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Hook-content Formulae for Symplectic and Orthogonal Tableaux

Peter S. Campbell and Anna Stokke

Abstract. By considering the specialisation $s_\lambda(1, q, q^2, \dots, q^{n-1})$ of the Schur function, Stanley was able to describe a formula for the number of semistandard Young tableaux of shape λ in terms of the contents and hook lengths of the boxes in the Young diagram. Using specialisations of symplectic and orthogonal Schur functions, we derive corresponding formulae, first given by El Samra and King, for the number of semistandard symplectic and orthogonal λ-tableaux.

1 Introduction

To each partition λ with at most n parts there corresponds an irreducible polynomial representation of the general linear group $\mathrm{GL}(n)$ over the field of complex numbers. Indeed, this representation has a basis indexed by semistandard Young tableaux of shape λ with entries from $\{1, 2, \dots, n\}$. The number of semistandard λ-tableaux is therefore equal to the dimension of the representation, and this is given by Weyl's dimension formula [12].

However, a more combinatorial description of the number of semistandard λ-tableaux was derived by Stanley ([8]) using Weyl's character formula. The character corresponding to the partition λ is the Schur function $s_\lambda(x_1, \dots, x_n)$, and Stanley showed that its specialisation $s_\lambda(1, q, q^2, \dots, q^{n-1})$ could be expressed as a product involving the hook lengths and contents of the boxes in the diagram for λ. This provides a generating function for the semistandard λ-tableaux with entries in $\{1, 2, \dots, n\}$ and, in particular, taking $q = 1$ yields a formula for the number of such tableaux.

A similar situation exists for the classical groups $\mathrm{Sp}(2n)$ and $\mathrm{O}(m)$ over the complex numbers. Semistandard symplectic and orthogonal λ-tableaux, which index bases for the irreducible polynomial representations associated with λ for $\mathrm{Sp}(2n)$ and $\mathrm{O}(m)$ respectively, have been introduced by various authors (see, for instance, [3, 5, 7, 11]). El Samra and King ([2]) were able to manipulate Weyl's dimension formula to produce formulae for the number of semistandard symplectic and odd orthogonal λ-tableaux in terms of hook lengths and contents.

The aim of this paper is to adapt Stanley's approach using Weyl's character formula to these cases. We obtain expressions for the specialisations

$$sp_{\lambda,2n}(q, q^3, q^5, \dots, q^{2n-1}),\ o_{\lambda,2n+1}(q^2, q^4, q^6, \dots, q^{2n}),\ \text{and}\ o_{\lambda,2n}(q, q^3, q^5, \dots, q^{2n-1})$$

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of the symplectic, odd orthogonal, and even orthogonal Schur functions. These give generating functions for the semistandard symplectic and orthogonal tableaux of shape λ with entries in the sets $\{1,\overline{1},2,\overline{2},\dots,n,\overline{n}\}$ or $\{1,\overline{1},2,\overline{2},\dots,n,\overline{n},\infty\}$ as appropriate, and we recover the formulae from [2] as a special case by setting $q=1$.

The specialisations in the symplectic and odd orthogonal cases have previously been studied by Koike ([6]), who was primarily concerned with the quantum dimensions of the irreducible modules as opposed to the associated tableaux. Using the principal specialisation of the Weyl character formula, Koike ([6]) also obtained formulae like the ones in which we are interested. By using different specialisations and by using an approach that is more direct and closer to the approach taken by Stanley, we obtain these results much more easily, and the result we obtain is different in the even orthogonal case.

We begin with a preliminary section that recalls the basic combinatorics of Young tableaux. The following section describes the symplectic and orthogonal tableaux that we will be considering. In the remaining sections we derive the generating functions for the symplectic and orthogonal cases.

2 Young Tableaux

A *partition* of a positive integer r is a k-tuple $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_k)$ of weakly decreasing non-negative integers such that $\sum_{i=1}^k \lambda_i = r$. The non-zero λ_i in the k-tuple are called the *parts* of λ. The *Young diagram* of shape λ is the subset of $\mathbb{Z}^2$ defined by

$$[\lambda]=\{(i,j)\mid i,j\in\mathbb{N},\ 1\le i\le k,\ 1\le j\le\lambda_i\}.$$

This is represented in the plane by arranging r boxes in k left-justified rows with the i-th row containing λ_i boxes. The *conjugate* of λ is the partition $\lambda^t=(\lambda_1^t,\lambda_2^t,\dots,\lambda_s^t)$, where λ_i^t is the number of boxes in the i-th column of the Young diagram of shape λ. One obtains a λ-tableau by filling $[\lambda]$ with entries from a set $\{1,2,\dots,n\}$, where n is a positive integer. A λ-tableau is *semistandard* if the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom.

