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DIFFERENTIAL FORMS ON STRATIFIED SPACES

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Abstract

First, we extend the notion of stratified spaces to diffeology. Then we characterise the subspace of stratified differential forms, or zero-perverse forms in the sense of Goresky–MacPherson, which can be extended smoothly into differential forms on the whole space. For that we introduce an index which outlines the behaviour of the perverse forms on the neighbourhood of the singular strata.

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1. Introduction

Stratified spaces were introduced in the founding papers of Mather [16], Thom [22] and Whitney [23]. From a pure differential point of view, the standard definition of stratified spaces is unsatisfactory. The simple juxtaposition of the topological structure of the global space, with the relatively independent smooth structures of the strata that constitute the space, is perturbing, especially when it comes to Cartan calculus on stratified spaces, for which a global smooth structure is obviously needed.

As differential geometers we expect a unique smooth structure on the whole object that captures at the same time its global smooth structure, even stratified, and the individual structure of each stratum. Diffeology is a good candidate for such a framework [10], because it can mix a global singular smooth structure with individual characteristics of the strata. Since the Cartan calculus is well developed in diffeology, it will apply straightforwardly on stratified spaces.

Diffeology has already been used to solve questions involving singularities and smooth structures. Examples include dense foliations [4, 13] or orbifolds [12, 14] and, combined with differential forms, to integrate general closed 1- and 2-forms with any countable group of periods [10, Sections 8.29 and 8.42] or in symplectic reduction with singular orbits [11]. The dual approach to the global smooth structure was also used to investigate differential structures by Pflaum [17] and Śniatycki [20].

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As we shall see in Section 2, it is not difficult to adapt the ordinary definition of stratified spaces to diffeology. It is basically a diffeological space, with a stratification for the D-topology (Sections 2.1 and 2.3). That definition leaves a large degree of freedom on the balance between the diffeology and the stratification, since there can be more than one diffeology on a space that gives the same D-topology. That leads us to single out a subcategory of diffeological spaces for which the stratification is defined by its geometry, that is, by the action of its pseudo-group of local diffeomorphisms (Section 2.4). In this case, the stratification is completely encrypted in the diffeology: the strata are the connected components of the orbits of the pseudo-group of local diffeomorphisms. That defines a subcategory of diffeology we can call *stratified diffeology*. Manifolds with corners are simple examples of such stratified diffeology (see [8]). We leave the study of this subcategory of *geometric stratified spaces* for a later time.

We now return to the question of Cartan calculus on stratified spaces. Since diffeological spaces have a well-defined De Rham complex, stratified diffeological spaces inherit this complex immediately. In the literature on stratified spaces, there already exists a notion of 'stratified (differential) form' as the form of (general) perversity p, according to Goresky and MacPherson [5, 6], and defined precisely by Brylinski [3] and reconsidered by Brasselet *et al.* [1]. The aim of these authors is to establish a pairing between a complex of singular intersection chains and the complex of stratified forms with perversity p. These complexes of perverse forms are also involved in the computation of the equivariant intersection cohomology, originally by Brylinski, but also by Brion [2].

A natural question is to compare these two classes of objects: differential forms versus stratified forms, beginning with perversity 0. That is the case we treat in this paper. Stratified forms are defined only on the regular part of a stratified space, with some conditions on the neighbourhood of the strata, while differential forms are defined on the whole space [10, Section 6.28]. The question is then to characterise the stratified forms that are the restriction of differential forms.

For that purpose we introduce an index that counts the number of different differential forms defined on each strata (precisely, on the universal coverings of the strata) by a given stratified form (Section 3.1). We show that if the form has index 1 for any stratum, then the stratified form extends to a differential form for the diffeology involved (Section 3.2).

Conversely, assuming that for any two points in the regular subspace there always exists a smooth path connecting them that cuts the singular subspace into a finite number of points, we show that the restriction of a 0-perverse differential form has its index constant and equal to 1 for each stratum (Section 3.2). The condition we introduce here seems to be not optimal, but it is natural on the most common stratified spaces, for example on semi-algebraic sets.

2. Stratified spaces

In this section we recall the standard definition of stratified spaces, in the style of Kloeckner's survey [15]. Then we recall the notion of diffeology and the associated smooth category, leading to a natural version of stratified spaces in diffeology.

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2.1. Basic stratified spaces. A stratification on a diffeological space X is a partition $\mathcal{S}$ of X into *strata*,

$$X = \bigcup_{S \in \mathcal{S}} S, \quad \text{with } S \neq S' \ \Rightarrow\ S \cap S' = \varnothing,$$

satisfying the *boundary condition*

$$S \cap \bar{S}' \neq \varnothing \Rightarrow S \subset \bar{S}',$$

where $\bar{S}'$ represents the closure of S′ for the D-topology of X [10, Section 2.8].

The boundary condition can be formulated as follows: the closure of a stratum is a union of strata. In the usual case, where strata are manifolds, the strata are organised by dimension and define a filtration:

$$X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X.$$

The *subsets of the filtration* X_j are the unions of all the strata with dimension less than or equal to a given dimension, let us say n_j. The subset $X_j - X_{j-1}$ is open in X_j and its components are the strata of dimension n_j. The subset $X_k - X_{k-1}$ is called the *regular part* of X and is denoted by X_{reg}. It is the union of the strata of maximal dimension. The subset X_{k-1} is called the *singular part* and is denoted by X_{sing}.

