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LIMIT THEOREMS FOR A DIFFUSION PROCESS WITH A ONE-SIDED BROWNIAN POTENTIAL

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Abstract

We consider a diffusion process $X(t)$ with a one-sided Brownian potential starting from the origin. The limiting behavior of the process as time goes to infinity is studied. For each $t > 0$, the sample space describing the random potential is divided into two parts, $\tilde{A}_t$ and $\tilde{B}_t$, both having probability $\frac{1}{2}$, in such a way that our diffusion process $X(t)$ exhibits quite different limiting behavior depending on whether it is conditioned on $\tilde{A}_t$ or on $\tilde{B}_t$ ($t \to \infty$). The asymptotic behavior of the maximum process of $X(t)$ is also investigated. Our results improve those of Kawazu, Suzuki, and Tanaka (2001).

Keywords: Random environment; diffusion process; occupation time

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1. Introduction

In [4] a diffusion process $\{X(t),\ t \geq 0\}$ with a one-sided Brownian potential was studied, and it was shown that the limit distribution of $t^{-1/2}X(t)$ as $t \to \infty$ exists and is given by

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x + \frac{1}{2}\delta_0(\mathrm{d}x),$$

the support being $[0, \infty)$. The long-time behavior of $X(t)$ is diffusive (in the sense that a limit distribution exists under the Brownian scaling) with probability $\frac{1}{2}$ and subdiffusive with the remaining probability $\frac{1}{2}$.

In this paper we treat the same model and give much more precise statements. In fact we prove, among other things, that $(\log t)^{-2}X(t)$ has a limit distribution with probability $\frac{1}{2}$; for the precise meaning of this, see Theorem 1.5.

Let us describe our model, following [4]. We denote by $\mathbb{W}$ the space of continuous functions w defined on $\mathbb{R}$ and vanishing identically on $[0, \infty)$. Let P be the Wiener measure on $\mathbb{W}$, namely the probability measure on $\mathbb{W}$ such that $\{w(-x),\ x \geq 0, \mathrm{P}\}$ is a Brownian motion with time parameter x. By Ω we denote the space of real-valued, continuous functions defined on $[0, \infty)$. For $\omega \in \Omega$, we write $X(t) \equiv X(t, \omega) \equiv \omega(t)$, the value of ω at t. For $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, we let $\mathrm{P}_w^{x_0}$ be the probability measure on Ω such that $\{X(t),\ t \geq 0,\ \mathrm{P}_w^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_w = \frac{1}{2}\mathrm{e}^{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-w(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

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starting from x_0. We define the probability measure $\mathcal{P}^{x_0}$ on $\mathbb{W} \times \Omega$ by

$$\mathcal{P}^{x_0}(\mathrm{d}w\,\mathrm{d}\omega) = \mathrm{P}(\mathrm{d}w)\,\mathrm{P}_w^{x_0}(\mathrm{d}\omega).$$

We regard $\{X(t),\ t \geq 0,\ \mathcal{P}^{x_0}\}$ as a process defined on the probability space $(\mathbb{W} \times \Omega, \mathcal{P}^{x_0})$ and call it a diffusion process with a one-sided Brownian potential. Our aim is to clarify the limiting behavior of $\{X(t),\ t \geq 0,\ \mathcal{P}^0\}$ as $t \to \infty$.

For the case in which $w(x)$ does not vanish identically for $x \geq 0$ or, more precisely, the case in which $\{w(x),\ x \geq 0,\ \mathrm{P}\}$ and $\{w(-x),\ x \geq 0,\ \mathrm{P}\}$ are independent Brownian motions, the corresponding diffusion process, $\{X(t),\ t \geq 0,\ \mathcal{P}^{x_0}\}$, was introduced by Brox [1] and Schumacher [5] as a diffusion analogue of Sinai's random walk [6]. In [1] and [5] it was proved that $\{(\log t)^{-2}X(t),\ t \geq 0,\ \mathcal{P}^0\}$ has a nondegenerate limit distribution.

We begin by presenting the result of [4]. Let $\mathcal{M}$ be the space of probability laws on Ω and let ρ be the Prokhorov metric on $\mathcal{M}$. Let $\{X(t),\ t \geq 0,\ \mathcal{P}^0\}$ be a diffusion process with a one-sided Brownian potential. Set

$$X_\lambda(t) = \lambda^{-1/2}X(\lambda t), \qquad t \geq 0,$$

for a constant $\lambda > 0$, and denote by $\mathrm{P}_\lambda(w) \in \mathcal{M}$ the probability law of the process $\{X_\lambda(t),\ t \geq 0,\ \mathrm{P}_w^0\}$. Also, denote by $\mathrm{P}_\mathrm{N} \in \mathcal{M}$ the probability law of the identically vanishing process, and by $\mathrm{P}_\mathrm{R} \in \mathcal{M}$ the probability law of the reflecting Brownian motion on $[0, \infty)$ starting from 0.

Theorem 1.1. ([4].) *For any ε such that $0 < \varepsilon < \rho(\mathrm{P}_\mathrm{N}, \mathrm{P}_\mathrm{R})/2$,*

$$\lim_{\lambda \to \infty} \mathrm{P}\{\rho(\mathrm{P}_\lambda(w), \mathrm{P}_\mathrm{N}) < \varepsilon\} = \tfrac{1}{2},$$
$$\lim_{\lambda \to \infty} \mathrm{P}\{\rho(\mathrm{P}_\lambda(w), \mathrm{P}_\mathrm{R}) < \varepsilon\} = \tfrac{1}{2}.$$

Our present results (namely Theorems 1.2 and 1.3, stated below) imply Theorem 1.1. To state the theorems, we introduce some notation.

For $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, the diffusion process $\{X(t),\ t \geq 0,\ \mathrm{P}_w^{x_0}\}$ can be constructed from a Brownian motion via time change and scale change [3, p. 165]. See [4] for the explicit representation of this diffusion process. The scale function of the process is given by

$$S(x) = \int_0^x \mathrm{e}^{w(y)}\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

If $S(x) \to -\infty$ (as $x \to -\infty$), then the diffusion process $\{X(t),\ t \geq 0,\ \mathrm{P}_w^{x_0}\}$ is recurrent and, hence, conservative. By restricting the whole space $\mathbb{W}$ to the set of ws satisfying $S(x) \to -\infty$ (as $x \to -\infty$), which still has P-measure 1, we may assume that the diffusion process $\{X(t),\ t \geq 0,\ \mathrm{P}_w^{x_0}\}$ is recurrent for any w.