For a λ-tableau T, let $a_i(T)$ denote the number entries equal to i in T. The *weight* of T is the monomial in the variables $x_1,x_2,\dots,x_n$ defined by $\operatorname{wt}(T)=\prod_{i=1}^n x_i^{a_i(T)}$. The *Schur function* corresponding to λ is

$$s_\lambda(x_1,x_2,\dots,x_n)=\sum_T \operatorname{wt}(T),$$

where the sum runs over all semistandard λ-tableaux T with entries in $\{1,2,\dots,n\}$.

Each box in $[\lambda]$ has an associated *hook* that consists of that box, all boxes to the right of it in that row, and all boxes below it in that column. The *hook length* of the box is then the number of boxes in its hook. Specifically, for $(i,j)\in[\lambda]$ we have $h(i,j)=\lambda_i+\lambda_j^t-i-j+1$.

Using the Schur function $s_\lambda(x_1, x_2, \dots, x_n)$, Stanley ([8]) obtained the following formula for the number of semistandard λ-tableaux with entries in the set $\{1, 2, \dots, n\}$:

$$s_\lambda(1^n) = \prod_{(i,j)\in[\lambda]} \frac{n + c(i,j)}{h(i,j)},$$

where $c(i, j) = j - i$ is the *content* of the (i, j)-th box. More generally, he proved that for an indeterminate q

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{b(\lambda)} \prod_{(i,j)\in[\lambda]} \frac{[n + c(i,j)]}{[h(i,j)]}, \tag{2.1}$$

where $b(\lambda) = \sum_{i=1}^{k}(i-1)\lambda_i$ and $[i] = q^i - 1$. If we let $|T|$ denote the sum of the entries in the tableau T, then the coefficient of q^i in $s_\lambda(1, q, \dots, q^{n-1})$ is the number of semistandard λ-tableaux with entries in $\{1, 2, \dots, n\}$ that have $|T| = i + r$. Consequently, (2.1) can be interpreted as providing a generating function for such tableaux.

3 Symplectic and Orthogonal Tableaux

Throughout, fix positive integers r and n and let λ be a partition of r into at most n parts. We will consider a set of $2n$ symbols $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$ with the ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$. A *semistandard symplectic tableau* (see [3]) of shape λ is a λ-tableau T with entries from $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$ that is semistandard in the usual sense and satisfies the additional property that the entries in the i-th row of T are greater than or equal to i for each i.

The weight of a symplectic λ-tableau T is defined by $\mathrm{wt}(T) = \prod_{i=1}^{n} x_i^{a_i(T) - a_{\overline{i}}(T)}$, where, as in the previous section, $a_i(T)$ is equal to the number of entries equal to i that appear in T. Then $sp_{\lambda,2n}(x_1, x_2, \dots, x_n) = \sum_T \mathrm{wt}(T)$, where the sum runs over all semistandard symplectic λ-tableaux and is the symplectic Schur function corresponding to λ. It is the character of the irreducible polynomial $\mathrm{Sp}(2n)$-module with highest weight λ [4].

Example The semistandard symplectic λ-tableaux for $\lambda = (1, 1)$ and $n = 2$ are as follows:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 1 \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{2} \\ \hline 2 \\ \hline\end{array}\,.$$

The corresponding symplectic Schur function is

$$sp_{\lambda,2n}(x_1, x_2) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} + 1.$$

For the even orthogonal tableaux we will use the same set as in the symplectic case, while for the odd tableaux we use $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n} < \infty$ and set $\overline{\infty} = \infty$. Let α_i and β_i denote the number of entries that are at most $\overline{i}$ in the first and second columns of a tableau T, respectively, and let $T_{i,j}$ denote the entry in the (i, j)-th box of T. A *semistandard even orthogonal* or *odd orthogonal tableau* (see [5]) of

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shape λ is a semistandard λ-tableau T with entries from the set $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$ or $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}, \infty\}$, respectively, such that for each $1 \le i \le n$:

(i) $\alpha_i + \beta_i \le 2i$;

(ii) if $\alpha_i + \beta_i = 2i$ with $\alpha_i > \beta_i$ and $T_{\alpha_i,1} = \overline{i}$ and $T_{\beta_i,2} = i$, then $T_{\alpha_i - 1,1} = i$;

(iii) if $\alpha_i + \beta_i = 2i$ with $\alpha_i = \beta_i = i$ and $T_{\alpha_i,1} = i$ and $T_{\alpha_i,j} = \overline{i}$, then $T_{\alpha_i - 1,j} = i$.