For all $j > 0$, the subset $X' = X_j$ of the filtration is itself a stratified space with regular part $X'_{\text{reg}} = X_j - X_{j-1}$. The subset X_0 is the union of the strata of minimal dimension. It is a stratified space without singular part.

In the following we shall assume that:

- the space X is connected, Hausdorff and metrisable;
- the regular part is an open dense subset;
- equipped with the subset diffeology, the strata are locally closed manifolds;
- the number of strata is finite.

In a future work it could be possible to ease these conditions.

2.2. Diffeology and diffeological spaces. A *diffeology* on a set X is the choice of a set $\mathcal{D}$ of parametrisations in X which satisfies the following axioms.

(1) Covering: $\mathcal{D}$ contains the constant parametrisations.

(2) Locality: Let P be a parametrisation in X. If for all $r \in \text{dom}(P)$ there is an open neighbourhood V of r such that $P \restriction V \in \mathcal{D}$, then $P \in \mathcal{D}$.

(3) Smooth compatibility: For all $P \in \mathcal{D}$, for all $F \in C^\infty(V, \text{dom}(P))$, where V is a Euclidean domain, $P \circ F \in \mathcal{D}$.

We recall that a parametrisation is a map defined on an open subset of a Euclidean space. A set X equipped with a diffeology is a *diffeological space*. The elements of $\mathcal{D}$ are called *plots* of the diffeological space.

Smooth maps. A map $f : X \to X'$ is said to be *smooth* if for any plot P in X, $f \circ P$ is a plot in X′. If f is smooth, bijective and its inverse f^{-1} is smooth, then f is said to be a *diffeomorphism*.

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Diffeological spaces and smooth maps constitute the category *diffeology* whose isomorphisms are diffeomorphisms.

Subset diffeology and subspaces. Let A be a subset of a diffeological space X. The plots in X which take their values in A are a diffeology called *subset diffeology*. Equipped with this diffeology, A is said to be a *subspace* of X.

Local smooth maps. The finest topology on X such that the plots are continuous is called the D-topology. A map $f\colon \mathrm{A} \to \mathrm{X}'$, where A is a subset of X, is said to be a *local smooth* map if A is a D-open subset of X and f is smooth for the subset diffeology. We denote by $\mathrm{C}^\infty_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ the set of local smooth maps from X to X′.

Actually, f is local smooth if and only if, for all plots P in X, the composite $f \circ \mathrm{P}\colon \mathrm{P}^{-1}(\mathrm{A}) \to \mathrm{X}'$ is a plot. That implies in particular that A is D-open, by definition of the D-topology.

Local diffeomorphisms. We say that $f\colon \mathrm{A} \to \mathrm{X}'$ is a local diffeomorphism if f, as well as its inverse $f^{-1}\colon f(\mathrm{A}) \to \mathrm{X}$, are local smooth injective maps. We denote by $\mathrm{Diff}_{\mathrm{loc}}(\mathrm{X}, \mathrm{X}')$ the set of local diffeomorphisms from X to X′.

Diffeological fibration. We say that a smooth projection $\pi\colon \mathrm{T} \to \mathrm{B}$ between diffeological spaces is a diffeological fibration, or T is a diffeological fibre bundle over B, if for all plots $\mathrm{P}\colon \mathrm{U} \to \mathrm{B}$, the pullback $\mathrm{pr}_1\colon \mathrm{P}^*(\mathrm{T}) \to \mathrm{U}$ is locally trivial (see [9] and [10, Sections 8.8 and 8.9]).

2.3. Locally fibred stratified spaces. Consider a diffeological space X equipped with a stratification $\mathcal{S}$. The stratification is *locally fibred* if there exists a *tube system* $\{\pi_{\mathrm{S}}\colon \mathrm{TS} \to \mathrm{S})\}_{S \in \mathcal{S}}$ (see Figure 1) such that:

(1) TS is an open neighbourhood of S, called a *tube* over S;

(2) the map $\pi_{\mathrm{S}}\colon \mathrm{TS} \to \mathrm{S}$ is a smooth retraction which is a diffeological fibration, with fibres the stratified spaces;

(3) for all $x \in \mathrm{TS} \cap \mathrm{TS}' \cap \pi_{\mathrm{S}'}^{-1}(\mathrm{TS})$, one has $\pi_{\mathrm{S}}(\pi_{\mathrm{S}'}(x)) = \pi_{\mathrm{S}}(x)$.

