ad}(\mathsf{c}) - 2\mathrm{ad}(\mathsf{c})\mathrm{ad}(\alpha))(h_3) = \mathrm{ad}(f_3)(h_3).$$

This justifies $\frac{1}{27}$ in the second formula. $\square$

Lemma 6.5 provides a different way to compute $\boldsymbol{b}$. Namely, $\tilde{\eta} - \boldsymbol{b}\eta$ has to act on $\mathfrak{g}$ as a scalar multiple of $\mathrm{ad}(f_3)$. Check

$$\Big(\tilde{\eta} - \frac{25}{108}\eta\Big)(\mathsf{a}) = \Big(-\frac{2}{9} - \frac{25}{108} \times \frac{42}{5}\Big)\mathrm{ad}(f_3)(\mathsf{a}) = \frac{-13}{6}\mathrm{ad}(f_3)(\mathsf{a}), \tag{6.2}$$

$$\Big(\tilde{\eta} - \frac{25}{108}\eta\Big)(h_3) = \Big(\frac{1}{18} - \frac{5 \times 48}{108}\Big)\mathrm{ad}(f_3)(h_3) = \frac{-39}{18}\mathrm{ad}(f_3)(h_3) = \frac{-13}{6}\mathrm{ad}(f_3)(h_3).$$

In order to compute $R(1)$ and $R(2)$, state first that according to (6.2), $\frac{-13}{6}\mathrm{ad}(f_3) \otimes e_3^2 f_3$ is a summand of $\mathsf{m}(\tilde{H})$. This indicates that if $\mathsf{m}(\tilde{H})$ is written as an element of $\mathcal{S}^4(\mathfrak{g})$, then it has a term $\frac{-13}{3}e_3^2 f_3^2$, which is a summand of $\frac{-13}{12}\Delta_2^2$. Thus $\mathsf{m}(\tilde{H}) = \frac{-13}{12}\Delta_2^2$. In terms of Lemma 5.4, we have

$$\mathsf{m}(\mathcal{H}^2) = \frac{3!}{4!} \times 4\Big(-2c_1 + \frac{1}{3}c_1\Big)\mathcal{H} = \frac{20}{3}\mathcal{H},$$

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since $c_1 = -4$ in our case. Making use of Theorem 3.11, we obtain the main result of this section:

$$S_2 = \varpi\big(\Delta_6 - \tfrac{25}{108}\Delta_2^3\big)[-1] - \tfrac{65}{4}\varpi(\tau^2\Delta_2^2[-1])\cdot 1 - \tfrac{325}{3}\varpi(\tau^4\Delta_2[-1])\cdot 1 \tag{6.3}$$

is an element of $\mathfrak{z}(\hat{\mathfrak{g}})$. Furthermore, $S_1 = \mathcal{H}[-1]$ and S_2 form a complete set of Segal–Sugawara vectors for $\mathfrak{g}$.

7. The orthogonal case

Now suppose that $\mathfrak{g} = \mathfrak{so}_n \subset \mathfrak{gl}_n$ with $n \geqslant 7$. A suitable matrix realisation of $\mathfrak{g}$ uses the elements $F_{ij} = E_{ij} - E_{j'i'}$ with $i, j \in \{1, \dots, n\}$, $i' = n - i + 1$. We will be working with the coefficients $\Phi_{2k} \in \mathcal{S}^{2k}(\mathfrak{g})^{\mathfrak{g}}$ of

$$\det(I_n - q(F_{ij}))^{-1} = 1 + \Phi_2 q^2 + \Phi_4 q^4 + \cdots + \Phi_{2k} q^{2k} + \cdots .$$

The generating invariants of this type appeared in [MY19, § 3] in connection with the symmetrisation map and they can be used in (2.4) as well. In [Mol18, MY19], the elements Φ_{2k} are called *permanents*, but they are not the permanents of matrices in the usual sense. Set $\mathfrak{h} = \langle F_{jj} \mid 1 \leqslant j \leqslant \ell\rangle_{\mathbb{C}}$.

In general, $\det(I_n - qA)^{-1} = \det(I_n + qA + q^2A^2 + \cdots)$ for $A \in \mathfrak{gl}_n$. In particular, $\Phi_{2k}|_{\mathfrak{h}}$ is equal to the homogeneous part of degree $2k$ of

$$\prod_{j=1}^{\ell}(1 + F_{jj}^2 + F_{jj}^4 + F_{jj}^6 + \cdots).$$

By the construction, $\mathsf{m}(\Phi_{2k})$ is a polynomial function on $(\Lambda^2\mathfrak{g} \oplus \mathfrak{g})^* \cong \Lambda^2\mathfrak{g} \oplus \mathfrak{g}$. Set

$$\boldsymbol{f} = \mathsf{m}(\Phi_{2k})|_{\Lambda^2\mathfrak{g}\oplus\mathfrak{h}} \quad \text{and write} \quad \boldsymbol{f} = \sum_{\nu}^{L} \xi_\nu \otimes \boldsymbol{H}_\nu,$$

where $\boldsymbol{H}_\nu \in \mathcal{S}^{2k-3}(\mathfrak{h})$ are linearly independent monomials in $\{F_{jj}\}$ and $\xi_\nu \in \Lambda^2\mathfrak{g}$. Note that each Φ_{2k} is an invariant of $\mathrm{Aut}(\mathfrak{g})$. Since Φ_{2k} is an element of $\mathfrak{h}$-weight zero, each ξ_ν is also of weight zero. Hence one can say that $\boldsymbol{f}$ is an invariant of $W(\mathfrak{g}, \mathfrak{h})$.

Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be an involution such that $\mathfrak{g}_0 = \mathfrak{g}^\sigma \cong \mathfrak{so}_{n-1}$, $\sigma(F_{11}) = -F_{11}$, i.e. $F_{11} \in \mathfrak{g}_1$, and $\sigma(F_{ss}) = F_{ss}$ for $\ell \geqslant s > 1$. Then $\mathfrak{g}_{0,F_{11}} := (\mathfrak{g}_0)_{F_{11}} \cong \mathfrak{so}_{n-2}$. Such an involution σ is not unique and we fix it by assuming that

$$\mathfrak{g}_1 = \langle F_{1i} + F_{i'1} \mid 1 < i < n\rangle_{\mathbb{C}} \oplus \mathbb{C}F_{11}. \tag{7.1}$$

The centraliser $\mathfrak{g}_{1,F_{11}}$ of F_{11} in $\mathfrak{g}_1$ is equal to $\mathbb{C}F_{11}$. This property defines an involution of *rank one*. Set $\mathfrak{h}_0 = \langle F_{ss} \mid \ell \geqslant s > 1\rangle_{\mathbb{C}}$.