Let $\mathrm{wt}(T)$ denote the weight of the tableau T as defined above. The *even orthogonal* and *odd orthogonal Schur functions* corresponding to λ are defined to be $o_{\lambda,m}(x_1, x_2, \dots, x_n) = \sum_T \mathrm{wt}(T)$, where $m = 2n$ or $2n+1$ and the sum runs over the semistandard even orthogonal and odd orthogonal λ-tableaux respectively. These are then the characters for the irreducible polynomial $\mathrm{O}(2n)$ and $\mathrm{O}(2n+1)$-modules of highest weight λ [7].

Example Let $\lambda = (1, 1)$ and $n = 2$. The semistandard odd orthogonal tableaux of shape λ are

$$\boxed{\begin{array}{c}1\\ \hline 2\end{array}},\ \boxed{\begin{array}{c}1\\ \hline \overline{2}\end{array}},\ \boxed{\begin{array}{c}\overline{1}\\ \hline 2\end{array}},\ \boxed{\begin{array}{c}\overline{1}\\ \hline \overline{2}\end{array}},\ \boxed{\begin{array}{c}1\\ \hline \infty\end{array}},\ \boxed{\begin{array}{c}\overline{1}\\ \hline \infty\end{array}},\ \boxed{\begin{array}{c}2\\ \hline \infty\end{array}},\ \boxed{\begin{array}{c}\overline{2}\\ \hline \infty\end{array}},\ \boxed{\begin{array}{c}1\\ \hline \overline{1}\end{array}},\ \boxed{\begin{array}{c}2\\ \hline \overline{2}\end{array}}.$$

The odd orthogonal Schur function is

$$o_{\lambda,2n+1}(x_1, x_2) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} + x_1 + x_1^{-1} + x_2 + x_2^{-1} + 2.$$

4 Generating Function for Semistandard Symplectic Tableaux

Our aim is to produce an analogue of (2.1) for the symplectic Schur function, and we start with the determinantal formula [1, Equation 24.18]

$$sp_{\lambda,2n}(x_1, x_2, \dots, x_n) = \frac{|x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i-1}|_{i,j=1}^n}{|x_j^{n-i+1} - x_j^{-n+i-1}|_{i,j=1}^n}. \tag{4.1}$$

Let q be an indeterminate, and for a positive integer i define $\langle i\rangle = q^i - q^{-i}$ with $\langle i\rangle! = \langle 1\rangle\langle 2\rangle \cdots \langle i-1\rangle\langle i\rangle$. When necessary, we will also set $\langle 0\rangle! = \langle 0\rangle = 1$. The following result is a generalisation of [1, Exercise 24.20].

Lemma 4.1 *Let λ be a partition with at most n parts and set $\mu_i = \lambda_i + n - i$. Then*

$$sp_{\lambda,2n}(q, q^3, q^5, \dots, q^{2n-1}) = \frac{\prod_{i=1}^n \langle \mu_i + 1\rangle \prod_{1\le i<j\le n} \langle \mu_i - \mu_j\rangle\langle \mu_i + \mu_j + 2\rangle}{\prod_{i=1}^n \langle 2i-1\rangle!}. \tag{4.2}$$

Proof Let d and d' denote the denominator and numerator of (4.1) respectively after setting $x_j = q^{2j-1}$. Elementary row operations allow us to rewrite the denominator as

$$d = (-1)^{n(n-1)/2} \left| (q^{2j-1} + q^{1-2j})^{i-1} \right|_{i,j=1}^n \prod_{j=1}^n \left(q^{2j-1} - q^{1-2j} \right).$$

The determinant in this expression is the determinant of the transpose of a Vandermonde matrix

$$\left|(q^{2j-1}+q^{1-2j})^{i-1}\right|_{i,j=1}^{n}=\prod_{1\le i<j\le n}\left[(q^{2j-1}+q^{1-2j})-(q^{2i-1}+q^{1-2i})\right].$$