The *locally cone-like* stratified spaces are in this category, as are essentially all kinds of stratified spaces (following Siebenmann [19]). However, in diffeology, this wording is ambiguous. Indeed, not all kinds of diffeological spaces that look like cones are equivalent, even if they share the same D-topology, as the following example shows. Consider the cone

$$C = \left\{(x, y, z) \in \mathbf{R}^3 \;\middle|\; z = \sqrt{x^2 + y^2}\right\}.$$

We can equip C with the subset diffeology and also with the diffeology of the *cone over the circle* S^1, that is, the pushforward of the diffeology of the cylinder, by

$$\pi\colon \mathrm{S}^1 \times [0, \infty) \to C \quad \text{with } \pi(u, t) = (tu, t).$$

Then the parametrisation defined on $\mathbf{R}$ by

$$\gamma\colon\ s \mapsto e^{-1/s^2} \begin{pmatrix} \cos 1/s^2 \\ \sin 1/s^2 \\ 1 \end{pmatrix} \quad \text{if } s \neq 0 \text{ and } \gamma(0) = 0$$

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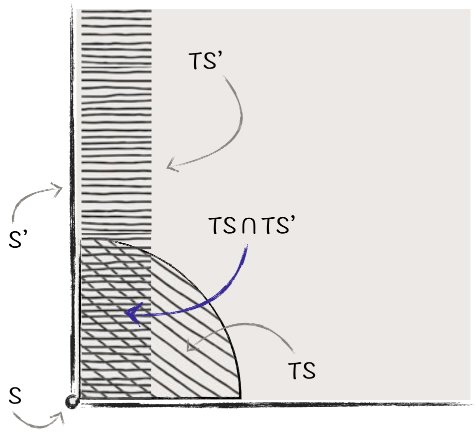

FIGURE 1. Tubes around strata.

is a plot for C embedded in $\mathbf{R}^3$ but not for the cone over the circle, because the parametrisation $s \mapsto (\cos 1/s^2, \sin 1/s^2)$ does not converge for $s = 0$.

2.4. Formal and geometric stratifications. The general definition of a diffeological space leaves some room between the topological and the smooth conditions. The same partition of a space can be a stratification for different diffeologies, since different diffeologies can have the same D-topology, as the following examples show.

EXAMPLE 2.1. Consider the real line $\mathbf{R}$, equipped with the standard diffeology. We define the strata

$$\mathsf{S}_- = (-\infty, 0), \quad \mathsf{S}_0 = \{0\} \quad \text{and} \quad \mathsf{S}_+ = (0, +\infty)$$

and the tube system

$$\mathsf{TS}_\pm = \mathsf{S}_\pm \ \text{ with } \pi_{\mathsf{S}_\pm} : x \mapsto x \quad \text{and} \quad \mathsf{TS}_0 = \mathbf{R} \ \text{ with } \pi_{\mathsf{S}_0} : x \mapsto 0.$$

One can check that this describes a locally fibred stratified space.

EXAMPLE 2.2. Consider the positive right angle L in $\mathbf{R}^2$ made up of points (x, y) such that $(x, y \geq 0)$ and $(x = 0 \text{ or } y = 0)$:

$$\mathsf{L} = \{(0, y) \mid y \geq 0\} \cup \{(x, 0) \mid x \geq 0\}.$$

We equip L with the subset diffeology of $\mathbf{R}^2$. Define the strata

$$\mathsf{S}_- = \{(0, y) \mid y > 0\}, \quad \mathsf{S}_+ = \{(x, 0) \mid x > 0\} \quad \text{and} \quad \mathsf{S}_0 = \{(0, 0)\}$$

and the tube system

$$\mathsf{TS}_\pm = \mathsf{S}_\pm \quad \text{with } \pi_{\mathsf{S}_\pm} : (x, y) \mapsto (x, y) \quad \text{and} \quad \mathsf{TS}_0 = \mathsf{L} \quad \text{with } \pi_{\mathsf{S}_0} : (x, y) \mapsto (0, 0).$$

One can check that this describes a locally fibred stratified space.

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Let $f: \mathsf{L} \to \mathbf{R}$ be the bijection defined by $f(x, y) = x - y$. Then equip $\mathbf{R}$ with the pushforward of the diffeology of L by f. Denote it by $\mathbf{R}_{\mathsf{L}}$. The map f sends the stratification of L onto the stratification of $\mathbf{R}$ from Example 2.1. Obviously, the diffeology of $\mathbf{R}_{\mathsf{L}}$ does not coincide with the standard diffeology, since it is strictly finer, but it induces the same D-topology. We have then two identical smooth stratifications on the same set, but equipped with different diffeologies.

The main difference between the two previous examples is the balance between the stratification and the action of the pseudo-group of local diffeomorphisms. In the second example the stratification is given by the action of the pseudo-group of diffeomorphisms: the strata are the connected components of its orbits [7]. *A contrario*, in the first example the pseudo-group of local diffeomorphisms is transitive and the stratification is transparent for the local diffeomorphisms—it has no structural geometric frame. This suggests a specification for *geometric stratified spaces* in diffeology.

Every diffeological space X admits a natural partition $\mathcal{S}$ in connected components of the orbits of the pseudo-group $\mathrm{Diff_{loc}(X)}$ of local diffeomorphisms. These components are called the Klein strata [10, Section 1.42]. We can single out the spaces X for which this partition is a stratification or, more precisely, a local fibred stratification. We shall talk in this case of *geometric stratification*, when the stratification is given by the action of the pseudo-group of local diffeomorphisms, and of *formal stratification* in the opposite case.