For $\omega \in \Omega$, we write

$$a(t) \equiv a(t, \omega) = \int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s, \qquad t \geq 0, \tag{1.1}$$

where $\mathbf{1}_A$ denotes the indicator function of the (generic) set A. Then, for any $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, we have

$$\mathrm{P}_w^{x_0}\Big\{\lim_{t \to \infty} a(t) = \infty\Big\} = 1,$$

since the diffusion process $\{X(t),\ t \geq 0,\ \mathrm{P}_w^{x_0}\}$ is recurrent. In what follows, we reduce Ω so that it equals the set of ωs satisfying $a(t) \to \infty$ (as $t \to \infty$). For $\lambda > 0$ and $\omega \in \Omega$, we let

$$a_\lambda(t) \equiv a_\lambda(t, \omega) = \int_0^t \mathbf{1}_{(0,\infty)}(X_\lambda(s))\, \mathrm{d}s, \qquad t \geq 0.$$

Since $a_\lambda(t) = \lambda^{-1} a(\lambda t) \to \infty$ (as $t \to \infty$), we can define

$$a_\lambda^{-1}(t) = \inf\{s > 0 \colon a_\lambda(s) > t\}, \qquad t \geq 0,$$

the right-continuous inverse function of $a_\lambda(t)$. We also let

$$G_\lambda(t) = X_\lambda(a_\lambda^{-1}(t)), \qquad t \geq 0.$$

Then $\{G_\lambda(t),\ t \geq 0, \mathrm{P}_w^0\}$ is a reflecting Brownian motion on $[0, \infty)$ starting from 0.

For $w \in \mathbb{W}$ and $a \in \mathbb{R}$, we let

$$\sigma(a) \equiv \sigma(a, w) = \sup\{x < 0 \colon w(x) = a\},$$

and introduce two subsets A and B of $\mathbb{W}$ as follows:

$$A = \{w \in \mathbb{W} \colon \sigma(\tfrac{1}{2}) > \sigma(-\tfrac{1}{2})\},$$
$$B = \{w \in \mathbb{W} \colon \sigma(\tfrac{1}{2}) < \sigma(-\tfrac{1}{2})\}.$$

Each of these subsets has P-measure $\frac{1}{2}$. For $w \in \mathbb{W}$ and $\lambda > 0$, we define $w_\lambda \in \mathbb{W}$ by

$$w_\lambda(x) = \lambda^{-1} w(\lambda^2 x), \qquad x \in \mathbb{R}.$$

Then

$$\{w_\lambda, \mathrm{P}\} \overset{\mathrm{D}}{=} \{w, P\}, \tag{1.2}$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. For each $\lambda > 0$, we also introduce two further subsets A_λ and B_λ of $\mathbb{W}$ as follows:

$$A_\lambda = \{w \in \mathbb{W} \colon w_\lambda \in A\},$$
$$B_\lambda = \{w \in \mathbb{W} \colon w_\lambda \in B\}.$$

Each of these also has P-measure $\frac{1}{2}$, by (1.2).

In the following theorems, $\mathrm{P}\{\cdots \mid \cdot\}$ denotes the conditional probability. We write $\tilde{A}_\lambda = A_{\log \lambda}$ and $\tilde{B}_\lambda = B_{\log \lambda}$.

Theorem 1.2. *For any $T > 0$ and $\varepsilon > 0$,*

$$\lim_{\lambda \to \infty} \mathrm{P}\Big\{\mathrm{P}_w^0\Big\{\sup_{0 \leq t \leq T} |X_\lambda(t) - G_\lambda(t)| < \varepsilon\Big\} > 1 - \varepsilon \;\Big|\; \tilde{A}_\lambda\Big\} = 1.$$

For $w \in \mathbb{W}$, we let

$$\zeta \equiv \zeta(w) = \sup\Big\{x < 0 \colon w(x) - \min_{x \leq y \leq 0} w(y) = 1\Big\},$$

$$M \equiv M(w) = \begin{cases} \sigma(\frac{1}{2}) & \text{if } w \in A, \\ \zeta(w) & \text{if } w \in B, \end{cases}$$

$$V \equiv V(w) = \min_{x \geq M} w(x).$$

We also define $b \equiv b(w)$ in $(M, 0)$ by $w(b) = V$. Note that b is determined uniquely by w (P-almost surely).

Theorem 1.3. *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathrm{P}\{\mathrm{P}_w^0\{|(\log t)^{-2}X(t) - b(w_{\log t})| < \varepsilon\} > 1-\varepsilon \mid \tilde{B}_t\} = 1.$$

To state the result on the maximum process of $X(t)$, we let

$$H(w) = \max_{M\le x\le 0} w(x).$$

Note that $H(w) = \frac{1}{2}$ if $w \in A$ and $0 < H(w) < \frac{1}{2}$ if $w \in B$.

Theorem 1.4. *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathcal{P}^0\left\{\left|\frac{\log \max_{0\le s\le t} X(s)}{\log t} - H(w_{\log t})\right| > \varepsilon\right\} = 0.$$

Our present results, together with those of [4], immediately imply the following theorem.

Theorem 1.5. *In each of the following instances, the distribution of* $\tilde{X}_t$ *under* $\mathcal{P}^0$ *tends to a limit distribution as* $t \to \infty$*, as described.*

- $\tilde{X}_t = t^{-1/2}X(t)$*: limit distribution is* $\mu_{\mathrm{I}}(\mathrm{d}x) = \frac{1}{2}\sqrt{2/\pi}\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x + \frac{1}{2}\delta_0(\mathrm{d}x)$*; support is* $[0,\infty)$.
- $\tilde{X}_t = (\log t)^{-2}X(t)$*: limit distribution is* $\mu_{\mathrm{II}}(\mathrm{d}x) = \mathrm{P}\{(b \in \mathrm{d}x) \cap B\} + \frac{1}{2}\delta_\infty(\mathrm{d}x)$*; support is* $(-\infty, 0) \cup \{\infty\}$.
- $\tilde{X}_t = t^{-1/2}\max_{0\le s\le t} X(s)$*: limit distribution is*

$$\mu_{\mathrm{III}}(\mathrm{d}x) = \tfrac{1}{2}\mathrm{P}_{\mathrm{R}}\left\{\max_{0\le s\le 1} X(s) \in \mathrm{d}x\right\} + \tfrac{1}{2}\delta_0(\mathrm{d}x);$$

  support is $[0,\infty)$.
- $\tilde{X}_t = \log(\max_{0\le s\le t} X(s))/\log t$*: limit distribution is* $\mu_{\mathrm{IV}}(\mathrm{d}x) = \mathrm{P}\{H \in \mathrm{d}x\}$*; support is* $(0, \frac{1}{2}]$.
- $\tilde{X}_t = (\log t)^{-2}\min_{0\le s\le t} X(s)$*: limit distribution is* $\mu_{\mathrm{V}}(\mathrm{d}x) = \mathrm{P}\{M \in \mathrm{d}x\}$*; support is* $(-\infty, 0)$.

Moreover, the Laplace transforms of the distributions of b, H*, and* M *appearing in the definitions of* μ_{II}, μ_{IV}*, and* μ_{V} *are as follows. For* $\xi > 0$,

$$\mathrm{E}[\mathrm{e}^{\xi b}, B] = \frac{\sinh(\sqrt{2\xi}/2)}{\sqrt{2\xi}\cosh\sqrt{2\xi}},$$

$$\mathrm{E}[\mathrm{e}^{\xi H}, A] = \tfrac{1}{2}\mathrm{e}^{\xi/2},$$

$$\mathrm{E}[\mathrm{e}^{\xi H}, B] = \int_0^{1/2} \mathrm{e}^{\xi x}\,\mathrm{d}x,$$

$$\mathrm{E}[\mathrm{e}^{\xi M}, A] = \frac{\sinh(\sqrt{2\xi}/2)}{\sinh\sqrt{2\xi}},$$

$$\mathrm{E}[\mathrm{e}^{\xi M}, B] = \frac{\sinh(\sqrt{2\xi}/2)}{(\sinh\sqrt{2\xi})(\cosh\sqrt{2\xi})}.$$

Here $\mathrm{E}[\cdot, A]$ *denotes the expectation with respect to* P *on the set* A.