By the construction, the map m is $\mathrm{Aut}(\mathfrak{g})$-equivariant. Here the group $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ acts on $\mathfrak{so}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ via conjugation. In particular, σ acts as $-\mathrm{id}$ on $\mathfrak{g}_0 \wedge \mathfrak{g}_1 \subset \mathfrak{so}(\mathfrak{g})$ and as id on the subspaces $\Lambda^2\mathfrak{g}_0$ and $\Lambda^2\mathfrak{g}_1$. For the future use, record: $\mathsf{m}(F_{ii}^3) = F_{ii}$ and if $i \neq j, j'$, then $\mathsf{m}(F_{ii}F_{jj}^2)$ acts as id on $F_{ij} = -F_{j'i'}$, $F_{ij'} = -F_{ji'}$, as $-\mathrm{id}$ on $F_{ji} = -F_{i'j'}$, $F_{j'i} = -F_{i'j}$, and as zero on all other elements F_{uw}. In particular, $\mathsf{m}(F_{ii}F_{jj}^2) \notin \mathfrak{g}$ if $i \notin \{j, j'\}$.

Lemma 7.1. *Suppose that $\boldsymbol{H}_\nu = F_{11}^{\beta_1} \dots F_{\ell\ell}^{\beta_\ell}$ and $\xi_\nu \neq 0$. Then there is exactly one odd β_j with $1 \leqslant j \leqslant \ell$. Furthermore, if β_1 is odd, then*

$$\xi_\nu \in \langle (F_{1i} - F_{i'1}) \wedge (F_{1i'} + F_{i1}) \mid 1 < i < n\rangle_{\mathbb{C}} \oplus \langle F_{11} \wedge F_{ss} \mid 1 < s \leqslant \ell\rangle_{\mathbb{C}}.$$

Proof. Without loss of generality assume that β_j is odd for $j \leqslant u$ and is even for $j > u$. Let $\sigma_j \in \mathrm{Aut}(\mathfrak{g})$ with $2 \leqslant j \leqslant u$ be an involution of rank one such that $\sigma_j(F_{jj}) = -1$ and $\sigma_j(F_{ss}) = F_{ss}$

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for $s \neq j, j'$. Following the case of $\sigma_1 = \sigma$, fix σ_j by setting

$$\sigma_j(F_{ji} + F_{i'j}) = -F_{ji} - F_{i'j} \quad \text{for } i \notin \{j, j'\}.$$

As we have already mentioned, $\sigma_j(\Phi_{2k}) = \Phi_{2k}$ for each j. Thereby $\mathsf{m}(\Phi_{2k})$ is a σ_j-invariant as well. At the same time $\sigma_j(\boldsymbol{H}_\nu) = -\boldsymbol{H}_\nu$ by the construction. Hence $\sigma_j(\xi_\nu) = -\xi_\nu$ for each $1 \leqslant j \leqslant u$.

The above discussion has clarified, how involutions σ_j act on $\Lambda^2\mathfrak{g} = \mathsf{m}(\mathcal{S}^3(\mathfrak{g}))$. In particular, we must have $\xi_\nu \in \mathfrak{g}_0 \wedge \mathfrak{g}_1$. We know also that ξ_ν is an element of $\mathfrak{h}$-weight zero and that $F_{11} \in \mathfrak{h}$. Recall that $\mathfrak{g}_{1,F_{11}} = \mathbb{C}F_{11}$. The decomposition $\mathfrak{g}_0 = \mathfrak{g}_{0,F_{11}} \oplus [F_{11}, \mathfrak{g}_1]$ indicates that

$$\xi \in \mathfrak{g}_{0,F_{11}} \wedge F_{11} \oplus [F_{11}, \mathfrak{g}_1] \wedge \mathfrak{g}_1.$$

Both summands here are $\mathfrak{h}$-stable. Furthermore, $(\mathfrak{g}_{0,F_{11}} \wedge F_{11})^{\mathfrak{h}}$ is spanned by $F_{ss} \wedge F_{11}$ with $\ell \geqslant s > 1$.

The subspace $[F_{11}, \mathfrak{g}_1]$ is spanned by $F_{1i} - F_{i'1}$, where $1 < i < n$. For each i, the element of the opposite $\mathfrak{h}_0$-weight in $\mathfrak{g}_1$ is $F_{1i'} + F_{i1}$. Note that $(F_{1i} + F_{i'1}) \wedge (F_{1i'} - F_{i1})$ is an eigenvector of F_{11} if and only if $i = i'$. Thus, $([F_{11}, \mathfrak{g}_1] \wedge \mathfrak{g}_1)^{\mathfrak{h}}$ is a linear span of

$$\Xi(i) := (F_{1i} + F_{i'1}) \wedge (F_{1i'} - F_{i1}) + (F_{1i'} + F_{i1}) \wedge (F_{1i} - F_{i'1})$$

with $1 < i \leqslant i'$.

If $u > 1$, then $u \geqslant 3$. The involution σ_2 acts on $F_{1i} \pm F_{i'1}$ as id if $2 < i < n-1$. Therefore ξ_ν has to be a linear combination of $F_{22} \wedge F_{11}$ and $\Xi(2)$. At the same time, σ_3 acts as id on both these vectors. This contradiction proves that $u = 1$. $\square$

Remark. Lemma 7.1 is valid for any homogeneous $\Phi \in \mathcal{S}(\mathfrak{g})^{\mathrm{Aut}(\mathfrak{g})}$.