Diffeological spaces that admit a geometric stratification form naturally a full subcategory in *diffeology*, which we call *stratified diffeology*. Obviously, diffeomorphisms between diffeological spaces respect their natural stratifications. Manifolds with corners are the first example of such geometric stratified diffeological spaces [8].

3. Differential forms

In this section we give a necessary and sufficient condition for a stratified differential form, defined on the regular part of some locally fibred stratified space X, to be the restriction of a differential form in the sense of diffeology, defined on the whole space.

Stratified differential form. A *stratified differential form* on a stratified space is any differential form with perversity 0, according to Goresky and MacPherson [5, 6] and defined by Brylinski in [3]. The following definition makes this precise.

Definition 3.1. A *stratified k-form* on a locally fibred stratified space X is a differential k-form α defined on the regular part $\mathrm{X_{reg}} \subset \mathrm{X}$ such that, for every stratum $\mathrm{S} \in \mathcal{S}$, for all points $x \in \mathrm{TS} \cap \mathrm{X_{reg}}$ and for all vectors $\xi \in \ker \mathrm{D}(\pi_{\mathrm{S}})_x$,

$$\alpha_x(\xi) = 0 \quad \text{and} \quad d\alpha_x(\xi) = 0, \tag{$\clubsuit$}$$

where $d\alpha_x(\xi)$ is the contraction of $d\alpha_x$ with ξ (denoted equivalently by $i_\xi(d\alpha_x)$ or sometimes by $\xi \bullet d\alpha_x$).

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The space of stratified differential k-forms α is denoted by $\Omega^k_{\bar{0}}[X]$. Note that α belongs *a priori* to $\Omega^k(X_{\rm reg})$.

Differential forms in diffeology. Differential forms in diffeology were introduced by Souriau [21] and developed in [10].

Definition 3.2. A *differential k-form* α on a diffeological space X is a mapping that associates with every plot $P\colon U \to X$ a smooth k-form $\alpha(P)$ on U such that, for any smooth parametrisation F in U,

$$\alpha(P \circ F) = F^*(\alpha(P)).$$

3.1. Strata forms associated with a form of perversity 0. Let X be a diffeological space equipped with a stratification $\mathcal{S}$ and a tube system $\{\pi_S\colon TS \to S\}_{S\in\mathcal{S}}$, as described above. Let $\alpha \in \Omega^k_{\bar{0}}[X]$, $S \in \mathcal{S}$ and $pr\colon \tilde{S} \to S$ be its universal covering. Consider the restriction

$$\pi_S \upharpoonright TS \cap X_{\rm reg}\colon TS \cap X_{\rm reg} \to S.$$

This is a fibre bundle over S. Let F denote a fibre. With α, we associate a set $\mathcal{A}_S$ of differential k-forms on $\tilde{S}$, indexed by the connected components of F:

$$\mathcal{A}_S = \{\bar{\alpha}_a\}_{a\in\pi_0(F)}, \quad \text{with } \bar{\alpha}_a \in \Omega^k(\tilde{S}), \tag{$\heartsuit$}$$

as described below.

Step 1. Let $pr\colon \tilde{S} \to S$ be the universal covering of S. The strata are always assumed to be connected. Let $pr^*(TS \cap X_{\rm reg})$ be the pullback of $\pi_S \upharpoonright TS \cap X_{\rm reg}$ by pr. That is,

$$pr^*(TS \cap X_{\rm reg}) = \{(\tilde{x}, y) \in \tilde{S} \times TS \cap X_{\rm reg} \mid pr(\tilde{x}) = \pi_S(y)\}$$

(see Figure 2). Let $pr_1\colon\ pr^*(TS \cap X_{\rm reg}) \to \tilde{S}$ and $pr_2\colon\ pr^*(TS \cap X_{\rm reg}) \to TS \cap X_{\rm reg}$ be the first and second projections:

$$\begin{array}{ccc}
pr^*(TS \cap X_{\rm reg}) & \xrightarrow{\ pr_2\ } & TS \cap X_{\rm reg} \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_S} \\
\tilde{S} & \xrightarrow[\ pr\ ]{} & S
\end{array}$$

The exact homotopy sequence of pr_1 gives

$$\pi_1(\tilde{S}) = \{0\} \to \pi_0(F) \to \pi_0(pr^*(TS \cap X_{\rm reg})) \to \pi_0(\tilde{S}) = \{\tilde{S}\}.$$

Thus,

$$\pi_0(F) \simeq \pi_0(pr^*(TS \cap X_{\rm reg})).$$

Now, since pr_1 is a fibration with fibre F, each connected component a of F defines a connected component of $pr^*(TS \cap X_{\rm reg})$ over $\tilde{S}$. The total space $pr^*(TS \cap X_{\rm reg})$ is diffeomorphic to the sum of these connected components over $\tilde{S}$, that is,

$$pr^*(TS \cap X_{\rm reg}) = \coprod_{a\in\pi_0(F)} \{pr^*(TS \cap X_{\rm reg})\}_a.$$