Let $\eta(t, x)$ be the local time at x of the Brox–Schumacher diffusion process. Hu and Shi [2] showed that, for any $x \in \mathbb{R}$,

$$\frac{\log \eta(t, x)}{\log t} \xrightarrow{\mathrm{D}} U \wedge \hat{U}, \qquad t \to \infty,$$

where '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution and U and $\hat{U}$ are independent random variables uniformly distributed in (0, 1).

For our diffusion process $\{X(t),\ t \geq 0,\ \mathcal{P}^0\}$ with a one-sided Brownian potential, Z. Shi (private communication (2000)) informed us that the same method based on the second Ray–Knight theorem as in [2] can be used to show that

$$\frac{\log a(t)}{\log t} \xrightarrow{\mathrm{D}} 1 \wedge (2U), \qquad t \to \infty,$$

where U is a random variable uniformly distributed in (0, 1).

In Section 3 we investigate the asymptotic behavior of the occupation time $a(t)$ as $t \to \infty$.

2. Preliminaries

For $\lambda > 0$, $w \in \mathbb{W}$, and $x_0 \in \mathbb{R}$, let $\mathrm{P}^{x_0}_{\lambda w}$ be the probability measure on Ω such that $\{X(t),\ t \geq 0,\ \mathrm{P}^{x_0}_{\lambda w}\}$ is a diffusion process with generator

$$\mathcal{L}_{\lambda w} = \frac{1}{2} \mathrm{e}^{\lambda w(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{-\lambda w(x)} \frac{\mathrm{d}}{\mathrm{d}x} \right),$$

starting from x_0. Denote by $\mathrm{E}^{x_0}_{\lambda w}$ the expectation with respect to $\mathrm{P}^{x_0}_{\lambda w}$. The following lemma was proved in [1].

Lemma 2.1. ([1].) *For any $\lambda > 0$ and $w \in \mathbb{W}$,*

$$\{X(t),\ t \geq 0, \mathrm{P}^0_{\lambda w_\lambda}\} \stackrel{\mathrm{D}}{=} \{\lambda^{-2} X(\lambda^4 t),\ t \geq 0, \mathrm{P}^0_w\}.$$

In preparation for the proofs of Theorem 1.2 and Theorem 1.3, we present the following theorems.

Theorem 2.1. *Let $\mu \equiv \mu(\lambda) = \lambda^{1/4} \log \lambda$. Then, for any $T > 0$ and $\varepsilon > 0$,*

$$\lim_{\lambda \to \infty} \mathrm{P}\Big\{ \mathrm{P}^0_{\mu w_\mu} \Big\{ \sup_{0 \leq t \leq T} |X(t) - G(t)| < \varepsilon \Big\} > 1 - \varepsilon \ \Big|\ A_{\log \lambda} \Big\} = 1,$$

where $G(t) = X(a^{-1}(t))$ and $a^{-1}(t) = \inf\{s > 0 \colon a(s) > t\}$, the right-continuous inverse function of $a(t)$ defined in (1.1).

Theorem 2.2. *Let r be a real-valued function of $\lambda > 0$ such that $r(\lambda) \to 1$ (as $\lambda \to \infty$). Then there exists a subset $B^\#$ of B, with $\mathrm{P}\{B \setminus B^\#\} = 0$, such that, for any $w \in B^\#$ and $\varepsilon > 0$,*

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda w}\{|X(\mathrm{e}^{\lambda r(\lambda)}) - b(w)| < \varepsilon\} = 1.$$

We remark that to prove Theorem 2.2 it is enough to show the following proposition.

Proposition 2.1. *There exists a subset $B^{\#}$ of B, with $\mathrm{P}\{B \setminus B^{\#}\} = 0$, such that for any $w \in B^{\#}$ the following holds: there exists a $\delta > 0$ such that, for any r_1 and r_2 satisfying $1 - \delta < r_1 < r_2 < 1 + \delta$ and any $\varepsilon > 0$,*

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \mathrm{P}^0_{\lambda w}\{|X(\mathrm{e}^{\lambda r}) - b(w)| < \varepsilon\} = 1. \tag{2.1}$$

In Section 4 we prove Theorem 2.1 and Theorem 1.2, in Section 5 we prove Proposition 2.1 and Theorem 1.3, and in Section 6 we prove Theorem 1.4.

3. Asymptotic behavior of the occupation time $a(t)$ as $t \to \infty$

In this section we investigate the limiting behavior of $\{t^{-1}a(t),\ t \ge 0,\ \mathcal{P}^0\}$ as $t \to \infty$. To do so, we need two lemmas. The first, Lemma 3.1, which is needed for the proof of Theorem 2.1, will be proved in Section 4.

Lemma 3.1. *There exists a subset $A^{\#}$ of A, with $\mathrm{P}\{A \setminus A^{\#}\} = 0$, such that, for any $w \in A^{\#}$ and $T > 0$,*

$$\lim_{\lambda\to\infty} \mathrm{E}^0_{\lambda w}\bigg[\frac{1}{T\mathrm{e}^{\lambda}} \int_0^{T\mathrm{e}^{\lambda}} \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg] = 1. \tag{3.1}$$

The second lemma, Lemma 3.2, can be obtained from Proposition 2.1.

Lemma 3.2. *There exists a subset $B^{\#}$ of B, with $\mathrm{P}\{B \setminus B^{\#}\} = 0$, such that, for any $w \in B^{\#}$,*

$$\lim_{\lambda\to\infty} \mathrm{E}^0_{\lambda w}\bigg[\frac{1}{\mathrm{e}^{\lambda}} \int_0^{\mathrm{e}^{\lambda}} \mathbf{1}_{(-\infty,0)}(X(s))\,\mathrm{d}s\bigg] = 1.$$

The main result in this section is as follows.

