HOLOMORPHIC CURVES IN THE ORTHOGONAL TWISTOR SPACE

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A complete description of holomorphic curves in the Hermitian symmetric space $SO(6)/U(3)$ is given in terms of orthogonal differential invariants.

0. INTRODUCTION

The Hermitian symmetric space $SO(2n)/U(n)$ is naturally identified with the twistor space $\mathcal{T}(\mathbb{R}^{2n})$, the space of orthogonal complex structures on $\mathbb{R}^{2n}$. Thus a curve in $SO(2n)/U(n)$ can be thought of as a 1-parameter family of complex structures on $\mathbb{R}^{2n}$, and the study of holomorphic curves in $SO(2n)/U(n)$ pertains to the deformation problem of complex structures.

The case of $SO(6)/U(3)$ is particularly appealing as this space is symmetric space isomorphic to $\mathbb{C}P^3$, albeit via a complicated isomorphism. (The space $SO(4)/U(2)$ is symmetric space isomorphic to $\mathbb{C}P^1$, and the geometry of curves in $SO(4)/U(2)$ is trivial.) The study of holomorphic curves in $SO(6)/U(3)$ yields a new perspective on the study of holomorphic curves, and more generally minimal surfaces, in $\mathbb{C}P^3$ (see [2]).

In the present paper we give a complete description of holomorphic curves in $SO(6)/U(3)$ in terms of orthogonal differential invariants. Given a Riemann surface $M$ we derive a system of partial differential equations with 3 unknown functions, $(\tau_i)$, on $M$. These partial differential equations (Section 4 (II - 3)) are the integrability conditions in the following sense: given a solution $(\tau_i)$ one can manufacture a holomorphic curve by integrating a Frobenius system. To put it another way, a solution to the integrability conditions determines a holomorphic curve constructively up to integration involving ordinary differential equations only. Moreover, every holomorphic curve in $SO(6)/U(3)$ arises in this manner.

1. THE SPACE OF ORTHOGONAL COMPLEX STRUCTURES ON $\mathbb{R}^{2n}$

Let $\mathcal{T}(\mathbb{R}^{2n})$ denote the space of orientation preserving orthogonal complex structures on $\mathbb{R}^{2n}$. More precisely,

$$\mathcal{T}(\mathbb{R}^{2n}) = \{ J \in \text{Aut}^+ (\mathbb{R}^{2n}) : J^2 = -\text{id}, \langle Jv, Jw \rangle = \langle v, w \rangle, v, w \in \mathbb{R}^{2n} \}.$$
where $\text{Aut}^+ (\mathbb{R}^{2n})$ denotes the set of orientation preserving automorphisms of $\mathbb{R}^{2n}$. The space $T(\mathbb{R}^{2n})$ can be identified with the Hermitian symmetric space $SO(2n)/U(n)$ as we shall see below.

Let $i: U(n) \rightarrow SO(2n)$ be the Lie group monomorphism induced by the identification

$$\mathbb{R}^{2n} = \mathbb{C}^n, (x^\alpha) \mapsto (z^1 + iz^2, \cdots, z^{2n-1} + iz^{2n}).$$

More explicitly, if one writes

$$Z = X + iY \in U(n), X, Y \text{ real matrices},$$

then

$$i(Z) = S \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} S;$$

where $S = (\varepsilon_1, \varepsilon_{n+1}, \varepsilon_2, \varepsilon_{n+2}, \cdots, \varepsilon_n, \varepsilon_{2n})$ and $\varepsilon_i$ denotes the column vector with 1 at the $i$th entry and zeros elsewhere.

We put $H = i(U(n)) \subset SO(2n)$; we shall occasionally confuse $H$ with $U(n)$. Let $A = (A^\alpha) = (A_1, \ldots, A_{2n}) \in SO(2n)$, and consider the assignment

$$A \mapsto J_A \in T(\mathbb{R}^{2n})$$

given by the prescription

$$J_A(A_{2i-1}) = A_{2i}, \quad J_A(A_{2i}) = -A_{2i-1}, \quad 1 \leq i \leq n,$$

that is, the matrix of $J_A$ with respect to the basis $(A^\alpha)$ is

$$J_n = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ & \ddots & \ddots \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}.$$

This assignment induces a bijection

$$SO(2n)/H \rightarrow T(\mathbb{R}^{2n})$$

as it is easily seen that $J_A = J_B$ if and only if the cosets $AH$ and $BH$ coincide.

In what follows we recall the root space decomposition of the Lie algebra $o(2n)$. First define $e^\alpha_\beta$ to be the $2n \times 2n$ with 1 at the $(\alpha, \beta)$-entry and zeros elsewhere. We
then define the following matrices in $\mathfrak{o}(2n)$ ($\mathfrak{o}(2n)$ is thought of as the set of all $2n \times 2n$ real skew-symmetric matrices):

\[
\begin{align*}
F_i &= \varepsilon_{2i-1}^{2i-1}, \\
E_{ij} &= \varepsilon_{2i-1}^{2i-1} + \varepsilon_{2j}^{2j-1} - \varepsilon_{2i}^{2i-1} - \varepsilon_{2j}^{2j}, \\
F_{ij} &= \varepsilon_{2j-1}^{2i-1} + \varepsilon_{2i}^{2j-1} - \varepsilon_{2i}^{2i-1} - \varepsilon_{2j}^{2j}, \\
E'_{ij} &= \varepsilon_{2i-1}^{2i-1} + \varepsilon_{2i}^{2j-1} - \varepsilon_{2i}^{2j} - \varepsilon_{2j}^{2i}, \\
F'_{ij} &= \varepsilon_{2j-1}^{2i-1} + \varepsilon_{2i}^{2j-1} - \varepsilon_{2i}^{2j} - \varepsilon_{2j}^{2i},
\end{align*}
\]