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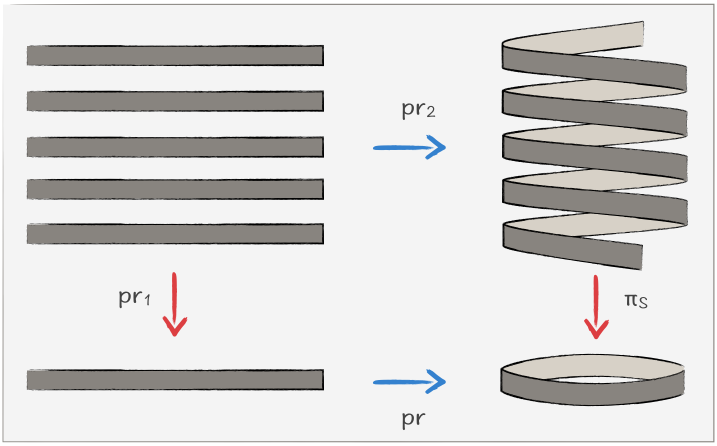

FIGURE 2. Pullback of $\pi_S \restriction TS \cap X_{reg}$ by pr.

The restriction of pr_1 to the component $\{pr^*(TS \cap X_{reg})\}_a$ is a fibre bundle with fibre $a \in \pi_0(F)$.

Step 2. Consider the k-forms

$$\tilde{\alpha} = pr_2^*(\alpha) \in \Omega^k(pr^*(TS \cap X_{reg})) \quad \text{and} \quad \tilde{\alpha}_a = \tilde{\alpha} \restriction \{pr^*(TS \cap X_{reg})\}_a,$$

with $a \in \pi_0(F)$. The $\tilde{\alpha}_a$ are restrictions of $\tilde{\alpha}$ to the connected components of $pr^*(TS \cap X_{reg})$. The forms $\tilde{\alpha}_a$ defined on $\{pr^*(TS \cap X_{reg})\}_a$ satisfy the condition (♣) of perversity 0. Indeed, $D(pr_2)$ maps $\ker(D(pr_1))$ to $\ker(D(\pi_S))$. Now, since $\{pr^*(TS \cap X_{reg})\}_a$ is connected, the condition (♣) means exactly that $\tilde{\alpha}_a$ is basic, that is, there exists a k-form $\bar{\alpha}_a$ on the covering $\tilde{S}$ such that

$$\tilde{\alpha}_a = pr_1^*(\bar{\alpha}_a) \quad \text{and then } \mathcal{A}_S = \{\bar{\alpha}_a\}_{a \in \pi_0(F)}.$$

In other words,

$$pr_2^*(\alpha) \restriction \{pr^*(TS \cap X_{reg})\}_a = pr_1^*(\bar{\alpha}_a). \tag{♠}$$

THEOREM 3.3. *Let X be a diffeological space equipped with a stratification $\mathcal{S}$. There exists a differential k-form α_S on the stratum S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$ if and only if $\bar{\alpha}_a = \bar{\alpha}_b$ for all $a, b \in \pi_0(F)$. In this case, $\bar{\alpha}_a = pr^*(\alpha_S)$ for all $a \in \pi_0(F)$.*

Introducing the *index* of the form $\alpha \in \Omega^*_{\bar{0}}[X]$ at the stratum S, as the number

$$\nu_S(\alpha) = card(\mathcal{A}_S),$$

we can paraphrase the theorem as follows: *there exists a differential k-form α_S on the stratum S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$ if and only if $\nu_S(\alpha) = 1$.*

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Proof of Theorem 3.3. Assume that there exists α_S such that $\alpha \restriction TS = \pi_S^*(\alpha_S)$. Let

$$\mathrm{pr}_{1,a} = \mathrm{pr}_1 \restriction \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_a \quad \text{and} \quad \mathrm{pr}_{2,a} = \mathrm{pr}_2 \restriction \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_a.$$

Then $\pi_S \circ \mathrm{pr}_{2,a} = \mathrm{pr} \circ \mathrm{pr}_{1,a}$ and $\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_{2,a}^*(\pi_S^*(\alpha_S)) = \mathrm{pr}_{1,a}^*(\mathrm{pr}^*(\alpha_S))$. But

$$\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_{1,a}^*(\bar{\alpha}_a),$$

so

$$\mathrm{pr}_{1,a}^*(\bar{\alpha}_a) = \mathrm{pr}_{1,a}^*(\mathrm{pr}^*(\alpha_S)).$$

Since $\mathrm{pr}_{1,a}$ is a fibration, $\bar{\alpha}_a = \mathrm{pr}^*(\alpha_S)$, that is, $\bar{\alpha}_a = \bar{\alpha}_b$ for all $a, b \in \pi_0(F)$.

Conversely, assume that $\bar{\alpha}_a = \bar{\alpha}_b$ for all $a, b \in \pi_0(F)$ and denote by $\bar{\alpha}$ this differential form on $\tilde{S}$. We want to prove that $\underline{k}^*(\bar{\alpha}) = \bar{\alpha}$ for all $k \in \pi_1(S)$. First, note that $\mathrm{pr}(\underline{k}(\tilde{x})) = \mathrm{pr}(\tilde{x})$ and so $\pi_1(S)$ acts on $\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})$ by

$$\text{for all } k \in \pi_1(S), \quad \underline{k}(\tilde{x}, y) = (\underline{k}(\tilde{x}), y),$$

where $\underline{k}$ denotes indifferently the two actions of k. We need three facts and a lemma.