Rewriting $(q^{2j-1}+q^{1-2j})-(q^{2i-1}+q^{1-2i})=(q^{j-i}-q^{i-j})(q^{i+j-1}-q^{1-i-j})$ then gives

$$d=(-1)^{n(n-1)/2}\prod_{i=1}^{n}\prod_{k=1}^{2i-1}(q^{k}-q^{-k})=(-1)^{n(n-1)/2}\prod_{i=1}^{n}\langle 2i-1\rangle!.$$

For the numerator we proceed similarly:

$$d'=\left|(q^{\mu_i+1}+q^{-\mu_i-1})^{2(j-1)}\right|_{i,j=1}^{n}\prod_{i=1}^{n}(q^{\mu_i+1}-q^{-\mu_i-1})$$

with

$$\left|(q^{\mu_i+1}+q^{-\mu_i-1})^{2(j-1)}\right|_{i,j=1}^{n}=\prod_{1\le i<j\le n}\left[(q^{\mu_j+1}+q^{-\mu_j-1})^2-(q^{\mu_i+1}+q^{-\mu_i-1})^2\right].$$

Here $(q^{\mu_j+1}+q^{-\mu_j-1})^2-(q^{\mu_i+1}+q^{-\mu_i-1})^2=(q^{\mu_j-\mu_i}-q^{\mu_i-\mu_j})(q^{\mu_i+\mu_j+2}-q^{-\mu_i-\mu_j-2})$, so we obtain

$$d'=(-1)^{n(n-1)/2}\prod_{i=1}^{n}\langle\mu_i+1\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j+2\rangle,$$

and the result follows. ■

We need to identify the right-hand side of (4.2) as a suitable product over the boxes in the diagram. First, we consider the contribution from the hook lengths. Although this is identical to the result for $\mathrm{GL}(n)$, for completeness we provide a proof.

Lemma 4.2 *Let λ be a partition of at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]}\langle h(i,j)\rangle=\frac{\prod_{i=1}^{n}\langle\mu_i\rangle!}{\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle}.$$

Proof Consider only the i-th row of the diagram. The hook lengths strictly decrease from left to right along the row, so it is enough to show that for each $1\le j\le\lambda_i$ we have $h(i,j)\le\mu_i$, but that $h(i,j)\ne\mu_i-\mu_\ell$ for any $i<\ell\le n$. Let $\ell=\lambda_j^t$ so that $h(i,j)=(\lambda_i-i)+(\ell-j)+1$. Then $\lambda_{\ell+1}<j<\lambda_\ell+1$ implies that $\mu_i-\mu_\ell<h(i,j)<\mu_i-\mu_{\ell+1}$, where this gives $\mu_i-\mu_n<h(i,j)\le\mu_i$ for $\ell=n$. ■

In the case of $\mathrm{GL}(n)$ the content of a box is defined independently of the partition. For symplectic tableaux, however, we take (see [10])

$$r_\lambda(i,j)=\begin{cases}\lambda_i+\lambda_j-i-j+2 & \text{if } i>j,\\ i+j-\lambda_i^t-\lambda_j^t & \text{if } i\le j.\end{cases}$$

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Lemma 4.3 *Let λ be a partition of at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]} \langle 2n + r_\lambda(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle \mu_i + 1\rangle! \prod_{1\le i<j\le n}\langle \mu_i + \mu_j + 2\rangle}{\prod_{i=1}^{n}\langle 2i-1\rangle!}. \tag{4.3}$$

Proof Let $\lambda = (\lambda_1, \dots, \lambda_k)$ so that $k = \lambda_1^t$ and consider $\lambda' = (\lambda_2 - 1, \dots, \lambda_k - 1)$, the partition obtained by removing the hook with corner $(1,1)$ from the diagram for λ. The content of the (i,j)-th box in $[\lambda']$ is $r_{\lambda'}(i,j) = r_\lambda(i+1, j+1)$, so we may calculate (4.3) by induction on the number of parts of λ. We begin by examining the entries of the boxes lying in the $(1,1)$-hook of λ.