where $1 \leq i < j \leq n$. Now put

\[
V_i = \mathbb{R} - \text{span}\{F_i\}, \quad V_{ij} = \mathbb{R} - \text{span}\{E_{ij}, F_{ij}\}, \quad V'_{ij} = \mathbb{R} - \text{span}\{E'_{ij}, F'_{ij}\}.
\]

The root spaces of $\mathfrak{o}(2n)$ relative to the standard maximal torus $T = S\mathcal{O}(2) \cap S\mathcal{O}(2n)$ ($S\mathcal{O}(2)$ is diagonally included in $S\mathcal{O}(2n)$) are precisely

\[
\bigoplus V_i = t, \quad V_{ij}, \quad V'_{ij},
\]

where $t$ is the Lie algebra of $T$, the trivial root space.

The Lie algebra of $H \subset S\mathcal{O}(2n)$ is given by

\[
\mathfrak{h} = \{X \in \mathfrak{o}(2n): \quad Xj_n + j_nX = 0\}
\]

so that

\[
\mathfrak{h} = t \oplus \sum V_{ij}.
\]

Consequently

\[
m = \bigoplus V'_{ij}
\]

is the orthogonal complement to $\mathfrak{h}$ relative to the Killing form. Via $\pi_*e$, where $\pi$ is the projection $S\mathcal{O}(2n) \to S\mathcal{O}(2n)/H$, $m$ gets identified with the tangent space at the identity coset of $S\mathcal{O}(2n)/H$:

\[
m = T_0(S\mathcal{O}(2n)/H), \quad 0 = H.
\]

Let $\Omega = \left(\Omega^g_0\right)$ denote the $\mathfrak{o}(2n)$-valued Maurer–Cartan form of $S\mathcal{O}(2n)$. We then have the decomposition

\[
\Omega = \Omega_h + \Omega_m, \quad \Omega_m = \sum \Omega_{V'_{ij}}.
\]
where
\[ \Omega'_{ij} = \frac{1}{2} \left[ (\Omega_{2j-1}^{2i-1} - \Omega_{2j}^{2i}) \otimes E'_{ij} + (\Omega_{2j}^{2i} + \Omega_{2j-1}^{2i-1}) \otimes F'_{ij} \right] \text{ (no sum)}. \]

We also put
\[ \Theta'_{ij} = \frac{1}{2} \left[ (\Omega_{2j-1}^{2i-1} - \Omega_{2j}^{2i}) + \sqrt{-1}(\Omega_{2j-1}^{2i} + \Omega_{2j}^{2i-1}) \right], \quad 1 \leq i < j \leq n. \]

Recall that the space \( SO(2n)/U(n) \) has, up to conjugacy, exactly one integrable almost complex structure; this complex structure is characterised by letting the pullbacks of \( (\Theta'_{ij}) \) (by a local section of \( SO(2n) \to SO(2n)/H \)) span the space of type \( (1, 0) \) forms on \( SO(2n)/H \).

Any invariant metric on \( SO(2n)/H \) is given by the pullback of the symmetric product
\[ c \cdot \sum \Theta'_{ij} \cdot \overline{\Theta'_{ij}}, \quad c > 0. \]

We shall use the metric coming from \( c = 1 \).

2. The linear isotropy representation

In this section we explicitly compute the linear isotropy representation
\[ \rho : U(n) \to GL(m) = GL(T_0(SO(2n)/H)). \]

For \( Z \in U(n) \) one has the inner automorphism
\[ \text{Inn}_z : SO(2n) \to SO(2n), \quad g \mapsto zyz^{-1}, \]
where \( z = i(Z) \in H \). The map \( \text{Inn}_z \) fixes the origin \( 0 = H \in SO(2n)/H \) and we obtain the automorphism
\[ \text{Inn}_{z=0} : m \to m, \quad m = T_0(SO(2n)/H). \]

The assignment
\[ Z \in U(n) \mapsto \text{Ad}(Z)|_m = \text{Inn}_{z=0} \in GL(m) \]
is nothing but the adjoint representation of \( U(n) \) restricted to the invariant subspace \( m \subset \mathfrak{o}(2n) \) \( (\text{Ad}(H)m \subset m) \). Then \( \rho = \text{Ad}|_m \).