- The action of $\pi_1(S)$ on $\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})$ is free. Every element k in $\pi_1(S)$ acts by diffeomorphism. In particular, the action exchanges the connected components.
- The two actions of $\pi_1(S)$ intertwine pr_1, that is, $\mathrm{pr}_1 \circ \underline{k} = \underline{k} \circ \mathrm{pr}_1$.
- The projection pr_2 is invariant by $\pi_1(S)$, that is, $\mathrm{pr}_2 \circ \underline{k} = \mathrm{pr}_2$.

Lemma 3.4. *Suppose that $k \in \pi_1(S)$. If $\underline{k}$ sends the component relative to $a \in \pi_0(F)$ onto the component relative to b, then $\bar{\alpha}_a = \underline{k}^*(\bar{\alpha}_b)$.*

Proof. First of all, $\tilde{\alpha}$ is invariant under the action of $\pi_1(S)$. Indeed, $\underline{k}^*(\mathrm{pr}_2^*(\alpha)) = \mathrm{pr}_2^*(\alpha)$, that is, $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$. Suppose that $\underline{k}$ maps the component relative to a onto the component relative to b:

$$\underline{k}\colon \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_a \to \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_b.$$

Let $j_i\colon \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_i \to \mathrm{pr}^*(TS \cap X_{\mathrm{reg}})$ be the inclusion map of the ith component for all $i \in \pi_0(F)$. Then $\underline{k} \circ j_a = j_b \circ \underline{k}$. Hence, $(\underline{k} \circ j_a)^*(\tilde{\alpha}) = (j_b \circ \underline{k})^*(\tilde{\alpha})$, that is, $j_a^*(\underline{k}^*(\tilde{\alpha})) = \underline{k}^*(j_b^*(\tilde{\alpha}))$. But $\underline{k}^*(\tilde{\alpha}) = \tilde{\alpha}$, so $j_a^*(\tilde{\alpha}) = \underline{k}^*(j_b^*(\tilde{\alpha}))$, that is, $\tilde{\alpha}_a = \underline{k}^*(\tilde{\alpha}_b)$, where $\tilde{\alpha}_i = \tilde{\alpha} \restriction \{\mathrm{pr}^*(TS \cap X_{\mathrm{reg}})\}_i$. Since $\tilde{\alpha}_i = \mathrm{pr}_1^*(\bar{\alpha}_i)$, it follows that $\mathrm{pr}_1^*(\bar{\alpha}_a) = \underline{k}^*(\mathrm{pr}_1^*(\bar{\alpha}_b))$. Now, $\underline{k} \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ \underline{k}$, so $\mathrm{pr}_1^*(\bar{\alpha}_a) = \mathrm{pr}_1^*(\underline{k}^*(\bar{\alpha}_b))$. Since pr_1 is a fibration, finally $\bar{\alpha}_a = \underline{k}^*(\bar{\alpha}_b)$. □

As a corollary, if $\bar{\alpha}_a = \bar{\alpha}_b$ (denoted by $\bar{\alpha}$), then $\bar{\alpha}$ is invariant by $\pi_1(S)$. Hence, $\bar{\alpha}$ is basic with respect to the group $\pi_1(S)$ and there exists $\alpha_S \in \Omega^k(S)$ such that $\bar{\alpha} = \mathrm{pr}^*(\alpha_S)$. On the one hand, from the commutative diagram $\pi_S \circ \mathrm{pr}_2 = \mathrm{pr} \circ \mathrm{pr}_1$ and so $\mathrm{pr}_2^*(\pi_S^*(\alpha_S)) = \mathrm{pr}_1^*(\mathrm{pr}^*(\alpha_S)) = \mathrm{pr}_1^*(\bar{\alpha})$. On the other hand, (♠) gives $\mathrm{pr}_{2,a}^*(\alpha) = \mathrm{pr}_1^*(\bar{\alpha}_a)$. But $\bar{\alpha}_a = \bar{\alpha}_b = \bar{\alpha}$ and so $\mathrm{pr}_2^*(\alpha) = \mathrm{pr}_1^*(\bar{\alpha})$. Therefore, $\mathrm{pr}_2^*(\pi_S^*(\alpha_S)) = \mathrm{pr}_2^*(\alpha)$ and, because pr_2 is a fibration, $\alpha \restriction TS = \pi_S^*(\alpha_S)$. This completes the proof. □

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3.2. Stratified differential forms as differential forms. Let X be a diffeological space equipped with a locally fibred stratification, as described above. In terms of the index function ν defined in Section 3.1, we have the following theorem.