Theorem 3.1. *For any $\varepsilon > 0$,*

$$\lim_{t\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_w\bigg\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s > 1 - \varepsilon\bigg\} > 1 - \varepsilon \;\bigg|\; \tilde{A}_t\bigg\} = 1, \tag{3.2}$$

$$\lim_{t\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_w\bigg\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s < \varepsilon\bigg\} > 1 - \varepsilon \;\bigg|\; \tilde{B}_t\bigg\} = 1. \tag{3.3}$$

Proof. We prove (3.2); we can prove (3.3) in the same way by using Lemma 3.2. By Lemma 3.1 we have, for $w \in A^{\#}$,

$$\lim_{t\to\infty} \mathrm{E}^0_{(\log t)w}\bigg[\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg] = 1.$$

Here $A^{\#}$ is a subset of A with $\mathrm{P}\{A \setminus A^{\#}\} = 0$. Therefore, it follows that

$$\lim_{t\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_{(\log t)w}\bigg\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s > 1 - \varepsilon\bigg\} > 1 - \varepsilon \;\bigg|\; A\bigg\} = 1$$

for any $\varepsilon > 0$. Moreover, (1.2) and Lemma 2.1 imply that

$$\lim_{t\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_w\bigg\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X((\log t)^4 s))\,\mathrm{d}s > 1 - \varepsilon\bigg\} > 1 - \varepsilon \;\bigg|\; A_{\log t}\bigg\} = 1. \tag{3.4}$$

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Changing the variable of the integral in (3.4) yields

$$\lim_{u\to\infty} \mathrm{P}\left\{\mathrm{P}^0_w\left\{\frac{1}{u}\int_0^u \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s > 1-\varepsilon\right\} > 1-\varepsilon \;\middle|\; A_{\log t}\right\} = 1, \tag{3.5}$$

where $t \equiv t(u)$ is determined by $u = t(\log t)^4$.

Let us prove that

$$\lim_{u\to\infty} \mathrm{P}\{A_{\log t(u)} \ominus A_{\log u}\} = 0, \tag{3.6}$$

where $A_1 \ominus A_2 = (A_1 \cap A_2^{\mathrm{c}}) \cup (A_1^{\mathrm{c}} \cap A_2)$ for sets A_1 and A_2. We note that

$$r = \frac{\log t(u)}{\log u} \to 1 \quad \text{as } u \to \infty. \tag{3.7}$$

Using the fact that $w_{\lambda_1\lambda_2} = (w_{\lambda_1})_{\lambda_2}$ for $\lambda_1, \lambda_2 > 0$, we have

$$\begin{aligned}\mathrm{P}\{A_{\log t(u)} \ominus A_{\log u}\} &= \mathrm{E}[|\,\mathbf{1}_A(w_{\log t(u)}) - \mathbf{1}_A(w_{\log u})|] \\ &= \mathrm{E}[|\,\mathbf{1}_A((w_{\log u})_r) - \mathbf{1}_A(w_{\log u})|],\end{aligned} \tag{3.8}$$

where E denotes the expectation with respect to P. By (1.2), the right-hand side of (3.8) is equal to

$$\mathrm{E}[|\,\mathbf{1}_A(w_r) - \mathbf{1}_A(w)|],$$

which converges to 0 as $u \to \infty$, due to (3.7). This proves (3.6). From (3.6) it follows that $\mathrm{P}\{\cdots \mid A_{\log t(u)}\} \to 1$ (as $u \to \infty$) is equivalent to $\mathrm{P}\{\cdots \mid A_{\log u}\} \to 1$ (as $u \to \infty$). Hence, by (3.5), we obtain (3.2).

Corollary 3.1. *The probability distribution of* $t^{-1}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s$ *under* $\mathcal{P}^0$ *converges to* $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ *as* $t \to \infty$.

4. Proofs of Theorem 2.1 and Theorem 1.2

In this section we prove Theorem 2.1 and Theorem 1.2. First we introduce a lemma from [4]. For $\omega \in \Omega$, let

$$\tau(a) \equiv \tau(a,\omega) = \inf\{t > 0\colon X(t) = a\}, \qquad a \in \mathbb{R}.$$

Lemma 4.1. ([4].) *Let* $w \in \mathbb{W}$ *and* $a < 0$. *Assume that* $w(a) > w(x)$ *for all* $x > a$. *Then, for any* $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(J-\varepsilon)} < \tau(a) < \mathrm{e}^{\lambda(J+\varepsilon)}\} = 1,$$

where $J = \max\{J_0, 2w(a)\}$ *and* $J_0 = w(a) - \min\{w(x)\colon x \geq a\}$.

Next we prove Lemma 3.1.

Proof of Lemma 3.1. Let $w \in A$. In this case we note that $V > -\frac{1}{2}$. First we choose r_0 and r_1 satisfying $-V < r_0 < r_1 < \frac{1}{2}$. We then see that

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{r_0\lambda}) < \tau(\sigma(r_1))\} = \lim_{\lambda\to\infty} \frac{\int_{\sigma(r_1)}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x}{\int_{\sigma(r_1)}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{r_0\lambda}} = 1, \tag{4.1}$$

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since $\lim_{\lambda\to\infty} \lambda^{-1} \log \int_{\sigma(r_1)}^{0} \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x = r_1 > r_0$. Moreover, an application of Lemma 4.1 with $a = \sigma(r_1)$ gives

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(2r_1-\varepsilon)} < \tau(\sigma(r_1)) < \mathrm{e}^{\lambda(2r_1+\varepsilon)}\} = 1 \tag{4.2}$$

for any $\varepsilon > 0$. Combining (4.1) and (4.2) yields

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{r_0\lambda}) < \mathrm{e}^{\lambda\theta}\} = 1 \quad \text{for some } \theta \in (0, 1). \tag{4.3}$$

Next we choose a $\rho > \frac{1}{2}$ satisfying

$$\min\{w(x) \colon \sigma(\rho) \le x \le \sigma(\tfrac{1}{2})\} > V. \tag{4.4}$$

(Note that the set of $w \in A$ for which there is no $\rho > \frac{1}{2}$ satisfying (4.4) is P-negligible.) By applying Lemma 4.1 with $a = \sigma(\rho)$, we have

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(2\rho-\varepsilon)} < \tau(\sigma(\rho)) < \mathrm{e}^{\lambda(2\rho+\varepsilon)}\} = 1$$

for any $\varepsilon > 0$. Therefore, for any $T > 0$,

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\sigma(\rho)) > T\mathrm{e}^{\lambda}\} = 1. \tag{4.5}$$

Now we define $m_{\lambda w}$, a probability measure on $I_\lambda = [\sigma(\rho), \mathrm{e}^{r_0\lambda}]$, by

$$m_{\lambda w}(E) = \frac{\int_{E\cap[\sigma(\rho),0]} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \int_{E\cap(0,\mathrm{e}^{r_0\lambda}]} \mathrm{d}x}{\int_{\sigma(\rho)}^{0} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{r_0\lambda}}$$

for any Borel set E in I_λ. This is the invariant probability measure for the reflecting $\mathcal{L}_{\lambda w}$-diffusion process on I_λ. Note that

$$\lim_{\lambda\to\infty} m_{\lambda w}((0, \mathrm{e}^{r_0\lambda}]) = \lim_{\lambda\to\infty} \frac{\mathrm{e}^{r_0\lambda}}{\int_{\sigma(\rho)}^{0} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{r_0\lambda}} = 1 \tag{4.6}$$

since $\lim_{\lambda\to\infty} \lambda^{-1} \log \int_{\sigma(\rho)}^{0} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x = -V < r_0$.