In what follows we take \( n = 3 \). Recall that a basis of \( m \) is given by
\[ E'_{12}, E'_{13}, E'_{23}, F'_{12}, F'_{13}, F'_{23} \in \mathfrak{o}(6). \]
\( \text{Ad}_Z(E_{ij}') \) is computed from the matrix multiplication

\[
i(Z) \cdot E_{ij}' \cdot i(Z)^{-1}.
\]

Written out more fully,

\[
\text{Ad}_Z (E_{ij}') = \left( S \cdot \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \cdot S \right) E_{ij}' \left( S \cdot \begin{bmatrix} tX & tY \\ -tY & tX \end{bmatrix} \cdot S \right).
\]

Similarly one computes \( \text{Ad}_Z (F_{ij}') \). We want to write \( \text{Ad}_Z (E_{ij}') \) and \( \text{Ad}_Z (F_{ij}') \) as linear combinations in \( (E_{ij}', F_{ij}') \). For this we define 3 × 3 matrices \( J_{ij} \) by the formula:

\[
J_{ij} = \epsilon_j^i - \epsilon_i^j \in o(3), \quad 1 \leq i < j \leq 3.
\]

Note that

\[
^t S \cdot E_{ij}' \cdot S = \begin{bmatrix} J_{ij} & 0 \\ 0 & -J_{ij} \end{bmatrix}, \quad ^t S \cdot F_{ij}' \cdot S = \begin{bmatrix} 0 & J_{ij} \\ J_{ij} & 0 \end{bmatrix}.
\]

Computations reveal that

\[
\text{Ad} (Z) : E_{ij}' \mapsto \{X J_{ij} \cdot ^t X - Y J_{ij} \cdot ^t Y\}_1 + \{X J_{ij} \cdot ^t Y + Y J_{ij} \cdot ^t X\}_2,
\]

\[
F_{ij}' \mapsto \{-X J_{ij} \cdot ^t Y - Y J_{ij} \cdot ^t X\}_1 + \{X J_{ij} \cdot ^t X - Y J_{ij} \cdot ^t Y\}_2,
\]

where \{·\}_1 means identify \( J_{kt} \) with \( E_{kt}' \), and \{·\}_2 means identify \( J_{kt} \) with \( F_{kt}' \).

It is more convenient to rewrite the above using complex notation. So we write

\[
Z = (x^i_j + \sqrt{-1} y^i_j) = (z^i_j) \in U(3),
\]

and compute \( \rho(Z) \) with respect to the complex basis

\[
E_{12}' + iF_{12}', \quad E_{13}' + iF_{13}', \quad E_{23}' + iF_{23}',
\]

so that \( \rho(Z) \in GL(3, \mathbb{C}) \). (In other words, we have chosen an identification of \( m \) with \( \mathbb{C}^3 \).) We find that

\[
\rho(Z) = \begin{bmatrix} Z_{33} & Z_{32} & Z_{31} \\ Z_{23} & Z_{22} & Z_{21} \\ Z_{13} & Z_{12} & Z_{11} \end{bmatrix} \in GL(3, \mathbb{C}),
\]

where \( Z_{ij} \) denotes the \((i, j)\)-minor of \( Z \). We see that \( \rho(Z) \) is related to the adjoint matrix of \( Z \) in a simple way. (The adjoint matrix of \( Z \) is the transpose of the cofactor matrix of \( Z \) so that \( Z \) times its adjoint matrix is the determinant of \( Z \).) More precisely, we let

\[
\delta = \begin{bmatrix} 1 & \quad 0 \\ 0 & -1 \end{bmatrix}.
\]
Then

\[
\rho(Z) = \det(Z)\delta \cdot \overline{Z} \cdot \delta \in GL(3, \mathbb{C}).
\]

**REMARK.** More generally, consider the linear isotropy representation

\[\rho: U(n) \to GL(N, \mathbb{C}),\]

where \(m\) is identified with \(\mathbb{C}^N, N = n(n-1)/2\), via the lexicographical ordering of the root basis as in the \(SO(6)/U(3)\) case. Identify \(\mathbb{C}^N\) with \(\Lambda^2(\mathbb{C}^n)\), the space of 2-vectors in \(\mathbb{C}^n\). Calculations then show that

\[\rho(Z)(\varepsilon_i \wedge \varepsilon_j) = Z_i \wedge Z_j,\]

where \(Z = (Z_i) \in U(n)\), and \((\varepsilon_i)\) are the canonical basis vectors of \(\mathbb{C}^n\). Extending linearly over the basis \(\{\varepsilon_i \wedge \varepsilon_j: 1 \leq i < j \leq n\}\) of \(\mathbb{C}^N\) one obtains the matrix representation \(\rho\). However, the orbit structure of this action is very complicated for large \(n\).

3. The Frenet Frame Along a Holomorphic Curve

Let \(M\) be a Riemann surface and consider a holomorphic map

\[f: M \to SO(6)/H.\]

We let \(e: U \subset M \to SO(6)\) denote a smooth local section of the \(U(3)\)-principal bundle

\[f^{-1}SO(6) \to M.\]

The holomorphy of \(f\) is reflected by the fact that the forms

\[e^*\Theta^{ij}, \quad 1 \leq i < j \leq 3,\]

are all of type \((1, 0)\) on \(M\).

**INDEX CONVENTION.** \((12) = 1, (13) = 2, (23) = 3; 1 \leq i, j \leq 3, 1 \leq \alpha, \beta \leq 6.\)

Fix a Riemannian metric on \(M\) from its conformal class, say \(ds^2_M\). This means that we can locally write

\[ds^2_M = \varphi \cdot \overline{\varphi}\]

for some type \((1, 0)\) nonvanishing form \(\varphi\). The 1-form \(\varphi\) is called a unitary coframe.

We define complex valued local functions \((Z^i)\) on \(M\) by

\[e^*\Theta'^i = Z^i \varphi,\]
where \( \phi \) is a fixed unitary coframe.