Recall that $\mu_i = \lambda_i + n - i$ for each $1 \le i \le n$. It is clear that for $1 < i \le k$ we have $2n + r_\lambda(i,1) = \mu_1 + \mu_i + 2$. Further, for $1 \le j \le \lambda_1$ we may write $2n + r_\lambda(1,j) = (n-k) + (\mu_1 - h(1,j)) + 2$, where $h(1,j)$ is the corresponding hook length in $[\lambda]$. From the proof of Lemma 4.2 we know that, as we run along the first row, $h(1,j)$ will take on the values 1 up to μ_1 excluding those of the form $\mu_1 - \mu_\ell$ for $1 < \ell \le n$. Thus

$$\prod_{j=1}^{\lambda_1}\langle 2n + r_\lambda(1,j)\rangle = \frac{\langle (n-k) + \mu_1 + 1\rangle!}{\langle (n-k)+1\rangle \prod_{\ell=2}^{n}\langle (n-k) + \mu_\ell + 2\rangle}.$$

However, $\mu_i = n - i$ for all $k < i \le n$, so we may express $\langle (n-k) + \mu_1 + 1\rangle!$ in the numerator as the product of $\langle \mu_1 + \mu_{k+1} + 2\rangle \cdots \langle \mu_1 + \mu_n + 2\rangle$ and $\langle \mu_1 + 1\rangle!$. Consequently, the product over all the boxes in the hook is

$$\prod_{(i,j)\in[(\lambda_1, 1^{k-1})]} \langle 2n + r_\lambda(i,j)\rangle = \frac{\langle \mu_1 + 1\rangle! \prod_{j=2}^{n}\langle \mu_1 + \mu_j + 2\rangle}{\langle (n-k)+1\rangle! \prod_{j=2}^{n}\langle (n-k) + \mu_j + 2\rangle}. \tag{4.4}$$

To prove the base case suppose that the diagram for λ is a single hook; that is, $\lambda = (\lambda_1, 1^{k-1})$. This gives $\mu_i = n - i + 1$ for $1 < i \le k$ and $n - i$ for $i > k$. In particular,

$$\langle \mu_i + 1\rangle! \prod_{j=i+1}^{n} \langle \mu_i + \mu_j + 2\rangle = \begin{cases} \langle 2(n-i+2) - 1\rangle! / \langle (n-k) + \mu_i + 2\rangle & \text{if } 1 < i \le k, \\ \langle 2(n-i+1) - 1\rangle! & \text{if } k < i \le n, \end{cases}$$

so

$$\prod_{i=2}^{n}\langle \mu_i + 1\rangle! \prod_{2\le i<j\le n} \langle \mu_i + \mu_j + 2\rangle = \frac{\prod_{i=1}^{n}\langle 2i-1\rangle!}{\langle 2(n-k)+1\rangle! \prod_{i=2}^{k}\langle (n-k) + \mu_i + 2\rangle}.$$

Moreover, we can replace the factorial $\langle 2(n-k)+1\rangle!$ in the denominator by the product of $\langle (n-k) + \mu_{k+1} + 2\rangle \cdots \langle (n-k) + \mu_n + 2\rangle$ and $\langle (n-k)+1\rangle!$. Hence we find that (4.4) is equivalent to (4.3) in this case.

Now suppose that $[\lambda]$ is more than a single hook and let $\lambda' = (\lambda_2 - 1, \dots, \lambda_k - 1)$ as above. By induction we know that

$$\prod_{(i,j)\in[\lambda']} \langle 2n + r_{\lambda'}(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle \mu_i' + 1\rangle! \prod_{1\le i<j\le n}\langle \mu_i' + \mu_j' + 2\rangle}{\prod_{i=1}^{n}\langle 2i-1\rangle!},$$

where $\mu_i' = \lambda_i' + n - i$. Here we see that $\mu_i' = \mu_{i+1}$ for $1 \le i < k$ and μ_i for $k < i \le n$ with $\mu_k' = n - k$. We may therefore reexpress this as

$$\frac{\prod_{(i,j)\in[\lambda']} \langle 2n + r_\lambda(i+1, j+1)\rangle}{\langle (n-k)+1\rangle! \prod_{j=2}^{n} \langle (n-k) + \mu_j + 2\rangle} = \frac{\prod_{i=2}^{n} \langle \mu_i + 1\rangle! \prod_{2\le i<j\le n} \langle \mu_i + \mu_j + 2\rangle}{\prod_{i=1}^{n} \langle 2i-1\rangle!}.$$

Combining with (4.4) we obtain (4.3), and we are done. ■

Remark 4.4 In [9] the formula for the product of the hook lengths is derived by manipulating the diagram for λ. We add $k-i$ boxes to the i-th row of the diagram, fill the row with the numbers 1 up to $\mu_i - (n-k)$ starting from the right, and remove the columns $1 + \mu_j$ for $1 < j \le n$. The boxes remaining form the diagram for λ with the hook lengths in the appropriate places, while the boxes removed are precisely those containing $\mu_i - \mu_j$ for $1 \le i < j \le k$. For example, when $\lambda = (7, 5, 4, 1)$ and $n = 4$, we have $\mu = (10, 7, 5, 1)$ and obtain