Now fix $\boldsymbol{H} = \boldsymbol{H}_\nu = F_{11}^{2b_1-1} F_{22}^{2b_2} \dots F_{\ell\ell}^{2b_\ell}$ with $b_j \in \mathbb{Z}_{\geqslant 0}$ and $b_1 \geqslant 1$. The task is to compute $\xi = \xi_\nu$. Set $b_{j'} = b_j$ for $j \leqslant \ell$. In type B, set also $b_{\ell+1} = 0$. Below we list the terms Y_3 such that $Y_3\boldsymbol{H}$ is a summand of Φ_{2k}:

$$\begin{gathered} F_{11}^3, \quad F_{11}F_{jj}^2, \quad 2(b_1+1)(b_j+1)F_{11}F_{1j}F_{j1}, \quad (b_i+1)(b_j+1)F_{11}F_{ij}F_{ji}, \\ 2b_1(b_j+1)F_{1j}F_{j1}F_{jj}, \quad 2b_1(b_i+1)(b_j+1)F_{1i}F_{ij}F_{j1}, \end{gathered} \tag{7.2}$$

where $1 < i, j < n$ and $i \notin \{j, j'\}$, and also in F_{jj} we have $1 < j \leqslant \ell$. When computing m, one has to take into account the additional coefficients appearing from the powers of F_{ii}. For instance, in the case of F_{11}^3, this coefficient is $\binom{2b_1+2}{3}$, for $2b_1(b_j+1)F_{1j}F_{j1}F_{jj}$, the additional scalar factor is $2b_j + 1$.

We will show that ξ acts on F_{ij} as $c(i,j)F_{11}$ for some constant $c(i,j) \in \mathbb{C}$, compute these constants and see that all of them are equal. Note that $[F_{11}, F_{ij}] = 0$ if $i, j \notin \{1, n\}$.

Lemma 7.2. *We have $\xi(F_{11}) = 0$, furthermore $\xi(F_{ij}) = 0$ if $i, j \notin \{1, n\}$.*

Proof. By a direct computation, we show that indeed $\xi(F_{11}) = 0$. Some expressions in (7.2) act on F_{11} as zero by obvious reasons. If one takes into account that $F_{1j}F_{11}F_{j1} + F_{j1}F_{11}F_{1j}$ acts as $[F_{j1}, F_{j1}]$, this covers the first line of (7.2). The same argument takes care of $\mathsf{m}(F_{jj}F_{1j}F_{j1})$. It remains to calculate $\eta = \mathsf{m}(F_{1i}F_{ij}F_{j1})(F_{11})$. Here we have $6\eta = (F_{11} - F_{ii}) + (F_{jj} - F_{11})$. If we switch i and j, then the total sum is zero.

Since $(F_{ss} \wedge F_{11})(F_{11}) = F_{ss}$ up to a non-zero scalar, Lemma 7.1 now implies that

$$\xi \in \langle (F_{1i} - F_{i'1}) \wedge (F_{1i'} + F_{i1}) \mid 1 < i < n \rangle_{\mathbb{C}}.$$

Hence $\xi(F_{ij}) = 0$ if $i, j \notin \{1, n\}$. $\square$

LEMMA 7.3. *Suppose that $n = 2\ell$. Assume that $1 < u < n$. Then*

$$\xi(F_{1u}) = \frac{3!(2k-3)!}{(2k)!} C(1) F_{1u}$$

and $C(1)$ is equal to

$$\binom{2b_1+2}{3} + \frac{2}{3} b_1 \sum_{j=2}^{\ell} (2b_j+1)(b_j+1) + \frac{8}{3} b_1(b_1+1) \sum_{j=2}^{\ell} (b_j+1) + \frac{8}{3} b_1 \sum_{1<i<j\leqslant \ell} (b_i+1)(b_j+1).$$

Proof. Recall that $\mathsf{m}(F_{11}^3) = F_{11}$. This leads to the summand $\binom{2b_1+2}{3}$ of $C(1)$. Consider

$$\xi_1^{(j)} = \mathsf{m}(F_{11}F_{jj}^2) + \mathsf{m}(F_{1j}F_{j1}F_{jj} - F_{1j'}F_{j'1}F_{jj})$$

with $1 < j \leqslant \ell$. Here $\xi_1^{(j)}(F_{1u}) = \frac{1}{3}F_{1u}$ for $u \notin \{j, j'\}$ and $\xi_1^{(j)}(F_{1j}) = (1 - \frac{2}{3})F_{1j}$ as well as $\xi_1^{(j)}(F_{1j'}) = (1 - \frac{2}{3})F_{1j'}$. In $C(1)$, we have to add $\frac{1}{3}$ with the coefficients

$$2b_1 \binom{2b_j+2}{2} = 2b_1(b_j+1)(2b_j+1).$$

The next terms are $\xi_2^{(j)} = \mathsf{m}(F_{11}F_{1j}F_{j1})$ with $1 < j < n$. Here $\xi_2^{(j)}(F_{1u}) = \frac{1}{3}F_{1u}$ for $u \notin \{j, j'\}$. Furthermore, $\xi_2^{(j)}(F_{1j}) = \frac{2}{3}F_{1j}$ and $\xi_2^{(j)}(F_{1j'}) = 0$. Adding $\xi_2^{(j)}$ and $\xi_2^{(j')}$ with $j \leqslant \ell$ and recalling the coefficient of $\xi_2^{(j)}$, we obtain the summands $\frac{8}{3}(b_1+1)b_1(b_j+1)$.

Fix $1 < i, j < n$ with $i \notin \{j, j'\}$ and consider

$$\xi_3^{i,j} = \mathsf{m}(F_{1i}F_{ij}F_{j1}), \quad \xi_4^{i,j} = \mathsf{m}(F_{11}F_{ij}F_{ji}).$$