Put \( \tau_1 = \sum |Z^i|^2 \). It is routinely verified that \( \tau_1 \) is a globally defined smooth function on \( M \). Let \( \sum_1 \) denote the zero set of \( \tau_1 \); we shall mostly work away from the set \( \sum_1 \).

Since \( f \) is nonconstant, the function \( \tau_1 \) is not identically zero: since the pullback of the standard metric on \( SO(6)/U(3) \) is given by

\[
ds^2_1 = \tau_1 \phi \cdot \bar{\phi},
\]

we see that \( \tau_1(x) = 0 \) if and only if \( x \in M \) is a non-immersion point. In fact, we can say a lot more. But first we need to recall the notion of an analytic type function.

**Definition:** Let \( U \) be a domain in the Riemann surface \( M \). A \( \mathbb{C}^n \)-valued smooth function \( h = (h^i) \) on \( U \) is said to be of analytic type if for each point \( x \in U \), if \( z \) is a local holomorphic coordinate centred at \( x \), then

\[
h = z^b \tilde{h},
\]

where \( b \) is a positive integer and \( \tilde{h} \) is a smooth \( \mathbb{C}^n \)-valued function with \( \tilde{h}(x) \neq 0 \).

So if \( h \) is a function of analytic type on \( U \), then \( h \) is either identically zero or its zeros are isolated and of finite multiplicity (the integer \( b \) in the above definition).

It is known [1] that the functions of analytic type are exactly solutions of exterior equation

\[
dh = \Phi h \pmod{\phi},
\]

where \( \Phi \) is an \( n \times n \) matrix of complex valued 1-forms on \( U \) and \( \phi \) is a nowhere zero type \((1, 0)\) form on \( U \).

**Proposition.** The function \( \tau_1 : M \to \mathbb{R} \) is an analytic type function on \( M \).

**Proof:** We will show that the local function

\[
(Z^i) : U \subset M \to \mathbb{C}^3
\]

is of analytic type. Since \( \tau_1 = \sum |Z^i|^2 \), the rest follows. Exterior differentiation of both sides of the equations

\[
e^*\Theta^{ij} = Z^i \phi
\]

leads to

\[
dZ^i \equiv \Psi_j^i Z^j \pmod{\phi}:
\]

one uses the Maurer-Cartan structure equations of \( SO(6) \) and the equation

\[
d\phi = -\Theta \wedge \phi,
\]
where $\theta_C$ is the complex connection form. For example,

$$dZ^1 \equiv (\theta_C + i\omega_2^1 + i\omega_4^1)Z^1 + (i\omega_3^1 - \omega_6^1)Z^2 + (\omega_5^1 - i\omega_7^1)Z^3 \pmod{\varphi},$$

where $\omega = e^*\Omega$.

Note that $r_1 = \sqrt{r_1}$ is a continuous function on $M$ smooth away from its zeros.

Suppose $\tilde{e}: \tilde{U} \to SO(6)$ is another lifting of $f$. This means that

$$\tilde{e} = e \cdot k,$$

for some smooth function $k: U \cap \tilde{U} \to U(3)$. Define $\tilde{Z}^i$ by setting

$$\tilde{e}^*\theta^i = \tilde{Z}^i \varphi.$$

We then obtain the following

**Transformation Rule.**  
$$\tilde{Z}^i = \rho(k^{-1})(Z^i).$$

The above rule follows from the fact that

$$(e \cdot k)^*\Omega_m = \text{Ad}(k^{-1})e^*\Omega_m.$$

Consulting (*) in the preceding section it is clear that near a point in $M \setminus \Sigma_1$ we can make

$$Z^1 = r_1, \quad Z^2 = Z^3 = 0,$$

where $r_1 = \sqrt{r_1}$. Any lifting $e$ achieving this will be called a first order frame.

Let $e$ be any first order frame along $f$, and write $\omega = e^*\Omega$. We then have

(E1) \hspace{1cm} \frac{1}{2}[(\omega_3^1 - \omega_4^1) + i(\omega_3^2 + \omega_4^2)] = r_1 \varphi,

(E2) \hspace{1cm} \omega_2^1 = \omega_2^2, \quad \omega_5^1 = -\omega_6^1,

(E3) \hspace{1cm} \omega_5^3 = \omega_6^4, \quad \omega_5^4 = \omega_6^3.

Define a subgroup of $U(3)$ by

$$G_1 = \{k \in U(3): \rho(k)^i(1, 0, 0) = i(1, 0, 0)\}.$$ 

We find that

$$G_1 = \left\{ \begin{bmatrix} Z & 0 \\ 0 & \exp(i\theta) \end{bmatrix} \mid Z \in SU(2) \right\} \cong SU(2) \times U(1).$$
Let $e$ be a first order frame along $f$. Then any other first order frame along $f$ is given by $e \cdot k$, where $k$ is a $G_1$-valued local function on $M$.

The first order frames are constructed near a point $x \in M \setminus \Sigma_1$. Near a point in $\Sigma_1$ we can find a generalised first order frame $e$ such that

$$e^* \Theta'^2 = e^* \Theta'^3 = 0, \quad e^* \Theta'^1 = Z_1 \varphi, \quad Z_1 \text{ complex valued},$$

where $|Z_1|^2 = \tau_1$. To see this observe that $(Z^i)$ can be written as

$$z^i(\tilde{Z}^i),$$

where $z$ is a local holomorphic coordinate centred at $x$ and $(\tilde{Z}^i(x)) \neq 0$. Thus we can use the $H$-action to bring about

$$\tilde{Z}^1 = \tau_1 > 0, \quad \tilde{Z}^2 = \tilde{Z}^3 = 0.$$

Then $Z_1 = z^i \tilde{Z}^i$ does the trick.