10	**9**	8	7	6	**5**	4	**3**	2	1
7	**6**	5	4	3	**2**	1			
5	**4**	3	2	1					
1									

$\longrightarrow$

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

A similar method can be used to derive (4.4), the formula for the product of the contents in the $(1,1)$-hook. We add $k-1$ boxes to the arm of the hook and label the boxes in the following way: the leg of the hook, excluding $(1,1)$, is filled with the numbers $\mu_1 + \mu_j + 2$ for $1 < j \le k$ starting from the top and the arm of the hook, including $(1,1)$, with $2(n-k)+2$ up to $\mu_1 + (n-k) + 1$ starting from the left. We then remove the boxes in the arm at positions $\mu_j - (n-k) + 1$ for $1 < j \le k$. The eliminated boxes contain $\mu_j + (n-k) + 2$ for $1 < j \le k$ and the remaining boxes the values of $2n + r_\lambda(i,j)$ for the hook. We therefore have

$$\prod_{(i,j)\in[(\lambda_1, 1^{k-1})]} \langle 2n + r_\lambda(i,j)\rangle = \frac{\langle \mu_1 + (n-k) + 1\rangle!}{\langle 2(n-k)+1\rangle! \prod_{j=1}^{k} \langle (n-k) + \mu_j + 2\rangle}$$

and note that $\langle \mu_1 + (n-k) + 1\rangle! = \langle \mu_1 + 1\rangle! \langle \mu_1 + \mu_{k+1} + 2\rangle \cdots \langle \mu_1 + \mu_n + 2\rangle$, while $\langle 2(n-k)+1\rangle! = \langle (n-k)+1\rangle! \langle (n-k) + \mu_{k+1} + 2\rangle \cdots \langle (n-k) + \mu_n + 2\rangle$.

For example, $\lambda = (7, 5, 4, 1)$ with $n = 4$ and $\mu = (10, 7, 5, 1)$ gives

2	**3**	4	5	6	**7**	8	**9**	10	11
19									
17									
13									

$\longrightarrow$

2	4	5	6	8	10	11
19						
17						
13						

for the first hook. To continue, we consider the partition $\lambda' = (4, 3)$, where we still have $n = 4$ but now $k = 2$. Consequently, $\mu' = (7, 5, 1, 0)$, and we obtain

6	7	8	**9**	10
14				

$\longrightarrow$

6	7	8	10
14			

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for the second hook. Finally, for the third hook we use $\lambda'' = (2)$ with $\mu'' = (5, 2, 1, 0)$, which simply produces

8	9

Combining yields the complete diagram

2	4	5	6	8	10	11
19	6	7	8	10		
17	14	8	9			
13						

.

Theorem 4.5 *Let λ be a partition with at most n parts. Then*

$$sp_{\lambda,2n}(q, q^3, \ldots, q^{2n-1}) = \prod_{(i,j)\in[\lambda]} \frac{\langle 2n + r_\lambda(i, j)\rangle}{\langle h(i, j)\rangle}.$$

Proof The result follows from the previous three lemmas. ∎

Define $|T|$ to be the sum of the entries of the symplectic tableau T, where the symbol $\bar{i}$ is counted as $-i$. Let $r(T) = r_+(T) - r_-(T)$, where $r_+(T)$ and $r_-(T)$ are the number of boxes of T containing a symbol from the sets $\{1, \ldots, n\}$ and $\{\bar{1}, \ldots, \bar{n}\}$ respectively. Then

$$sp_{\lambda,2n}(q, q^3, q^5, \ldots, q^{2n-1}) = \sum_T q^{2|T|-r(T)},$$

where T runs over the semistandard symplectic λ-tableaux. As a special case, setting $q = 1$ recovers King and El-Samra's expression ([2]) for the dimension of the irreducible polynomial $\mathrm{Sp}(2n)$-module with highest weight λ.