THEOREM 3.5. *Let $\alpha \in \Omega^k_{\bar{0}}[\mathrm{X}]$. If $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathcal{S}$, then there exists a (unique) differential form $\underline{\alpha} \in \Omega^k(\mathrm{X})$ such that $\alpha = \underline{\alpha} \upharpoonright \mathrm{X}_{\mathrm{reg}}$. Conversely, let $\underline{\alpha} \in \Omega^k(\mathrm{X})$, with $\alpha = \underline{\alpha} \upharpoonright \mathrm{X}_{\mathrm{reg}} \in \Omega^k_{\bar{0}}[\mathrm{X}]$. If for any two points in $\mathrm{X}_{\mathrm{reg}}$ there is a smooth map joining them that cuts $\mathrm{X}_{\mathrm{sing}}$ into a finite number of points, then $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathcal{S}$.*

REMARK 3.6. The perversity condition applying on the entire tube around the strata is clearly too strong. Indeed, in the case of the simple example treated in Section 2.4, for 0-forms, that is, real functions, one gets only restrictions of constant functions. That is clearly insufficient. Instead, we should get all the smooth functions locally constant on the neighbourhood of the origin. We can obtain that result by weakening the condition of perversity and considering the germs of the tubes around the strata. That is, given a system of tubes, we should say that a form is 0-perverse if there exists a subsystem of tubes, made of restrictions of the original system around each stratum, for which the form is 0-perverse. With this condition the property of perversity becomes semilocal, which is more appropriate. This has been adopted, for example, in [17, page 23] and [18, page 83]. In the examples above, the stratified real functions are then the restrictions of any smooth function locally constant on the neighbourhood of the origin. Considering 1-forms, we would obtain differential forms that vanish locally around the origin.

Note that the smooth functions for the diffeology are obviously all functions on $\mathbf{R}$, in the first case, and contain at least all the restrictions of any smooth function to L, in the second case. We treated the general case of forms on corners in [8].

PROOF OF THEOREM 3.5. Let $\alpha \in \Omega^k_{\bar{0}}[\mathrm{X}]$, with $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathcal{S}$. Thanks to Section 3.1, for all $\mathrm{S} \in \mathcal{S}$, there exists $\alpha_{\mathrm{S}} \in \Omega^k(\mathrm{S})$ such that $\alpha \upharpoonright \mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}} = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})$. Let S and S′ be two strata such that $\mathrm{TS} \cap \mathrm{TS}' \neq \emptyset$. On $\mathrm{TS} \cap \mathrm{TS}' \cap \mathrm{X}_{\mathrm{reg}}$, which is open, $\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}) = \pi^*_{\mathrm{S}'}(\alpha_{\mathrm{S}'}) = \alpha$. Consider $x \in \mathrm{TS} \cap \mathrm{TS}'$ but $x \notin \mathrm{X}_{\mathrm{reg}}$, that is, $x \in \mathrm{X}_{\mathrm{sing}}$. Then $x \in \mathrm{S}'' \cap \mathrm{TS} \cap \mathrm{TS}'$, with $\mathrm{S}'' \subset \mathrm{X}_{\mathrm{sing}}$. Thus, $\mathrm{S}'' \cap \mathrm{TS} \neq \emptyset$, which implies in particular that $\mathrm{S} \subset \bar{\mathrm{S}}''$ and $\pi_{\mathrm{S}} \circ \pi_{\mathrm{S}''} = \pi_{\mathrm{S}}$ on $\mathrm{TS} \cap \mathrm{TS}'' \cap \pi^{-1}_{\mathrm{S}''}(\mathrm{TS})$, by the definition of locally fibred stratified spaces. Hence, $(\pi_{\mathrm{S}} \circ \pi_{\mathrm{S}''})^*(\alpha_{\mathrm{S}}) = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})$, that is, $\pi^*_{\mathrm{S}''}(\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})) = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})$. On the regular part, $\mathrm{TS} \cap \mathrm{TS}'' \cap \pi^{-1}_{\mathrm{S}''}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}}$ and $\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}) = \alpha = \pi^*_{\mathrm{S}''}(\alpha_{\mathrm{S}''})$. It follows that, $\pi^*_{\mathrm{S}''}(\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})) = \pi^*_{\mathrm{S}''}(\alpha_{\mathrm{S}''})$ on $\mathrm{TS} \cap \mathrm{TS}'' \cap \pi^{-1}_{\mathrm{S}''}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}}$, that is,

$$\pi^*_{\mathrm{S}''}(\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}) - \alpha_{\mathrm{S}''}) \upharpoonright \mathrm{TS} \cap \mathrm{TS}'' \cap \pi^{-1}_{\mathrm{S}''}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}} = 0.$$

But $\pi_{\mathrm{S}''} \upharpoonright \mathrm{TS} \cap \mathrm{TS}'' \cap \pi^{-1}_{\mathrm{S}''}(\mathrm{TS}) \cap \mathrm{X}_{\mathrm{reg}}$ is a submersion on $\mathrm{S}'' \cap \mathrm{TS}$ and so

$$\alpha_{\mathrm{S}''} \upharpoonright \mathrm{S}'' \cap \mathrm{TS} = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}).$$