An easy observation is that $\xi_4^{i,j}(F_{1u}) = 0$ if $u \notin \{i, i', j, j'\}$. Also $\xi_3^{i,j}(F_{1u}) = \frac{1}{6}F_{1u}$ in this case. Furthermore, $\xi_4^{i,j}(F_{1i}) = \frac{1}{2}F_{1i}$ and $\xi_4^{i,j}(F_{1i'}) = \frac{1}{2}F_{1i'}$. A more lengthy calculation brings

$$\xi_3^{i,j}(F_{1i}) = \tfrac{1}{6}F_{1i} - \tfrac{1}{6}((\mathrm{ad}(F_{1i})\mathrm{ad}(F_{j1}) + \mathrm{ad}(F_{j1})\mathrm{ad}(F_{1i}))(F_{1j}) = \tfrac{1}{6}F_{1i} - \tfrac{1}{6}F_{1i} = 0;$$

$$\xi_3^{i,j}(F_{1j}) = \tfrac{1}{6}F_{1j} + \tfrac{1}{6}\mathrm{ad}(F_{ij})(F_{1i}) = 0; \quad \xi_3^{i,j}(F_{1j'}) = \tfrac{1}{6}\mathrm{ad}(F_{1i})\mathrm{ad}(F_{j1})(F_{1i'}) = -\tfrac{1}{6}F_{1j'};$$

$$\xi_3^{i,j}(F_{1i'}) = \tfrac{1}{6}\mathrm{ad}(F_{ij})\mathrm{ad}(F_{1i})(F_{ji'}) = \tfrac{1}{6}\mathrm{ad}(F_{ij})(F_{jn}) = -\tfrac{1}{6}F_{1i'}.$$

Note that $\xi_4^{i,j} = \xi_4^{j,i} = \xi_4^{i',j'}$. Now fix $1 < i < j \leqslant \ell$ and consider

$$\xi_5^{i,j} = \xi_3^{i,j} + \xi_3^{j,i} + \xi_3^{i',j} + \xi_3^{j,i'} + \xi_3^{i,j'} + \xi_3^{j',i} + \xi_3^{i',j'} + \xi_3^{j',i'} + \xi_4^{i,j} + \xi_4^{i',j} + \xi_4^{i,j'} + \xi_4^{i',j'}.$$

Here $2b_1(b_i+1)(b_j+1)\xi_5^{i,j}$ is a summand of $((2k)!/3!(2k-3)!)\xi$. Moreover, $\xi_5^{i,j}(F_{1s}) = \frac{4}{3}F_{1s}$ for each $1 < s < n$. This justifies the last summand of $C(1)$. $\square$

Rearranging the expression for $C(1)$, one obtains

$$C(1) = \frac{2}{3} b_1 \left(\sum_{j=1}^{\ell} (b_j+1)(2b_j+1) + 4 \sum_{1\leqslant i<j\leqslant \ell} (b_i+1)(b_j+1) \right). \tag{7.3}$$

LEMMA 7.4. *Suppose that $n = 2\ell+1$. Assume that $1 < u < n$. Then*

$$\xi(F_{1u}) = \frac{3!(2k-3)!}{(2k)!} \tilde{C}(1) F_{1u}$$

and $\tilde{C}(1)$ is equal to

$$C(1) + \frac{4}{3} b_1(b_1+1) + \frac{4}{3} b_1 \sum_{1<i\leqslant \ell} (b_i+1) = C(1) + \frac{4}{3} b_1 \sum_{1\leqslant j\leqslant \ell} (b_j+1).$$

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Proof. We have to take care of the instances, where $j=\ell+1=j'$. Here $F_{jj}=0$, thereby also $\xi_1^{(j)}=0$. By a direct calculation, $\xi_2^{(j)}(F_{1u})=\frac{1}{3}F_{1u}$ for each u. Recall that $\xi_2^{(j)}$ corresponds to $Y_3=2(b_1+1)(b_j+1)F_{11}F_{1j}F_{j1}$ in (7.2) and that the additional scalar factor in this case is $2b_1$. Since $b_{\ell+1}=0$, we have to add $\frac{4}{3}b_1(b_1+1)$ to $C(1)$.

The calculations for $\xi_3^{i,j}, \xi_3^{j,i}$, and $\xi_4^{i,j}$ have to be altered. The modifications are:

$$\xi_4^{i,j}(F_{1j})=F_{1j}, \quad \xi_3^{i,j}(F_{1j})=-\tfrac{1}{3}F_{1j}, \quad \xi_3^{j,i}(F_{1j})=-\tfrac{1}{3}F_{1j},$$

and $\xi_5^{i,j}$ with $1<i<j=l+1$ has a simpler form, here

$$\xi_5^{i,j}=\xi_3^{i,j}+\xi_3^{j,i}+\xi_3^{i',j}+\xi_3^{j,i'}+\xi_4^{i,j}+\xi_4^{i',j}.$$

The coefficient of this $\xi_5^{i,j}$ in $((2k)!/3!(2k-3)!)\xi$ is $2b_1(b_i+1)$ and $\xi_5^{i,j}(F_{1u})=\frac{2}{3}F_{1u}$ for all u. This justifies the second additional summand. □

Proposition 7.5. *For $\mathfrak{g}=\mathfrak{so}_n$, we have $\mathsf{m}(\Phi_{2k})=R(k)\Phi_{2k-2}$, where*

$$R(k)=\frac{1}{k(2k-1)}\left(\binom{n}{2}+2n(k-1)+(k-1)(2k-3)\right).$$

Proof. According to Lemmas 7.3 and 7.4, there is $c(1)\in\mathbb{C}$ such that $\xi(F_{1u})=c(1)F_{1u}$ for each $1<u<n$. Since $\xi\in\mathfrak{so}(\mathfrak{g})$, we have also $\xi(F_{u1})=-c(1)F_{u1}$. Taking into account Lemma 7.2, we conclude that $\xi=c(1)F_{11}$.

Simplifying (7.3) and using Lemma 7.4, we obtain that

$$\begin{aligned} c(1)&=\frac{2}{3}b_1\frac{3!(2k-3)!}{(2k)!}\left(2\left(\sum_{j=1}^{\ell}b_j\right)^2+(4\ell-1)\left(\sum_{j=1}^{\ell}b_j\right)+\ell+2\ell(\ell-1)\right)\\ &=\frac{b_1}{k(2k-1)(k-1)}(2(k-1)^2+(4\ell-1)(k-1)+\ell(2\ell-1))\end{aligned}$$

in type D and that

$$c(1)=\frac{b_1}{k(2k-1)(k-1)}(2(k-1)^2+(4\ell-1)(k-1)+\ell(2\ell-1)+2(k-1)+2\ell)$$

in type B. In both cases, the scalars $c(1)/b_1$ depend only on k and ℓ. Making use of the action of $W(\mathfrak{g},\mathfrak{h})$, we can conclude now that $\mathsf{m}(\Phi_{2k})$ is a symmetric invariant and that it is equal to $R(k)\Phi_{2k-2}$ with $R(k)\in\mathbb{Q}$. More explicitly, $R(k)$ is equal to $2(k-1)(c(1)/2b_1)=(k-1)c(1)/b_1$.