Corollary 4.6 *The number of semistandard symplectic λ-tableaux with entries in the set $\{1, \bar{1}, 2, \bar{1}, \ldots, n, \bar{n}\}$ is*

$$sp_{\lambda,2n}(1, \ldots, 1) = \prod_{(i,j)\in[\lambda]} \frac{2n + r_\lambda(i, j)}{h(i, j)}.$$

5 Generating Function for Semistandard Orthogonal Tableaux

Although our approach for the orthogonal tableaux will be identical to that for the symplectic tableaux, there are important differences between the odd orthogonal and even orthogonal cases. For the odd orthogonal tableaux the relevant determinantal formula is ([1, Equation 24.28])

$$o_{\lambda,2n+1}(x_1, x_2, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i+1/2} - x_j^{-\lambda_i-n+i-1/2}|_{i,j=1}^n}{|x_j^{n-i+1/2} - x_j^{-n+i-1/2}|_{i,j=1}^n}.$$

However, the situation is more complicated in the case of even orthogonal tableaux (see [1, pp. 410–411] or [7, p. 356]). If λ has strictly fewer than n parts, then

$$o_{\lambda,2n}(x_1,x_2,\dots,x_n)=\frac{|x_j^{\lambda_i+n-i}+x_j^{-\lambda_i-n+i}|_{i,j=1}^n}{|x_j^{n-i}+x_j^{-n+i}|_{i,j=1}^n},$$

whereas for λ with exactly n parts

$$o_{\lambda,2n}(x_1,x_2,\dots,x_n)=2\frac{|x_j^{\lambda_i+n-i}+x_j^{-\lambda_i-n+i}|_{i,j=1}^n}{|x_j^{n-i}+x_j^{-n+i}|_{i,j=1}^n}. \tag{5.1}$$

Lemma 5.1 *Let λ be a partition with at most n parts and set $\mu_i=\lambda_i+n-i$. Then*

$$o_{\lambda,2n+1}(q^2,q^4,\dots,q^{2n})=\frac{\prod_{i=1}^n\langle 2\mu_i+1\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j+1\rangle}{\prod_{i=1}^n\langle 2i-1\rangle!}$$

and

$$o_{\lambda,2n}(q,q^3,q^5,\dots,q^{2n-1})=\frac{\prod_{i=1}^n\langle 2\mu_i\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j\rangle}{\prod_{i=1}^n\langle\mu_i\rangle\prod_{i=1}^{n-1}\langle 2i\rangle!}.$$

Proof The argument for the odd orthogonal case is analagous to the proof of Lemma 4.1, but some care needs to be taken in the even orthogonal case. Setting $x_j=q^{2j-1}$ means that we should factor 2 out of the last row of the denominator to obtain

$$d=2(-1)^{n(n-1)/2}\prod_{i=1}^{n-1}\langle 2i\rangle!.$$

Further, the determinant in both numerators can be expressed as

$$d'=(-1)^{n(n-1)/2}\prod_{i=1}^n(q^{\mu_i}+q^{-\mu_i})\prod_{1\le i<j\le n}\langle\mu_j-\mu_i\rangle\langle\mu_i+\mu_j\rangle.$$

When λ has exactly n parts we may replace $q^{\mu_i}+q^{-\mu_i}$ by $\langle 2\mu_i\rangle/\langle\mu_i\rangle$ for each i, and the additional factor of 2 in (5.1) cancels with the denominator. However, when λ has fewer than n parts we have $\mu_n=0$ so $q^{\mu_n}+q^{-\mu_n}=2$, while $\langle 2\mu_n\rangle/\langle\mu_n\rangle=1$. In this case,

$$d'=2(-1)^{n(n-1)/2}\prod_{i=1}^n\frac{\langle 2\mu_i\rangle}{\langle\mu_i\rangle}\prod_{1\le i<j\le n}\langle\mu_j-\mu_i\rangle\langle\mu_i+\mu_j\rangle,$$

and the result again holds. ■

For the orthogonal λ-tableaux, the content of the (i,j)-th box is (see [10])

$$r'_\lambda(i,j)=\begin{cases}\lambda_i+\lambda_j-i-j & \text{if } i\ge j,\\ i+j-\lambda_i^t-\lambda_j^t-2 & \text{if } i<j.\end{cases}$$

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Lemma 5.2 *Let λ be a partition with at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]} \langle 2n+1+r'_\lambda(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle \mu_i\rangle! \prod_{1\le i\le j\le n}\langle \mu_i+\mu_j+1\rangle}{\prod_{i=1}^{n}\langle 2i-1\rangle!}$$

and

$$\prod_{(i,j)\in[\lambda]} \langle 2n+r'_\lambda(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle \mu_i\rangle! \prod_{1\le i\le j\le n}\langle \mu_i+\mu_j\rangle}{\prod_{i=1}^{n}\langle \mu_i\rangle \prod_{i=1}^{n-1}\langle 2i\rangle!}.$$

Proof These can be derived in the same way as Lemma 4.3. ■

Note that, using an argument similar to the one at the beginning of the proof of Lemma 4.3, one can derive orthogonal versions of the formula (4.4). A pictorial method, as in Remark 4.4, can then be used to derive these formulae.