By the comparison theorem for one-dimensional diffusion processes, for $\lambda > 0$ we can construct diffusion processes $\{Y_\lambda(t),\ t \ge 0\}$ and $\{Z_\lambda(t),\ t \ge 0\}$, on a probability space $(\tilde{\Omega}, \tilde{\mathrm{P}})$, with the following properties:

$\{Y_\lambda(t),\ t \ge 0\}$ is a reflecting $\mathcal{L}_{\lambda w}$-diffusion process on $[\sigma(\rho), \infty)$ starting from $\mathrm{e}^{r_0\lambda}$. (4.7)

$\{Z_\lambda(t),\ t \ge 0\}$ is a reflecting $\mathcal{L}_{\lambda w}$-diffusion process on I_λ with initial distribution $m_{\lambda w}$.

$$\tilde{\mathrm{P}}\{Y_\lambda(t) \ge Z_\lambda(t) \text{ for all } t \ge 0\} = 1. \tag{4.8}$$

Since $\{Z_\lambda(t),\ t \ge 0\}$ is a stationary process with invariant probability measure $m_{\lambda w}$, by (4.6) we have

$$\lim_{\lambda\to\infty} \tilde{\mathrm{E}}\biggl[\frac{1}{T\mathrm{e}^{\lambda}} \int_0^{T\mathrm{e}^{\lambda}} \mathbf{1}_{(0,\infty)}(Z_\lambda(s))\,\mathrm{d}s\biggr] = 1 \tag{4.9}$$

for any $T > 0$. Here $\tilde{\mathrm{E}}$ denotes the expectation with respect to $\tilde{\mathrm{P}}$. Moreover, (4.8) and (4.9) imply that

$$\lim_{\lambda\to\infty} \tilde{\mathrm{E}}\bigg[\frac{1}{T\mathrm{e}^{\lambda}}\int_0^{T\mathrm{e}^{\lambda}} \mathbf{1}_{(0,\infty)}(Y_\lambda(s))\,\mathrm{d}s\bigg] = 1. \tag{4.10}$$

Let us prove (3.1). Using the strong Markov property of $\{X(t),\ t \ge 0,\ P^{\cdot}_{\lambda w}\}$, for any $T > 0$ we obtain

$$\begin{aligned}
&\mathrm{E}^0_{\lambda w}\bigg[\frac{1}{T\mathrm{e}^{\lambda}}\int_0^{T\mathrm{e}^{\lambda}} \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg] \\
&\quad \ge \mathrm{E}^0_{\lambda w}\bigg[\frac{1}{T\mathrm{e}^{\lambda}}\int_{\tau(\mathrm{e}^{r_0\lambda})}^{T\mathrm{e}^{\lambda}} \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s,\ \tau(\mathrm{e}^{r_0\lambda}) < \mathrm{e}^{\lambda\theta}\bigg] \\
&\quad \ge \mathrm{E}^{\mathrm{e}^{r_0\lambda}}_{\lambda w}\bigg[\frac{1}{T\mathrm{e}^{\lambda}}\int_0^{T\mathrm{e}^{\lambda}-\mathrm{e}^{\lambda\theta}} \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg]\mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{r_0\lambda}) < \mathrm{e}^{\lambda\theta}\}.
\end{aligned} \tag{4.11}$$

Owing to (4.3), (4.5), (4.7), and (4.10), we see that the right-hand side of (4.11) converges to 1 as $\lambda \to \infty$. Hence, we obtain (3.1).

We now present three lemmas in preparation for the proof of Theorem 2.1.

Lemma 4.2. *Let* $\mu \equiv \mu(\lambda) = \lambda^{1/4}\log\lambda$. *Then, for any* $T > 0$ *and* $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \mathrm{P}\Big\{\mathrm{P}^0_{\mu w_\mu}\Big\{\sup_{0\le t\le T} |t - a(t)| < \varepsilon\Big\} > 1 - \varepsilon \;\Big|\; A_{\log\lambda}\Big\} = 1.$$

Proof. Using Chebyshev's inequality, we have

$$\begin{aligned}
\mathrm{P}\Big\{\mathrm{P}^0_{\mu w_\mu}\Big\{\sup_{0\le t\le T} |t - a(t)| \ge \varepsilon\Big\} \ge \varepsilon,\ A_{\log\lambda}\Big\} &= \mathrm{P}\{\mathrm{P}^0_{\mu w_\mu}\{T - a(T) \ge \varepsilon\} \ge \varepsilon,\ A_{\log\lambda}\} \\
&\le \mathrm{P}\Big\{\frac{1}{\varepsilon}\,\mathrm{E}^0_{\mu w_\mu}[T - a(T)] \ge \varepsilon,\ A_{\log\lambda}\Big\}.
\end{aligned} \tag{4.12}$$

By Lemma 2.1, we have

$$\begin{aligned}
\mathrm{E}^0_{\mu w_\mu}[a(T)] &= \mathrm{E}^0_{\mu w_\mu}\bigg[\int_0^T \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg] \\
&= \mathrm{E}^0_{w}\bigg[\int_0^T \mathbf{1}_{(0,\infty)}(X_{\lambda(\log\lambda)^4}(s))\,\mathrm{d}s\bigg] \\
&= \mathrm{E}^0_{w}\bigg[\frac{1}{\lambda}\int_0^{T\lambda} \mathbf{1}_{(0,\infty)}(X_{(\log\lambda)^4}(s))\,\mathrm{d}s\bigg] \\
&= \mathrm{E}^0_{(\log\lambda)w_{\log\lambda}}\bigg[\frac{1}{\lambda}\int_0^{T\lambda} \mathbf{1}_{(0,\infty)}(X(s))\,\mathrm{d}s\bigg] \\
&= \mathrm{E}^0_{(\log\lambda)w_{\log\lambda}}\bigg[\frac{1}{\lambda}a(T\lambda)\bigg].
\end{aligned}$$

Therefore, the right-hand side of (4.12) is equal to

$$\mathrm{P}\bigg\{\frac{1}{\varepsilon}\,\mathrm{E}^0_{(\log\lambda)w_{\log\lambda}}\bigg[T - \frac{1}{\lambda}a(T\lambda)\bigg] \ge \varepsilon,\ A_{\log\lambda}\bigg\} = \mathrm{P}\bigg\{\frac{1}{\varepsilon}\,\mathrm{E}^0_{(\log\lambda)w}\bigg[T - \frac{1}{\lambda}a(T\lambda)\bigg] \ge \varepsilon,\ A\bigg\}.$$

This probability converges to 0 as $\lambda \to \infty$, since

$$\lim_{\lambda\to\infty} \mathrm{E}^0_{(\log\lambda)w}\left[\frac{1}{T\lambda}a(T\lambda)\right] = 1 \quad \text{for } w \in A^\#$$

by Lemma 3.1. Here $A^\#$ is a subset of A with $\mathrm{P}\{A \setminus A^\#\} = 0$. Hence, we obtain Lemma 4.2.