In type D, we have $2(k-1)^2+(4\ell-1)(k-1)=2n(k-1)+(k-1)(2k-3)$ and $\ell(2\ell-1)=\binom{n}{2}$. Quite similarly, in type B, we have $\ell(2\ell-1)+2\ell=\ell(2\ell+1)=\binom{n}{2}$ and

$$2(k-1)^2+(4\ell-1)(k-1)+2(k-1)=2n(k-1)+(k-1)(2k-3).$$

Therefore multiplying $c(1)$ with $(k-1)/b_1$ we obtain the desired formula for $R(k)$. □

We have

$$\begin{aligned} R(k)&=\frac{1}{k(2k-1)}\left(\binom{n}{2}+2n(k-1)+(k-1)(2k-3)\right)\\ &=\frac{1}{k(2k-1)}\left((k-1)(n+2k-3)+\frac{n}{2}(2k-2+n-1)\right)\\ &=\frac{(n+2k-3)(n+2k-2)}{2k(2k-1)}\end{aligned}$$

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and then

$$\binom{2k}{2r}\prod_{u=0}^{r-1}R(k-u)=\binom{2k}{2r}\prod_{u=1}^{r}\frac{(n+2k-2u)(n+2k-2u-1)}{(2k-2u+2)(2k-2u+1)}=\binom{n+2k-2}{2r}.$$

Iterating the map m, sf. (2.4) and Theorem 3.11, we obtain the following result.

Theorem 7.6. *For any $k\geqslant 2$,*

$$S_k=\varpi(\Phi_{2k})[-1]+\sum_{1\leqslant r<k}\binom{n+2k-2}{2r}\varpi(\tau^{2r}\Phi_{2k-2r}[-1])\cdot 1$$

is a Segal–Sugawara vector.

8. Applications and open questions

The Feigin–Frenkel centre can be used in order to construct commutative subalgebras of the enveloping algebra in finite-dimensional cases. There are two most remarkable instances.

8.1 Quantum Mishchenko–Fomenko subalgebras

Recall the construction from [Ryb06]. For any $\mu\in\mathfrak{g}^*$ and a non-zero $u\in\mathbb{C}$, the map

$$\varrho_{\mu,u}\colon \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])\to\mathcal{U}(\mathfrak{g}),\quad xt^d\mapsto u^dx+\delta_{d,-1}\mu(x),\quad x\in\mathfrak{g},\tag{8.1}$$

defines a G_μ-equivariant algebra homomorphism. The image of $\mathfrak{z}(\hat{\mathfrak{g}})$ under $\varrho_{\mu,u}$ is a commutative subalgebra $\tilde{\mathcal{A}}_\mu$ of $\mathcal{U}(\mathfrak{g})$, which does not depend on u [Ryb06, FFTL10]. Moreover, $\mathrm{gr}(\tilde{\mathcal{A}}_\mu)$ contains the *Mishchenko–Fomenko subalgebra* $\mathcal{A}_\mu\subset\mathcal{S}(\mathfrak{g})$ associated with μ, which is generated by all *μ-shifts* $\partial_\mu^m H$ of the $\mathfrak{g}$-invariants $H\in\mathcal{S}(\mathfrak{g})$. The main property of $\mathcal{A}_\mu$ is that it is *Poisson-commutative*, i.e. $\{\mathcal{A}_\mu,\mathcal{A}_\mu\}=0$ [MF78]. If $\mu\in\mathfrak{g}^*\cong\mathfrak{g}$ is *regular*, i.e. if $\dim\mathfrak{g}_\mu=\mathrm{rk}\,\mathfrak{g}$, then $\mathcal{A}_\mu$ is a maximal with respect to inclusion Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ [PY08] and hence $\mathrm{gr}(\tilde{\mathcal{A}}_\mu)=\mathcal{A}_\mu$. Several important properties and applications of (quantum) MF-subalgebras are discussed e.g. in [Vin91, FFR10].

Mishchenko–Fomenko subalgebras were introduced in [MF78], before the appearance of the Feigin–Frenkel centre. In [Vin91], Vinberg posed a problem of finding a *quantisation* of $\mathcal{A}_\mu$. A natural idea is to look for a solution given by the symmetrisation map ϖ. For $\mathfrak{g}=\mathfrak{gl}_n$, the elements $\varpi(\partial_\mu^m\Delta_k)\in\mathcal{U}(\mathfrak{g})$ with $1\leqslant k\leqslant n$, $0\leqslant m<k$ commute and therefore produce a solution to Vinberg's quantisation problem [Tar00, FM15, MY19].

Consider $\mathcal{F}[\bar a]=\varpi(F)[\bar a]\in\mathcal{U}(\hat{\mathfrak{g}}^-)$ corresponding to $F\in\mathcal{S}^m(\mathfrak{g})^{\mathfrak{g}}$ in the sense of (0.3). Set $p=|\{i\mid a_i=-1\}|$. Then

$$\langle\varrho_{\mu,u}(\mathcal{F}[\bar a])\mid u\in\mathbb{C}\backslash\{0\}\rangle_{\mathbb{C}}=\langle\varpi(\partial_\mu^lF)\mid 0\leqslant l\leqslant p\rangle_{\mathbb{C}}.\tag{8.2}$$

Combining (8.2) with (2.2), we conclude immediately that for $\mathfrak{g}=\mathfrak{gl}_n$, the algebra $\tilde{\mathcal{A}}_\mu$ is generated by $\varpi(\partial_\mu^m\Delta_k)$. This observation is not new, see [MY19, § 3] and in particular § 3.2 there for a historical overview and a more elaborated proof.