Theorem 5.3 *Let λ be a partition with at most n parts. Then*

$$o_{\lambda,2n+1}(q^2,q^4,q^6,\dots,q^{2n}) = \prod_{(i,j)\in[\lambda]} \frac{\langle 2n+1+r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}$$

and

$$o_{\lambda,2n}(q,q^3,q^5,\dots,q^{2n-1}) = \prod_{(i,j)\in[\lambda]} \frac{\langle 2n+r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}.$$

Let $|T|$ be the sum of the entries of the odd orthogonal tableau T, where $\bar{i}$ is counted as $-i$ and ∞ is omitted. Further, let $r(T) = r_+(T) - r_-(T)$, where $r_+(T)$ and $r_-(T)$ are the number of boxes in T containing symbols from the sets $\{1,2,\dots,n\}$ and $\{\bar{1},\bar{2},\dots,\bar{n}\}$, respectively. Then

$$o_{\lambda,2n+1}(q^2,q^4,\dots,q^{2n}) = \sum_T q^{2|T|}$$

and

$$o_{\lambda,2n}(q,q^3,q^5,\dots,q^{2n-1}) = \sum_T q^{2|T|-r(T)},$$

where the sums run over the odd and even semistandard orthogonal λ-tableaux respectively. Again, setting $q = 1$ yields King and El-Samra's expression ([2]) for the dimension of the irreducible polynomial $\mathrm{O}(m)$-module with highest weight λ.

Corollary 5.4 *The number of semistandard orthogonal λ-tableaux with entries in the set $\{1,\bar{1},2,\bar{2},\dots,n,\bar{n},\infty\}$ for $m = 2n+1$ or $\{1,\bar{1},2,\bar{2},\dots,n,\bar{n}\}$ for $m = 2n$ is*

$$o_{\lambda,m}(1,\dots,1) = \prod_{(i,j)\in[\lambda]} \frac{m+r'_\lambda(i,j)}{h(i,j)}.$$

Remark 5.5 The irreducible $O(2n+1)$-module of highest weight λ remains irreducible on restriction to the special orthogonal group $SO(2n+1)$, and the same is true for the irreducible $O(2n)$-module when λ has strictly fewer than n parts. However, if λ has exactly n parts, then the restriction decomposes as the direct sum of irreducible $SO(2n)$-modules of highest weights $\lambda^+ = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$ and $\lambda^- = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$, and in this case there is a corresponding definition of positive and negative even semistandard orthogonal λ-tableaux ([5]). The $SO(2n)$-modules have characters

$$so_{\lambda^\pm,2n}(x_1, \dots, x_n) = \frac{|x_j^{\lambda_i+n-i} + x_j^{-\lambda_i-n+i}|_{i,j=1}^n \pm |x_j^{\lambda_i+n-i} - x_j^{-\lambda_i-n+i}|_{i,j=1}^n}{|x_j^{n-i} + x_j^{-n+i}|_{i,j=1}^n},$$

so here it is more convenient to use the specialisation $x_j = q^{2(j-1)}$, since this eliminates the second term in the numerator above, and in both cases we obtain

$$so_{\lambda^\pm,2n}(1, q^2, q^4, \dots, q^{2n-2}) = \prod_{i=1}^{n-1} \frac{\langle 2i\rangle}{\langle i\rangle} \prod_{i=1}^{n} \frac{\langle \mu_i\rangle}{\langle 2\mu_i\rangle} \prod_{(i,j)\in[\lambda]} \frac{\langle 2n + r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle},$$

since $\mu_n > 0$. It is clear that when we set $q = 1$ the additional terms reduce to $1/2$ so we find that the number of positive or negative semistandard even orthogonal λ-tableaux with entries in the set $\{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$ is

$$so_{\lambda^\pm,2n}(1, \dots, 1) = \frac{1}{2} \prod_{(i,j)\in[\lambda]} \frac{2n + r'_\lambda(i,j)}{h(i,j)},$$

as one would expect.

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(Campbell)
e-mail: peter_s_campbell@yahoo.co.uk

(Stokke) *Department of Mathematics and Statistics, University of Winnipeg, Winnipeg, MB R3B 2E9*
e-mail: a.stokke@uwinnipeg.ca