Lemma 4.3. *Let* $\mu \equiv \mu(\lambda) = \lambda^{1/4}\log\lambda$. *Then, for any* $T > 0$ *and* $\varepsilon > 0$,

$$\lim_{\lambda\to\infty} \mathrm{P}\left\{\mathrm{P}^0_{\mu w_\mu}\left\{\sup_{0\le t\le T}|G(t) - G(a(t))| < \varepsilon\right\} > 1-\varepsilon \;\middle|\; A_{\log\lambda}\right\} = 1.$$

Proof. We note that, for any $w \in \mathbb{W}$ and $\lambda > 0$, $\{G(t),\ t \ge 0, \mathrm{P}^0_{\mu w_\mu}\}$ is a reflecting Brownian motion on $[0, \infty)$ starting from 0. Since, with probability 1, Brownian sample paths are locally Hölder continuous with exponent γ, for every $\gamma \in (0, \frac{1}{2})$, we obtain Lemma 4.3 from Lemma 4.2.

The following lemma plays an important role in the proof of Theorem 2.1.

Lemma 4.4. *Let* f *be a real-valued, continuous function of* $t \ge 0$ *with* $f(0) = 0$, *and let*

$$\alpha(t) = \int_0^t \mathbf{1}_{(0,\infty)}(f(s))\,\mathrm{d}s.$$

Take $T > 0$ *and assume that there exists a* $T_1 > T$ *such that* $\alpha(T_1) > T$. *Define the right-continuous inverse function of* $\alpha(t)$ *by*

$$\alpha^{-1}(t) = \inf\{s > 0\colon \alpha(s) > t\}, \qquad 0 \le t \le T,$$

and let

$$g(t) = f(\alpha^{-1}(t)), \qquad 0 \le t \le T.$$

Then

$$|g(t) - f(t)| = |g(t) - g(\alpha(t))| - \min\{f(t), 0\}, \qquad 0 \le t \le T. \tag{4.13}$$

Proof. First assume that $\alpha^{-1}(\alpha(t)) = t$. In this case we notice that $f(t) \ge 0$ and

$$|g(t) - f(t)| = |g(t) - f(\alpha^{-1}(\alpha(t)))| = |g(t) - g(\alpha(t))|,$$

which establishes (4.13).

Next assume that $\alpha^{-1}(\alpha(t)) \ne t$. In this case $\alpha^{-1}(\alpha(t)) > t$. Moreover, $f \le 0$ on the interval $[t, \alpha^{-1}(\alpha(t))]$ and $f(\alpha^{-1}(\alpha(t))) = 0$, i.e. $g(\alpha(t)) = 0$. Noting that $g(t) \ge 0$, we obtain

$$|g(t) - f(t)| = |g(t)| + |f(t)| = |g(t) - g(\alpha(t))| - \min\{f(t), 0\}.$$

This completes the proof of the lemma.

Proof of Theorem 2.1. By virtue of Lemma 4.4, we have

$$\begin{aligned}
&\mathrm{P}\left\{\mathrm{P}^0_{\mu w_\mu}\left\{\sup_{0\le t\le T}|G(t) - X(t)| \ge \varepsilon\right\} \ge \varepsilon,\ A_{\log\lambda}\right\} \\
&\quad\le \mathrm{P}\left\{\mathrm{P}^0_{\mu w_\mu}\left\{\sup_{0\le t\le T}|G(t) - G(a(t))| \ge \frac{\varepsilon}{2}\right\} \ge \frac{\varepsilon}{2},\ A_{\log\lambda}\right\} \\
&\qquad + \mathrm{P}\left\{\mathrm{P}^0_{\mu w_\mu}\left\{\inf_{0\le t\le T} X(t) \le -\frac{\varepsilon}{2}\right\} \ge \frac{\varepsilon}{2},\ A_{\log\lambda}\right\}.
\end{aligned} \tag{4.14}$$

The first term on the right-hand side of (4.14) converges to 0 as $\lambda \to \infty$, due to Lemma 4.3. The second term is bounded above by

$$\mathrm{P}\bigg\{\mathrm{P}^0_{\mu w_\mu}\bigg\{\inf_{0\le t\le T} X(t) \le -\frac{\varepsilon}{2}\bigg\} \ge \frac{\varepsilon}{2}\bigg\} = \mathrm{P}\bigg\{\mathrm{P}^0_{\mu w}\bigg\{\inf_{0\le t\le T} X(t) \le -\frac{\varepsilon}{2}\bigg\} \ge \frac{\varepsilon}{2}\bigg\},$$

which converges to 0 as $\lambda \to \infty$, by [4, Lemma 4.2]. This completes the proof of Theorem 2.1.

Proof of Theorem 1.2. By combining Theorem 2.1 and Lemma 2.1, we have

$$\lim_{\lambda\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_w\bigg\{\sup_{0\le t\le T} |X_{\mu^4}(t) - G_{\mu^4}(t)| < \varepsilon\bigg\} > 1-\varepsilon \;\bigg|\; A_{\log\lambda}\bigg\} = 1$$

or, equivalently,

$$\lim_{\nu\to\infty} \mathrm{P}\bigg\{\mathrm{P}^0_w\bigg\{\sup_{0\le t\le T} |X_{\nu}(t) - G_{\nu}(t)| < \varepsilon\bigg\} > 1-\varepsilon \;\bigg|\; A_{\log\lambda}\bigg\} = 1,$$

where $\lambda \equiv \lambda(\nu)$ is determined by $\nu = \lambda(\log\lambda)^4$. We obtain Theorem 1.2 by the same argument as in the proof of Theorem 3.1.

5. Proofs of Proposition 2.1 and Theorem 1.3

In this section we prove Proposition 2.1 and Theorem 1.3. We begin by introducing a lemma due to Brox [1]. Let $w \in \mathbb{W}$ and $\alpha < m < \beta < 0$. We call a triple of negative numbers $\Delta = (\alpha, m, \beta)$ a valley of w if the following conditions are satisfied.

(i) $w(\alpha) > w(x) > w(m)$ for all $x \in (\alpha, m)$ and $w(\beta) > w(x) > w(m)$ for all $x \in (m, \beta)$.

(ii) $w(\alpha) - w(m) > H_{\beta,m} := \sup\{w(y) - w(x) : m < y < x < \beta\}$ and $w(\beta) - w(m) > H_{\alpha,m} := \sup\{w(y) - w(x) : \alpha < x < y < m\}$.

For a valley $\Delta = (\alpha, m, \beta)$, we call $D(\Delta) = \{w(\alpha) - w(m)\} \wedge \{w(\beta) - w(m)\}$ the depth of Δ and $A(\Delta) = H_{\beta,m} \vee H_{\alpha,m}$ the inner directed ascent of Δ. A valley $\Delta = (\alpha, m, \beta)$ is said to contain x_0 if $\alpha < x_0 < \beta$.

Lemma 5.1. ([1].) *Let $w \in \mathbb{W}$ and let $\Delta = (\alpha, m, \beta)$ be a valley of w containing x_0. Then, for any r_1 and r_2 satisfying $A(\Delta) < r_1 < r_2 < D(\Delta)$ and any $\varepsilon > 0$,*

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \mathrm{P}^{x_0}_{\lambda w}\{|X(\mathrm{e}^{\lambda r}) - m| < \varepsilon\} = 1.$$

Let us now prove Proposition 2.1.