In [MY19, § 3.3], sets of generators $\{H_i\mid 1\leqslant i\leqslant\ell\}$ of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that

$$\tilde{\mathcal{A}}_\mu=\mathsf{alg}\langle\varpi(\partial_\mu^mH_i)\mid 1\leqslant i\leqslant\ell,\ 0\leqslant m<\deg H_i\rangle\tag{8.3}$$

are exhibited in types B, C, and D. We rejoice to say that in type C, $H_k=\Delta_{2k}$ in the notation of § 4. In the even orthogonal case, the set $\{H_i\}$ includes the Pfaffian. The other generators in types B and D are Φ_{2k} of § 7. Thus Theorems 4.4 and 7.6 provide a new proof of [MY19, Theorem 3.2].

In conclusion, we show that Proposition 5.6 confirms Conjecture 3.3 of [MY19] in type G_2.

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Proposition 8.1. *Let $\mathfrak{g}$ be a simple Lie algebra of type G_2. Let $\tilde{H} \in \mathcal{S}^6(\mathfrak{g})$ be a $\mathfrak{g}$-invariant satisfying (0.2), cf. Corollaries 5.7 and 6.4. Then $\tilde{\mathcal{A}}_\mu$ is generated by $\mu, \mathcal{H}$, and $\varpi(\partial_\mu^m \tilde{H})$ with $0 \leqslant m \leqslant 5$.*

Proof. We work with μ as with an element of $\mathfrak{g}$. Clearly $\partial_\mu \mathcal{H} = 2\sum(x_i,\mu)x_i = 2\mu$ and

$$\varrho_{\mu,u}(\mathcal{H}[b_1,b_2]) = u^{b_1+b_2}\mathcal{H} + (u^{b_2}\delta_{b_1,-1} + u^{b_1}\delta_{b_2,-1})\mu + \delta_{b_1,-1}\delta_{b_2,-1}(\mu,\mu).$$

Let S_2 be the Segal–Sugawara vector provided by (5.1) and (6.3). Set also $S_1 = \mathcal{H}[-1]$. Then $\{S_1, S_2\}$ is a complete set of Segal–Sugawara vectors. A general observation is that $\tilde{\mathcal{A}}_\mu$ is generated by $\{\varrho_{\mu,u}(S_\nu) \mid u \in \mathbb{C}^\times, \nu = 1,2\}$ [Mol18, Corollary 9.2.3]. We have already computed the images of S_1 and also of $\varpi(\tau^4\mathcal{H}[-1]){\cdot}1$. Similarly to (8.2), the images of $\varpi(\tilde{H})[-1]$ span $\langle\varpi(\partial_\mu^m H)\rangle_{\mathbb{C}}$. It remains to deal with

$$\mathcal{Y} = \varrho_{\mu,u}(\varpi(\tau^2\mathcal{H}^2[-1]){\cdot}1) = u^{-6}\mathcal{Y}_4 + u^{-5}\mathcal{Y}_3 + u^{-4}\mathcal{Y}_2 + u^{-3}\mathcal{Y}_1.$$

Here $\mathcal{Y}_1$ is proportional to μ; the term $\mathcal{Y}_2$ is a linear combination of $\mathcal{H}$ and μ^2. Furthermore, $\mathcal{Y}_3$ is a linear combination of $\mu\mathcal{H}$ and $\sum_i x_i\mu x_i$, therefore of $\mu\mathcal{H}$ and μ, cf. Lemma 5.4. Finally, $\mathcal{Y}_4$ is a linear combination of $\mathcal{H}^2$ and $\sum_{i,j} x_ix_jx_jx_i = \sum_i x_i\mathcal{H}x_i = \mathcal{H}^2$,

$$\sum_{i,j} x_ix_jx_ix_j = \mathcal{H}^2 + \sum_j c_1x_jx_j = \mathcal{H}^2 + c_1\mathcal{H}.$$

This completes the proof. □

8.2 Gaudin algebras

Recall that elements $S \in \mathfrak{z}(\hat{\mathfrak{g}})$ give rise to higher Hamiltonians of the Gaudin models, which describe completely integrable quantum spin chains [FFR94].

The underlying space of a Gaudin model is the direct sum of n-copies of $\mathfrak{g}$, and the Hamiltonians are the following sums

$$\mathcal{H}_k = \sum_{j\neq k} \frac{\sum_i x_i^{(k)}x_i^{(j)}}{z_k - z_j}, \quad 1 \leqslant k \leqslant n,$$

where $z_1,\dots,z_n$ are pairwise different complex numbers. Here $\{x_i^{(k)} \mid 1 \leqslant i \leqslant \dim\mathfrak{g}\}$ is an orthonormal basis for the k'th copy of $\mathfrak{g}$. These Gaudin Hamiltonians can be regarded as elements of $\mathcal{U}(\mathfrak{g})^{\otimes n}$ or of $\mathcal{S}(\mathfrak{g}\oplus\cdots\oplus\mathfrak{g})$. They commute (and hence Poisson-commute) with each other. Higher Gaudin Hamiltonians are elements of $\mathcal{U}(\mathfrak{g})^{\otimes n}$ that commute with all $\mathcal{H}_k$.

The construction of [FFR94] produces a *Gaudin subalgebra* $\mathcal{G}$, which consists of Gaudin Hamiltonians and contains $\mathcal{H}_k$ for each k. Let $\Delta\mathcal{U}(\hat{\mathfrak{g}}^-) \cong \mathcal{U}(\hat{\mathfrak{g}}^-)$ be the diagonal of $\mathcal{U}(\hat{\mathfrak{g}}^-)^{\otimes n}$. Then a vector $\bar{z} = (z_1,\dots,z_n) \in (\mathbb{C}^\times)^n$ defines a natural homomorphism $\rho_{\bar{z}}\colon \Delta\mathcal{U}(\hat{\mathfrak{g}}^-) \to \mathcal{U}(\mathfrak{g})^{\otimes n}$. In this notation, $\mathcal{G} = \mathcal{G}(\bar{z})$ is the image of $\mathfrak{z}(\hat{\mathfrak{g}})$ under $\rho_{\bar{z}}$. By the construction, $\mathcal{G} \subset (\mathcal{U}(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$. Since $\mathfrak{z}(\hat{\mathfrak{g}})$ is homogeneous in t, it is clear that $\mathcal{G}(\bar{z}) = \mathcal{G}(c\bar{z})$ for any non-zero complex number c.