By way of notation we put

$$\psi^1 = \frac{1}{2}(\omega^1 - \omega^2), \quad \psi^2 = \frac{1}{2}(\omega^2 + \omega^1), \quad \psi = \psi^1 + i\psi^2.$$

Note that $\psi$ is of type $(1, 0)$, and that

$$\psi = Z_1 \varphi,$$

and near a point in $M \setminus \Sigma_1$

$$\psi = \tau_1 \varphi.$$

We now exterior differentiate the relations (E1 - 3) and construct the second order frames. We will use the Maurer–Cartan structure equations of $SO(6)$. We will also need the structure equations for $(M, \varphi \cdot \bar{\varphi})$:

$$d \varphi = -\theta_C \wedge \varphi.$$

The purely imaginary 1-form $\theta_C$ is the complex connection form with respect to $\varphi$, and is nothing but $-i$ times the Levi–Civita connection form of $(M, (\varphi^i))$. Differentiating this equation one more time we obtain

$$d\theta_C = \frac{K}{2} \varphi \wedge \bar{\varphi}, \quad K = \text{the Gaussian curvature}. $$
We now exterior differentiate the left hand side of (E1):
\[ d(\psi^1 + i\psi^2) = i(\omega_2^1 + \omega_2^3) \wedge (\psi^1 + i\psi^2) = i(\omega_2^1 + \omega_2^3) \wedge r_1 \varphi. \]

On the other hand,
\[ d(r_1 \varphi) = dr_1 \wedge \varphi - r_1 \theta_{\mathcal{O}} \wedge \varphi = (dr_1 - r_1 \theta_{\mathcal{O}}) \wedge \varphi. \]

It follows that
\[ [d \log r_1 - i(\omega_2^1 + \omega_2^3) - \theta_{\mathcal{O}}] \wedge \varphi = 0. \]

Since \( d \log r_1 \) is real and \( i(\omega_2^1 + \omega_2^3) + \theta_{\mathcal{O}} \) is purely imaginary, we then must have
\[ (F1) \quad *d \log r_1 = i[\theta_{\mathcal{O}} + i(\omega_2^1 + \omega_2^3)], \]

where * is the Hodge operator of \((M, ds^2)\).

Exterior differentiation of the first equation in (E2) leads to
\[ (\dagger) \quad \omega_5^3 \wedge \varphi^1 + \omega_5^4 \wedge \varphi^2 = 0; \]
the second equation yields
\[ (\ddagger) \quad \omega_5^3 \wedge \varphi^1 - \omega_5^3 \wedge \varphi^2 = 0. \]

It follows from (\(\dagger, \ddagger\)) that
\[ \omega_5^4 = -* \omega_5^3, \quad \omega_5^2 = -* \omega_5^1, \]
and we may set
\[ \omega_5^1 - i\omega_5^3 = Z^1 \varphi, \quad \omega_5^3 - i\omega_5^4 = Z^2 \varphi \]
for some local complex valued functions \((Z^i)\) on \(M\).

Put \( \tau_2 = \sum |Z^i|^2 \). It is easily verified that \( \tau_2 \) is a well-defined smooth function on \(M\). A consideration similar to that given for \( \tau_1 \) shows that \( \tau_2 \) is an analytic type function on \(M\). Let \( \sum_2 \) denote the zero set of \( \tau_2 \).

Define tilded quantities \( \tilde{Z}^1, \tilde{Z}^2 \) by
\[ \tilde{\omega}_5^4 - i\tilde{\omega}_5^3 = \tilde{Z}^1 \varphi, \quad \tilde{\omega}_5^3 - i\tilde{\omega}_5^1 = \tilde{Z}^2 \varphi, \quad \tilde{\varphi} = e^{\ast \Omega}, \]
where \( e = e \cdot k, \) \( k \in G_1\)-valued, is another first order frame. Write
\[ k = (Z, e^{i\varphi}), \]
where \( Z \) is \( SU(2) \)-valued. We can write \( Z \) as
\[
Z = \begin{bmatrix} z_1^1 & z_1^2 \\ -z_2^1 & z_2^2 \end{bmatrix}, \quad z_j^i = z_j^i + iy_j^i.
\]

From the formula \( \bar{w} = i(k^{-1}) \cdot w \cdot i(k) \) we compute that
\[
t^i(\bar{Z}^i) = e^{-i\theta} \cdot tZ \cdot t(Z^i).
\]
We see from this that we can make \( Z^1 = \tau_2 > 0 \), and \( Z^2 = 0 \).

Summarising the preceding computation, we have

**Proposition.** Let \( f: M \to SO(6)/U(3) \) be a nonconstant holomorphic map. Near any point \( x \in M \setminus \{\Sigma_1 \cup \Sigma_2\} \) there exists a local lifting \( e \) into \( SO(6) \) such that in addition to \((E1 - 3)\) we have
\[
(\text{E4}) \quad \omega_5 = \omega_5^5 = 0,
\]
where, as usual, \( \omega = e^*\Omega \).