Proof of Proposition 2.1. Let $w \in B$. In this case $M = \zeta$ and $V < -\frac{1}{2}$. Let

$$V' \equiv V'(w) = \max_{x\ge b} w(x)$$

and define $b' \equiv b'(w)$ in $(b, 0)$ by $w(b') = V'$. Note that b' is determined uniquely by w (P-almost surely).

First we consider the case $V' - V > 1$. Let

$$c \equiv c(w) = \sup\{x < b' : w(x) = 0\}$$

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and define $\tilde{w} \in \mathbb{W}$ by

$$\tilde{w}(x) = \begin{cases} w(x) & \text{for } x \geq c, \\ -x + c & \text{for } x < c. \end{cases}$$

We can choose a $c' < c$ satisfying

$$\tilde{w}(b') < \tilde{w}(c') < \tfrac{1}{2},$$
$$\tilde{J} := \Big\{\tilde{w}(c') - \min_{c' \leq x \leq 0} \tilde{w}(x)\Big\} \vee 2\tilde{w}(c') < 1.$$

An application of Lemma 4.1 with $a = c'$ yields

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda \tilde{w}}\{\tau(c') < \mathrm{e}^{\lambda(\tilde{J}+\varepsilon)}\} = 1 \tag{5.1}$$

for any $\varepsilon > 0$. Since

$$\mathrm{P}^0_{\lambda \tilde{w}}\{\tau(c') < \mathrm{e}^{\lambda(\tilde{J}+\varepsilon)}\} \leq \mathrm{P}^0_{\lambda \tilde{w}}\{\tau(c) < \mathrm{e}^{\lambda(\tilde{J}+\varepsilon)}\} = \mathrm{P}^0_{\lambda w}\{\tau(c) < \mathrm{e}^{\lambda(\tilde{J}+\varepsilon)}\},$$

(5.1) implies that

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda w}\{\tau(c) < \mathrm{e}^{\lambda \theta_0}\} = 1 \quad \text{for some } \theta_0 \in (0, 1). \tag{5.2}$$

On the other hand, we see that $\Delta = (\zeta, b, b')$ is a valley of w of depth 1 containing c. Thus, there exists a negative number α such that $\alpha < \zeta$ and $\Delta' = (\alpha, b, b')$ is a valley of w containing c with $A(\Delta') < 1 < D(\Delta')$. (The set of $w \in B$ for which there is no α satisfying this condition is P-negligible [1].) Therefore, by Lemma 5.1, there exists a $\delta_0 > 0$ such that, for any r_1 and r_2 satisfying $1 - \delta_0 < r_1 < r_2 < 1 + \delta_0$ and any $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} \mathrm{P}^c_{\lambda w}\{|X(\mathrm{e}^{\lambda r}) - b| < \varepsilon\} = 1. \tag{5.3}$$

Using the strong Markov property of $\{X(t),\ t \geq 0, \mathrm{P}^{\cdot}_{\lambda w}\}$ and (5.2) and (5.3), we obtain (2.1) for any $\delta \in (0, \delta_0 \wedge (1 - \theta_0))$ in the case $V' - V > 1$.

Next we let $V' - V < 1$. In this case we note that $0 < V' < w(\zeta) < \frac{1}{2}$. Thus, we can choose a ρ' satisfying $V' < \rho' < w(\zeta)$, and note that $\sigma(\rho') < b$. Applying Lemma 4.1 with $a = \sigma(\rho')$ yields

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(\rho' - V - \varepsilon)} < \tau(\sigma(\rho')) < \mathrm{e}^{\lambda(\rho' - V + \varepsilon)}\} = 1$$

for any $\varepsilon > 0$. Since $\rho' - V < 1$, it follows that

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda w}\{\tau(\sigma(\rho')) < \mathrm{e}^{\lambda \theta_1}\} = 1 \quad \text{for some } \theta_1 \in (0, 1). \tag{5.4}$$

Also, we can choose a ρ satisfying

$$\begin{gathered} w(\zeta) < \rho < \tfrac{1}{2}, \\ \min\{w(x) \colon \sigma(\rho) \leq x \leq \zeta\} > V. \end{gathered} \tag{5.5}$$

(Note that the set of $w \in B$ for which there is no ρ satisfying (5.5) is P-negligible.) An application of Lemma 4.1 with $a = \sigma(\rho)$ yields

$$\lim_{\lambda \to \infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(\rho - V - \varepsilon)} < \tau(\sigma(\rho)) < \mathrm{e}^{\lambda(\rho - V + \varepsilon)}\} = 1 \tag{5.6}$$

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for any $\varepsilon > 0$. Let $\tau_\lambda = \tau(\sigma(\rho)) \wedge \tau(e^{\lambda/2})$; we then observe that

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\sigma(\rho)) < \tau(\mathrm{e}^{\lambda/2})\} = \lim_{\lambda\to\infty} \frac{\mathrm{e}^{\lambda/2}}{\int_{\sigma(\rho)}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{\lambda/2}} = 1, \tag{5.7}$$

because $\lim_{\lambda\to\infty} \lambda^{-1} \log \int_{\sigma(\rho)}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x = \rho < \frac{1}{2}$. By (5.6) and (5.7), for any $\varepsilon > 0$ we have

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda(\rho-V-\varepsilon)} < \tau_\lambda < \mathrm{e}^{\lambda(\rho-V+\varepsilon)}\} = 1.$$

Since $\rho - V > 1$, for any small $\delta_1 > 0$ we may consider the process $\{X(t),\ 0 \le t \le \mathrm{e}^{\lambda(1+\delta_1)},\ \mathrm{P}^0_{\lambda w}\}$ to be a reflecting $\mathcal{L}_{\lambda w}$-diffusion process on $I'_\lambda = [\sigma(\rho), \mathrm{e}^{\lambda/2}]$. We define $m'_{\lambda w}$, a probability measure on I'_λ, by

$$m'_{\lambda w}(E) = \frac{\int_{E\cap[\sigma(\rho),0]} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \int_{E\cap(0,\mathrm{e}^{\lambda/2}]} \mathrm{d}x}{\int_{\sigma(\rho)}^0 \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{\lambda/2}}$$

for any Borel set E in I'_λ. This is the invariant probability measure for the reflecting $\mathcal{L}_{\lambda w}$-diffusion process on I'_λ. Notice that, for any $\varepsilon > 0$ satisfying $[b-\varepsilon, b+\varepsilon] \subset [\sigma(\rho), 0]$,

$$\lim_{\lambda\to\infty} m'_{\lambda w}((b-\varepsilon, b+\varepsilon)) = \lim_{\lambda\to\infty} \frac{\int_{b-\varepsilon}^{b+\varepsilon} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x}{\int_{\sigma(\rho)}^0 \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{\lambda/2}} = 1, \tag{5.8}$$

since

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_{b-\varepsilon}^{b+\varepsilon} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x = -V > \frac{1}{2},$$

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \int_{(b-\varepsilon,b+\varepsilon)^{\mathrm{c}}\cap[\sigma(\rho),0]} \mathrm{e}^{-\lambda w(x)}\,\mathrm{d}x < -V.$$