Gaudin subalgebras have attracted a great deal of attention, see e.g. [CFR10] and references therein. They are closely related to quantum Mishchenko–Fomenko subalgebras and share some of their properties. In particular, for a generic $\bar{z}$, the action of $\mathcal{G}(\bar{z})$ on an irreducible finite-dimensional $(\mathfrak{g}\oplus\cdots\oplus\mathfrak{g})$-module $V(\lambda_1)\otimes\cdots\otimes V(\lambda_n)$ is diagonalisable and has a simple spectrum on the subspace of highest weight vectors of the diagonal $\mathfrak{g}$ [Ryb20]. Applying $\rho_{\bar{z}}$, one obtains explicit formulas for higher Gaudin Hamiltonians from explicit formulas for the generators of $\mathfrak{z}(\hat{\mathfrak{g}})$. In the following, we discuss which generators of $\mathfrak{z}(\hat{\mathfrak{g}})$ one has to consider.

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Let $\{S_1, \dots, S_\ell\}$ with $\mathrm{gr}(S_k) = H_k[-1]$ be a complete set of Segal–Sugawara vectors as in § 1.6. Set $\deg H_k =: d_k$. Assume that $z_k \neq z_j$ for $k \neq j$ and that $z_k \neq 0$ for all k. According to [CFR10, Proposition 1], $\mathcal{G}(\bar{z})$ has a set of algebraically independent generators $\{F_1, \dots, F_{\boldsymbol{B}(n)}\}$, where $\boldsymbol{B}(n) := ((n-1)/2)(\dim\mathfrak{g} + \ell) + \ell$, see also [Ryb06, Theorems 2&3]. Moreover, exactly $(n-1)d_k + 1$ elements among the F_j belong to $\langle \rho_{\bar{z}}(\tau^m(S_k)) \mid m \geqslant 0\rangle_{\mathbb{C}}$ [CFR10, Proposition 1]. Note that $\sum_{k=1}^{\ell} d_k$ equals $(\dim\mathfrak{g} + \ell)/2$. Furthermore, the symbols $\mathrm{gr}(F_j)$ are algebraically independent as well and $\deg\mathrm{gr}(F_j) = d_k$ if $F_j \in \langle \rho_{\bar{z}}(\tau^m(S_k)) \mid m \geqslant 0\rangle_{\mathbb{C}}$, see the proof of [CFR10, Proposition 1(2)] and [Ryb06, § 4].

Remark 8.2. If $\mathcal{A} \subset (\mathcal{S}(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$ is a Poisson-commutative algebra, then $\mathrm{tr.deg}\,\mathcal{A} \leqslant \boldsymbol{B}(n)$ by [MY19, Proposition 1.1]. Combining this with [BK76, Satz 5.7], we obtain that $\mathrm{tr.deg}\,\tilde{\mathcal{A}} \leqslant \boldsymbol{B}(n)$ for a commutative subalgebra $\tilde{\mathcal{A}} \subset (\mathcal{U}(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$. Thus $\mathcal{G}$ has the maximal possible transcendence degree. Arguing in the spirit of [PY08] and using the results of [Ryb06], one can show that $\mathcal{G}$ is also a maximal commutative subalgebra of $(\mathcal{U}(\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$ with respect to inclusion.

In the case $n = 2$, the application of our result looks particularly nice. Besides, this *two points* case has several features. Suppose that $n = 2$. Set $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}$. For $H \in \mathcal{S}^d(\mathfrak{l})$, $\xi^{(1)}$ in the first copy of $\mathfrak{g}$, $\eta^{(2)}$ in the second, and a non-zero $c \in \mathbb{C}$, write

$$H(\xi^{(1)} + c\eta^{(2)}) = H_{d,0}(\xi^{(1)}) + cH_{d-1,1}(\xi^{(1)}, \eta^{(2)}) + \cdots + c^{d-1}H_{1,d-1}(\xi^{(1)}, \eta^{(2)}) + c^d H_{0,d}(\eta^{(2)}). \tag{8.4}$$

Here $H_{d,0}$ belongs to the symmetric algebra of the first copy of $\mathfrak{g}$. The symbol of $\rho_{\bar{z}}(\tau^m(S_k))$ lies in $\langle (H_k)_{d_k-j,j} \mid 0 \leqslant j \leqslant d_k\rangle_{\mathbb{C}}$. Since we must have $d_k + 1$ linearly independent elements among these symbols, $\mathrm{gr}(\mathcal{G})$ is freely generated by $(H_k)_{d_k-j,j}$ with $1 \leqslant k \leqslant \ell$, $0 \leqslant j \leqslant d_k$.

The Lie algebra $\mathfrak{l}$ has the following symmetric decomposition

$$\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1, \quad \text{where } \mathfrak{l}_1 = \{(\xi, -\xi) \mid \xi \in \mathfrak{g}\} \tag{8.5}$$

and $\mathfrak{l}_0 = \Delta\mathfrak{g} = \{(\xi, \xi) \mid \xi \in \mathfrak{g}\}$ is the diagonal. Similarly to (8.4), one polarises $H \in \mathcal{S}^d(\mathfrak{l})$ with respect to the decomposition (8.5). Let $H_{[j,d-j]}$ with $0 \leqslant j \leqslant d$ be the arising components. Then $\langle H_{j,d-j} \mid 0 \leqslant j \leqslant d\rangle_{\mathbb{C}} = \langle H_{[j,d-j]} \mid 0 \leqslant j \leqslant d\rangle_{\mathbb{C}}$.

On the one side, the polynomials $(H_k)_{j,d_k-j}$ generate $\mathrm{gr}(\mathcal{G})$, on the other, the polynomials $(H_k)_{[j,d_k-j]}$ generate a Poisson-commutative subalgebra $\mathcal{Z} \subset \mathcal{S}(\mathfrak{l})$ related to the symmetric pair $(\mathfrak{l}, \mathfrak{l}_0)$, which has many interesting properties [PY21]. Thus, our discussion results in the following statement.