The totality of such frames, called the second order frames, is determined up to the structure group
\[
G_2 = \{ (Z, e^{i\theta}) \in SU(2) \times U(1) : e^{-i\theta} \cdot tZ \cdot t(1, 0) = t(1, 0) \}
\]
\[
= \{ (e^{i\theta}, e^{-i\theta}, e^{i\theta}) \in U(3) \} \cong U(1).
\]

**Theorem.** Suppose \( \tau_2(f) \equiv 0 \). Then \( f(M) \) is congruent to an open submanifold of \( SO(4)/U(2) \cong \mathbb{C}P^1 \).

**Proof:** \( \tau_2(f) \equiv 0 \) means that the bundle of first order frames along \( f \), denoted by \( L_1 \), is an integral manifold of the exterior system
\[
\Omega_B^A = 0, \quad 1 \leq A \leq 4, \quad 5 \leq B \leq 6,
\]
on \( SO(6) \). (So a first order frame along \( f \) is a local section of \( L_1 \to M \).) It follows that \( L_1 \) is a translate of \( SO(4) \times SO(2) \subset SO(6) \). Then \( f(M) \) is congruent to a submanifold of
\[
SO(4) \times SO(2)/(U(3) \cap SO(4) \times SO(2)) \cong SO(4)/U(2) \cong \mathbb{C}P^1.
\]

Hereafter we assume that \( \tau_2 \) is not identically zero.

We now exterior differentiate both sides of the equations in \((\text{E4})\), and construct the third order frames.
We obtain from the first equation of (E4)
\[ d(\omega_5^1 - i\omega_5^2) = i(\omega_6^5 - \omega_2^1) \wedge r_2 \varphi, \]
\[ d(r_2 \varphi) = dr_2 \wedge \varphi - r_2 \theta_C \wedge \varphi. \]
Consequently,
\[ \{d\log r_2 - \theta_C - i(\omega_6^5 - \omega_2^1)\} \wedge \varphi = 0. \]
Therefore
\[ \ast d\log r_2 = i(\theta_C + i(\omega_6^5 - \omega_2^1)). \]
The remaining two equations in (E4), upon exterior differentiation, yield
\[ \omega_3^1 \wedge \varphi^1 - \omega_3^2 \wedge \varphi^2 = 0, \quad \omega_4^1 \wedge \varphi^1 - \omega_4^2 \wedge \varphi^2 = 0. \]
Consequently, we can write
\[ \begin{align*}
\omega_3^1 &= a \varphi^1 + b \varphi^2, \\
\omega_4^1 &= b \varphi^1 + c \varphi^2, \\
\omega_3^2 &= a \omega_3^1, \\
\omega_4^2 &= a \omega_4^1,
\end{align*} \]
where \(a, b, c\) are some local functions on \(M\) with \(a + c = 2r_1\).

Define tilded quantities \(\tilde{a}, \tilde{b}, \tilde{c}\) using another second order frame \(\tilde{e} = e \cdot k\), where
\[ k = (e^{i\theta}, e^{-i\theta}, e^{i\theta}), \quad \theta \text{ a local function on } M. \]

We want to know how \(\tilde{a}, \tilde{b}, \tilde{c}\) are related to \((a, b, c)\).

Again using the formula
\[ \tilde{\omega} = i(k^{-1}) \cdot \omega \cdot i(k), \quad \omega = e^\ast \Omega, \quad \tilde{\omega} = \tilde{e}^\ast \Omega, \]
we compute that
\[ \tilde{a} = a \cdot \cos^2 \theta + c \cdot \sin^2 \theta - 2b \cdot \cos \theta \sin \theta, \]
\[ \tilde{b} = b \cdot \cos 2\theta + (a - c) \cdot \cos \theta \sin \theta, \]
\[ \tilde{c} = a \cdot \sin^2 \theta + c \cdot \cos^2 \theta + b \cdot \sin 2\theta. \]
If \(b\) does not vanish, then we can smoothly choose \(\theta\) so that
\[ \cotan 2\theta = (c - a)/2b \]
making \(\tilde{b} = 0\). All this leads to the third, and final, normal form
\[ \begin{align*}
\omega_3^1 + i\omega_3^2 &= a \varphi, \\
\omega_4^1 + i\omega_4^2 &= -ic \varphi, \\
\end{align*} \]
where \( a + c = 2r_1 \). Put \( r_3 = (a - c)/2 \).

The function \( \tau_3 = r_3^2 \) is an analytic type function on \( M \); we let \( \Sigma_3 \) denote the zero set of \( \tau_3 \).

The isotropy group \( G_3 \) is given by

\[
G_3 = \{(e^{i\theta}, e^{-i\theta}, e^{i\theta}) \in G_2 : \theta = n\pi/2, n \in \mathbb{Z}\} \cong \mathbb{Z}_4.
\]

It follows that near a point \( z \in M \setminus \{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3\} \) there is a more or less unique lifting

\[
e_f : U \subset M \to SO(6)
\]

achieving the normal forms (E1) through (E5). Such a lifting will be called a Frenet frame along \( f \).