Therefore, on $\mathrm{TS}'' \cap \mathrm{TS}$,

$$\pi^*_{\mathrm{S}''}(\alpha_{\mathrm{S}''}) = \pi^*_{\mathrm{S}''}(\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}})) = (\pi_{\mathrm{S}} \circ \pi_{\mathrm{S}''})^*(\alpha_{\mathrm{S}}) = \pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}).$$

Moreover, $\pi^*_{\mathrm{S}''}(\alpha_{\mathrm{S}''}) = \pi^*_{\mathrm{S}'}(\alpha_{\mathrm{S}'})$ on $\mathrm{TS}'' \cap \mathrm{TS}'$. Note that $\mathrm{TS}'' \cap \mathrm{TS}'$ and $\mathrm{TS}'' \cap \mathrm{TS}$ are two open neighbourhoods of x. Hence, $\pi^*_{\mathrm{S}}(\alpha_{\mathrm{S}}) = \pi^*_{\mathrm{S}'}(\alpha_{\mathrm{S}'})$ on an open neighborhood

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of x, which belongs in this case to $\mathrm{TS} \cap \mathrm{TS}' \cap \mathrm{X}_{\mathrm{sing}}$. Therefore, $\pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}}) = \pi_{\mathrm{S}'}^*(\alpha_{\mathrm{S}'})$ on $\mathrm{TS} \cap \mathrm{TS}'$. Since differential forms on diffeological spaces are local [10, Section 6.36], there exists a differential form $\underline{\alpha}$ defined on X such that $\underline{\alpha} \upharpoonright \mathrm{TS} = \pi_{\mathrm{S}}^*(\alpha_{\mathrm{S}})$ for all $\mathrm{S} \in \mathcal{S}$ and, in particular, $\alpha = \underline{\alpha} \upharpoonright \mathrm{X}_{\mathrm{reg}}$.

Now, let $\alpha = \underline{\alpha} \upharpoonright \mathrm{X}_{\mathrm{reg}}$, with $\underline{\alpha} \in \Omega^k(\mathrm{X})$. The pullback $\mathrm{pr}_1 : \mathrm{pr}^*(\mathrm{TS}) \to \tilde{\mathrm{S}}$ is a locally trivial fibre bundle. Let $\underline{\mathrm{F}}$ be its fibre and $\mathrm{F} = \underline{\mathrm{F}}_{\mathrm{reg}}$ be the regular part. The differential form $\mathrm{pr}_2^*(\underline{\alpha})$ is defined on the whole pullback $\mathrm{pr}^*(\mathrm{TS})$ and

$$\mathrm{pr}_2^*(\alpha) = \mathrm{pr}_2^*(\underline{\alpha}) \upharpoonright \mathrm{pr}^*(\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}).$$

For each component $a \in \pi_0(\mathrm{F})$, according to (♠), $\mathrm{pr}_2^*(\alpha) \upharpoonright \{\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\}_a = \mathrm{pr}_1^*(\bar{\alpha}_a)$, that is, $\mathrm{pr}_2^*(\underline{\alpha}) \upharpoonright \{\mathrm{TS} \cap \mathrm{X}_{\mathrm{reg}}\}_a = \mathrm{pr}_1^*(\bar{\alpha}_a)$. Now, let $y \in \mathrm{F}_a$ and $y' \in \mathrm{F}_b$ be two points in F belonging to two different connected components. There exists a smooth path $t \mapsto y_t$ in $\underline{\mathrm{F}}$ that connects y to y' and which cuts the singular subset $\underline{\mathrm{F}}_{\mathrm{sing}}$ in a finite number of points. The interval $(0, 1)$ is then divided into a finite set of open intervals denoted by I_a, where y_t belongs to the component $a \in \pi_0(\mathrm{F})$, separated by points belonging to $\underline{\mathrm{F}}_{\mathrm{sing}}$. Let $(r, y) \mapsto (\tilde{x}_r, y)$ be a local trivialisation of $\mathrm{pr}_1 : \mathrm{pr}^*(\mathrm{TS}) \to \tilde{\mathrm{S}}$. Then $(r, t) \mapsto (\tilde{x}_r, y_t)$ is a plot of $\mathrm{pr}^*(\mathrm{TS})$. On the open subset of (r, t) such that $t \in \mathrm{I}_a$,

$$\begin{aligned}\mathrm{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k &= \mathrm{pr}_1^*(\bar{\alpha}_a)((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k \\ &= \bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k),\end{aligned}$$

where $\underline{\alpha}$ is a k-form and the (u_i, ε_i) are tangent vectors at (t, r). But, for each r and $u_1 \ldots u_k$, the map

$$t \mapsto \mathrm{pr}_2^*(\underline{\alpha})((r,t) \mapsto (\tilde{x}_r, y_t))_{\binom{r}{t}} \begin{pmatrix} u_i \\ \varepsilon_i \end{pmatrix}_{i=1}^k$$

is smooth but constant on each I_a for $a \in \pi_0(\mathrm{F})$ with value $\bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k)$. Since $\underline{\mathrm{F}}$ is connected and is the closure of F,

$$\bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k) = \bar{\alpha}_b(r \mapsto \tilde{x}_r)_r(u_1)\ldots(u_k) \quad \text{for all } r, u_1 \ldots u_k. \qquad \square$$

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