Corollary 8.3 (cf. [PY21, Example 6.5]). *The two-points Gaudin subalgebra $\mathcal{G}(z_1, z_2)$ is a quantisation of the Poisson-commutative subalgebra $\mathcal{Z}$ associated with the symmetric pair $(\mathfrak{g} \oplus \mathfrak{g}, \Delta\mathfrak{g})$.*

Let us give more information on the issue of Corollary 8.3. Observe that $\mathcal{G}(z_1, z_2) = \mathcal{G}(z_1 - b, z_2 - b)$ if $b \in \mathbb{C}\backslash\{z_1, z_2\}$, see [CFR10, Proposition 1]. Hence

$$\mathcal{G}(\bar{z}) = \mathcal{G}\left(\frac{z_1 - z_2}{2}, \frac{z_2 - z_1}{2}\right) = \mathcal{G}(1, -1).$$

For $\rho = \rho_{1,-1}$, we have $\rho(\xi t^k) = \xi^{(1)} + (-1)^k \xi^{(2)}$, i.e. $\mathfrak{g}[-1]$, as well as any $\mathfrak{g}[-2k-1]$, is mapped into $\mathfrak{l}_1$ and each $\mathfrak{g}[-2k]$ is mapped into $\Delta\mathfrak{g}$. One can understand ρ as the map from $\mathcal{U}(\hat{\mathfrak{g}}^-)$ to $\mathcal{U}(\hat{\mathfrak{g}}^-)/(t^2 - 1) \cong \mathcal{U}(\mathfrak{l})$.

It is not difficult to see that $\mathrm{gr}(\rho(S_k)) = (H_k)_{[0,d_k]}$, $\mathrm{gr}(\rho(\tau(S_k))) = (H_k)_{[1,d_k-1]}$, and in general

$$\mathrm{gr}(\rho(\tau^m(S_k))) \in m!(H_k)_{[m,d_k-m]} + \langle (H_k)_{[j,d_k-j]} \mid j < m\rangle_{\mathbb{C}}$$

as long as $m \leqslant d_k$. This shows that indeed $\mathrm{gr}(\mathcal{G}) = \mathcal{Z}$ and that

$$\mathcal{G} = \mathsf{alg}\langle \rho(\tau^m(S_k)) \mid 1 \leqslant k \leqslant \ell, 0 \leqslant m \leqslant d_k \rangle. \tag{8.6}$$

Suppose that $H_1, \dots, H_\ell$ are homogeneous generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and for each k there is k' such that $0 \leqslant k' < k$ and $\mathsf{m}(H_k) \in \mathbb{C}H_{k'}$, where $H_0 = 0$. If $\mathfrak{g}$ is simple and classical, then explicit descriptions of such sets are contained in §§ 2, 4, 7.

Theorem 8.4. *If we keep the above assumption on the set $\{H_k\}$, then the two-points Gaudin subalgebra $\mathcal{G} \subset \mathcal{U}(\mathfrak{l})$ is generated by $\varpi((H_k)_{d_k - j, j})$ with $1 \leqslant k \leqslant \ell$, $0 \leqslant j \leqslant d_k$.*

Proof. For each k, let S_k be the Segal–Sugawara vector obtained from H_k by (2.4). We will show that $V_{\mathrm{sym}} := \langle \varpi((H_k)_{d_k - j, j}) \mid 1 \leqslant k \leqslant \ell,\ 0 \leqslant j \leqslant d_k \rangle_{\mathbb{C}}$ is equal to

$$V_{\mathcal{G}} := \langle \rho(\tau^m(S_k)) \mid 1 \leq k \leq \ell,\ 0 \leq m \leq d_k \rangle_{\mathbb{C}}.$$

Since $\dim V_{\mathcal{G}} = \dim V_{\mathrm{sym}}$, it suffices to prove the inclusion $V_{\mathcal{G}} \subset V_{\mathrm{sym}}$. We argue by induction on k. If $k = 1$, then $S_1 = \varpi(H_1[-1])$. Hence $\langle \rho(\tau^m(S_1)) \mid 0 \leqslant m \leqslant d_1 \rangle_{\mathbb{C}}$ is equal to

$$\langle \varpi((H_1)_{d_1 - j, j}) \mid 0 \leqslant j \leqslant d_1 \rangle_{\mathbb{C}}.$$

If $k \geqslant 2$ and $m \leqslant d_k$, then according to the structure of (2.4) and our condition on $\{H_1, \dots, H_\ell\}$, we have

$$\rho(\tau^m(S_k)) \in m! \varpi((H_k)_{[m, d_k - m]}) + V_{m,k},$$

where

$$V_{m,k} = \langle \varpi((H_k)_{[j, d_k - j]}) \mid j < m \rangle_{\mathbb{C}} \oplus \langle \varpi((H_{k'})_{j, d_{k'} - j}) \mid k' < k,\ 0 \leqslant j < d_{k'} \rangle_{\mathbb{C}}.$$

Thus $\rho(\tau^m(S_k)) \in V_{\mathrm{sym}}$ and we are done. □

8.3 Further directions

For all classical types, we find families of generators $\{H_k\}$ that behave well in terms of (0.2). The general picture is not complete yet, since the following question remains open.

Question 8.5. Does any exceptional Lie algebra $\mathfrak{g}$ poses a set of generators $\{H_k\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that each H_k satisfies (0.2)?

Proposition 5.6 takes care of type G_2. We have seen also some partial positive answers in other types.

Question 8.6. Are there homogeneous generators $\{H_k\}$ of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\mathsf{m}(H_k) = 0$ for each k?

The calculations in § 6 prove that in type G_2, the answer is negative. I would expect that the answer is negative in general.

As Example 3.7 shows, a set of generators $\{H_k\}$, where each H_k satisfies (0.2), is not unique. For the classical Lie algebras, there is a freedom of choice in degree 4 and there is also some freedom in degree 6.

It is quite possible that the condition (8.3) on the set $\{H_k\}$ is less restrictive than (0.2). However, we have no convincing evidence to this point.

Remark 8.7. Probably there are some intricate combinatorial identities hidden in (2.4). In order to reveal them, one has to understand the natural numbers $c(r, \bar{a})$ appearing in the proof of Lemma 2.2, the rational constants $c_{2,3}(j, p)$ of Lemma 3.2, as well as the scalars $C(\bar{a}^{(r)}, \bar{\gamma})$ of Proposition 3.4.

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