Exterior differentiation of both sides of the equations in (E5) leads to

\[
\begin{align*}
\text{(F1)} & \quad *d\log r_1 = i(\theta_C + i(\omega_1^1 + \omega_3^1)), \\
\text{(F3)} & \quad *d\log r_3 = i(\theta_C + i(\omega_2^1 - \omega_3^3)).
\end{align*}
\]

**REMARK.** Suppose \( \tau_3(f) \) is identically zero. Then one can show that \( f(M) \) lies in the image of a \( \mathbb{C}P^2 \subset \mathbb{C}P^3 \) under the symmetric space isomorphism

\[
\mathbb{C}P^3 \cong SO(6)/U(3).
\]

**THEOREM.** Let \( f : M \to SO(6)/U(3) \) be a nonconstant holomorphic map. Fix a conformal metric \( ds^2 = \varphi \cdot \bar{\varphi}, \varphi \in \text{type } (1,0) \), and define the differential invariants \( (\tau_i) \) as in the above. We then have

\[
\begin{align*}
\text{(11)} & \quad \Delta \log \tau_1 = K - 4\tau_1 + 2\tau_2, \\
\text{(12)} & \quad \Delta \log \tau_2 = K + 2(\tau_1 + \tau_3) - 4\tau_2, \\
\text{(13)} & \quad \Delta \log \tau_3 = K + 2\tau_2 - 4\tau_3,
\end{align*}
\]

away from the singular locus \( \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \subset M \).

**PROOF:** Exterior differentiate both sides of the equations in (F1 - 3) using

\[
\begin{align*}
d \ast d \log r_1 & = \frac{i}{2} \Delta \log r_1 \varphi \wedge \bar{\varphi}, \\
d\omega_1^1 & = \frac{i}{2} (a^2 + c^2 - 2r_2^2) \varphi \wedge \bar{\varphi}, \\
d\omega_2^1 & = iac \varphi \wedge \bar{\varphi}, \\
d\omega_3^3 & = i\tau_3 \varphi \wedge \bar{\varphi}.
\end{align*}
\]

We give an application.
**Corollary.** Suppose \( f: M \to SO(6)/U(3) \) is a holomorphic isometric immersion from a compact \( M \). Further suppose that \( K \geq 4/3 \), where \( K \) is the Gaussian curvature of \((M, ds^2)\). Then we must have \( K = 4/3 \).

**Proof:** \( K \geq 4/3 \) implies that

\[
\Delta \log (r_2^2 r_3) \geq 0.
\]

Thus \( \log (r_2^2 r_3) \) is a subharmonic function with singularities at the zeros of \( r_2 \) and \( r_3 \) where it goes to \(-\infty\). In particular, this function attains a maximum on \( M \). Now the maximum principle for subharmonic functions says that it must be a constant. \( \square \)

4. **The Integrability Conditions and the Associated PDE System**

In this section we summarise the frame construction by setting up a bijective correspondence between the holomorphic curves in \( SO(6)/U(3) \) and the solutions to the PDE system coming from (11-3).

**Definition:** We shall say that \( f: M \to SO(6)/U(3) \) is a nondegenerate curve if none of the \( \tau_i \)'s are identically zero. The map \( f \) will be called a regular curve if \( \tau_1 \cdot \tau_2 \cdot \tau_3 \) is never zero.

Observe that the regularity assumption is a global assumption.

Consider the following exterior differential system, denoted by \( \mathcal{S} \), defined on \( M \times SO(6) \) with independence condition \( \varphi \wedge \bar{\varphi} \neq 0 \):

\[
\begin{align*}
\Omega_1^1 + i\Omega_2^1 &= (r_1 + r_2)\varphi, \\
-\Omega_1^2 + i\Omega_2^2 &= (r_1 - r_2)\varphi, \\
\Omega_3^1 - i\Omega_3^2 &= \Omega_6^1 - i\Omega_6^2 = r_2 \varphi, \\
\Omega_4^1 - \Omega_4^2 &= \Omega_8^1 - \Omega_8^2 = \Omega_6^4 - \Omega_6^5 = 0. \\
\Omega_5^1 + \Omega_5^2 &= i\theta_C - *d\log r_1, \\
\Omega_6^3 - \Omega_6^2 &= i\theta_C - *d\log r_2, \\
\Omega_7^1 - \Omega_7^2 &= i\theta_C - *d\log r_3,
\end{align*}
\]

where \( \theta_C \) is the complex connection form of \( (M, \varphi \cdot \bar{\varphi}) \), and the \( r_i \)'s are any positive functions on \( M \) solving the PDE system

\[
\begin{align*}
(11) \quad \Delta \log r_1 &= K - 4r_1^2 + 2r_2^2, \\
(12) \quad \Delta \log r_2 &= K + 2(r_1^2 + r_2^2) - 4r_2^2, \\
(13) \quad \Delta \log r_3 &= K + 2r_2^2 - 2r_3^2.
\end{align*}
\]

**Theorem.** The set of regular holomorphic curves \( M \to SO(6)/U(3) \) is in bijective correspondence with the set of all solutions \((r_1, r_2, r_3)\) to the integrability conditions (11-3).
PROOF: Any regular curve certainly gives rise to such a solution: this is the content of the frame construction given in the preceding section. Conversely suppose we are given such a solution \((r_1)\). Counting the number of independent equations in \(S\) we see that \(S\) defines a two-dimensional distribution on \(M \times SO(6)\). Moreover, this distribution is completely integrable and, hence, defines a foliation on \(M \times SO(6)\). The independence condition \(\varphi \wedge \overline{\varphi} \neq 0\) implies that a leaf of this foliation can be written locally as

\[ U \to U \times SO(6), \quad z \mapsto (z, e(z)). \]

It is straightforward to verify that \(e(z)\) is a Frenet frame along \(f = \pi \circ e\), where \(\pi\) denotes the projection \(SO(6) \to SO(6)/U(3)\).

5. COMPACT CURVES

In this section we give the integrated version of the integrability conditions (II – 3) assuming that \(M\) is compact.

DEFINITION: Let \(M\) be a Riemann surface. A singular Hermitian metric on \(M\) is given locally as

\[ ds^2 = \psi \cdot \overline{\psi}, \]

where \(\psi\) is a type \((1, 0)\) smooth form of analytic type, that is, \(\psi\) can be written as the product of an analytic type function and a nowhere vanishing type \((1, 0)\) form. We can rewrite \(ds^2\) as

\[ ds^2 = h(z)dz \cdot d\overline{z}, \]

where \(h(z) \geq 0\) and \(z\) is a holomorphic coordinate. Moreover, we have

\[ h(z) = |z|^{2n} \tilde{h}(z), \]

where \(\tilde{h}(z)\) is never zero and \(n\) is a nonnegative integer. The integer \(n\) is the order of \(\psi\) at \(z = 0\) and we write \(\text{ord}_p \psi = n\). The singular divisor of \(ds^2\), denoted by \(D_\psi\), is defined to be the zero divisor of \(\psi\). So

\[ D_\psi = \sum \text{ord}_p(\varphi)p, \quad p \in M. \]

It is easy to see that \(D_\psi\) depends only on the singular metric, not on the particular choice of \(\psi\). The degree of \(D_\psi\) is locally finite, and is the total number of zeros of \(\psi\) counted with multiplicity.

Given a singular metric \(ds^2\) on \(M\) we have the usual Hermitian structure equations away from the support of the singular divisor:

\[ d\psi = -\theta_G \wedge \psi, \quad d\theta_G = -\frac{K}{2} \psi \wedge \overline{\psi} = (-iK) \cdot \text{the Kähler form}. \]

There is the
**Generalised Gauss–Bonnet–Chern Theorem.** Let \( M \) be a compact Riemann surface of genus \( g \) equipped with a singular metric \( \psi \bar{\psi} \). Then
\[
\frac{i}{2\pi} \int_M d\theta_C = 2 - 2g + \deg D_\psi.
\]

**Proof:** This follows from the usual Gauss–Bonnet–Chern theorem combined with the argument principle: one notes that \( d\theta_C \) is a multiple of \( \Delta \log h(dz \wedge \bar{dz}) \). \( \square \)

Given a nondegenerate holomorphic curve \( f: M \to SO(6)/U(3) \) we define the \( i \)th osculating metric to be
\[
\partial_i^2 = r_i \psi \cdot \bar{\psi}.
\]
These metrics are singular metrics. (Note that \( \partial_i^2 \) is just the induced metric.) We put
\[
\psi_i = \tau_i \psi,
\]
\[
\Lambda_i = \frac{i}{2} \psi_i \wedge \bar{\psi}_i = \text{the Kähler form of } (M, \partial_i^2),
\]
\[
\theta_i = \text{the complex connection form of } (M, \partial_i^2),
\]
\[
K_i = \text{the Gaussian curvature of } (M, \partial_i^2)
\]
so that
\[
d\psi_i = -\theta_i \wedge \psi_i, \quad d\theta_i = -iK_i \Lambda_i.
\]

Let \( e_f \) be a Frenet frame along \( f \), and put \( \omega = e^* \Omega \). Consulting the normal forms (E1–5) in Section 3 we compute that
\[
\theta_1 = i(\omega_1^2 + \omega_3^4),
\]
\[
\theta_2 = i(\omega_2^2 - \omega_1^4),
\]
\[
\theta_3 = i(\omega_2^4 - \omega_3^4).
\]
(For example, the first and third relations follow upon exterior differentiating the first two equations in (E5).) Exterior differentiation of these equations leads to
\[
d\theta_1 = 2i(-2\Lambda_1 + \Lambda_2),
\]
\[
d\theta_2 = 2i(\Lambda_1 - 2\Lambda_2 + \Lambda_3),
\]
\[
d\theta_3 = 2i(\Lambda_2 - 2\Lambda_3).
\]

**Theorem.** Let \( M = M_g \) denote a compact Riemann surface of genus \( g \), and consider a nondegenerate curve \( f: M \to SO(6)/U(3) \). Then
\[
2g - 2 = \#_i - 2d_i + d_{i+1},
\]
where \( \#_i = \deg D_\psi_i = \text{the total number of zeros of } r_i \),
\[
d_i = \frac{1}{\pi} \cdot \text{the area of } (M, \partial_i^2),
\]
\[
d_{-1} = d_4 = 0.
\]
Proof: We have
\[ \frac{i}{2\pi} \int_M d\theta_{i,C} = 2 - 2g + \#_i \]
from the generalised Gauss–Bonnet–Chern theorem. From the Wirtinger theorem
\[ \int_M \Lambda_i = \text{the area of } (M, ds_i^2) \]
and the result follows.

Remark. The relations in (P) correspond to the Plucker relations for algebraic curves in \( CP^3 \) [2] p.86–95).

References


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