Recall that $\sigma(\rho') < b < 0$. In the following, $\varepsilon > 0$ is chosen to be small enough that $\sigma(\rho') < b - \varepsilon$ and $b + \varepsilon < 0$. Let $\{X^{\mathrm{R}}_\lambda(t),\ t \ge 0\}$ be a reflecting $\mathcal{L}_{\lambda w}$-diffusion process on I'_λ with initial distribution $m'_{\lambda w}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathrm{P}})$. This is a stationary process. From (5.8), it follows that

$$\lim_{\lambda\to\infty} \tilde{\mathrm{P}}\{b - \varepsilon < X^{\mathrm{R}}_\lambda(0) < b + \varepsilon\} = 1 \tag{5.9}$$

and that, for any r_1 and r_2 satisfying $0 < r_1 < r_2$,

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \tilde{\mathrm{P}}\{b - \varepsilon < X^{\mathrm{R}}_\lambda(\mathrm{e}^{\lambda r}) < b + \varepsilon\} = 1. \tag{5.10}$$

By (5.9), (5.10), and the comparison theorem for one-dimensional diffusion processes, we deduce that

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \mathrm{P}^{b-\varepsilon}_{\lambda w}\{X(\mathrm{e}^{\lambda r}) < b + \varepsilon\} = 1, \tag{5.11}$$

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \mathrm{P}^{0}_{\lambda w}\{X(\mathrm{e}^{\lambda r}) > b - \varepsilon\} = 1, \tag{5.12}$$

for any r_1 and r_2 satisfying $0 < r_1 < r_2 < 1 + \delta_1$.

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Now, by (5.4), we notice that

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(b-\varepsilon) < \mathrm{e}^{\lambda\theta_1}\} = 1 \quad \text{for some } \theta_1 \in (0,1). \tag{5.13}$$

Choose any $\delta \in (0, \delta_1 \wedge (1-\theta_1))$. Then, by the strong Markov property of $\{X(t),\ t \geq 0, \mathrm{P}^{\cdot}_{\lambda w}\}$, (5.11), and (5.13), for any r_1 and r_2 satisfying $1-\delta < r_1 < r_2 < 1+\delta$ we obtain

$$\lim_{\lambda\to\infty} \inf_{r\in[r_1,r_2]} \mathrm{P}^0_{\lambda w}\{X(\mathrm{e}^{\lambda r}) < b+\varepsilon\} = 1. \tag{5.14}$$

Combining (5.12) and (5.14) yields (2.1) for $V' - V < 1$. This completes the proof of Proposition 2.1.

Proof of Theorem 1.3. Using Lemma 2.1 and (1.2), we have

$$\begin{aligned} &\mathrm{P}\{\mathrm{P}^0_w\{|\lambda^{-2}X(\mathrm{e}^{\lambda}) - b(w_\lambda)| < \varepsilon\} > 1-\varepsilon,\ B_\lambda\} \\ &\qquad = \mathrm{P}\{\mathrm{P}^0_{\lambda w}\{|X(\mathrm{e}^{\lambda r(\lambda)}) - b(w)| < \varepsilon\} > 1-\varepsilon,\ B\}, \end{aligned} \tag{5.15}$$

where $r(\lambda) = 1 - 4\lambda^{-1}\log\lambda$. The right-hand side of (5.15) converges to $\frac{1}{2}$ as $\lambda \to \infty$, by virtue of Theorem 2.2, which is derived from Proposition 2.1 as we remarked above. We hence obtain Theorem 1.3.

6. Proof of Theorem 1.4

We first present a lemma in preparation for the proof of Theorem 1.4.

Lemma 6.1. *Let r be a real-valued function of $\lambda > 0$ such that $r(\lambda) \to 1$ (as $\lambda \to \infty$). Then, for almost all $w \in \mathbb{W}$ (with respect to* P*) and any $\varepsilon > 0$,*

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\left\{\mathrm{e}^{\lambda(H-\varepsilon)} \leq \max_{0\leq s\leq \mathrm{e}^{\lambda r(\lambda)}} X(s) \leq \mathrm{e}^{\lambda(H+\varepsilon)}\right\} = 1.$$

Proof. We prove that, for almost all $w \in \mathbb{W}$,

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{\lambda(H-\varepsilon)}) < \mathrm{e}^{\lambda r(\lambda)} < \tau(\mathrm{e}^{\lambda(H+\varepsilon)})\} = 1, \tag{6.1}$$

which clearly implies the lemma. Let $w \in \mathbb{W}$ and, for any ε such that $0 < \varepsilon < H(w)$, let

$$M' = \begin{cases} \sigma(\frac{1}{2} - \varepsilon/2) & \text{if } w \in A, \\ \sup\left\{x < \sigma(-\frac{1}{2}) \colon w(x) - \min\limits_{x\leq y\leq\sigma(-1/2)} w(y) = 1-\varepsilon/2\right\} & \text{if } w \in B. \end{cases}$$

Then we see that

$$\begin{aligned} \lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{\lambda(H-\varepsilon)}) < \tau(M')\} &= \lim_{\lambda\to\infty} \frac{\int_{M'}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x}{\int_{M'}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{\lambda(H-\varepsilon)}} \\ &= 1, \end{aligned} \tag{6.2}$$

since

$$\lim_{\lambda\to\infty} \lambda^{-1}\log\int_{M'}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x = \max_{M'\leq x\leq 0} w(x) \geq H - \frac{\varepsilon}{2} > H - \varepsilon.$$

Moreover, by applying Lemma 4.1 with $a = M'$, we have

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(M') < \mathrm{e}^{\lambda r(\lambda)}\} = 1. \tag{6.3}$$

Combining (6.2) and (6.3) yields

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{\lambda(H-\varepsilon)}) < \mathrm{e}^{\lambda r(\lambda)}\} = 1. \tag{6.4}$$

Next, for any $\varepsilon > 0$ we let

$$M'' = \begin{cases} \sigma(\frac{1}{2} + \varepsilon/2) & \text{if } w \in A, \\ \sup\Big\{x < \sigma(-\frac{1}{2})\colon w(x) - \min\limits_{x\le y\le\sigma(-1/2)} w(y) = 1 + \varepsilon/2\Big\} & \text{if } w \in B. \end{cases}$$

Then we have

$$\begin{aligned} \lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(\mathrm{e}^{\lambda(H+\varepsilon)}) > \tau(M'')\} &= \lim_{\lambda\to\infty} \frac{\mathrm{e}^{\lambda(H+\varepsilon)}}{\int_{M''}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x + \mathrm{e}^{\lambda(H+\varepsilon)}} \\ &= 1, \end{aligned} \tag{6.5}$$

since

$$\lim_{\lambda\to\infty} \lambda^{-1}\log\int_{M''}^0 \mathrm{e}^{\lambda w(x)}\,\mathrm{d}x = \max_{M''\le x\le 0} w(x) \le H + \frac{\varepsilon}{2} < H + \varepsilon.$$

Moreover, an application of Lemma 4.1 with $a = M''$ yields

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\tau(M'') > \mathrm{e}^{\lambda r(\lambda)}\} = 1. \tag{6.6}$$

By (6.5) and (6.6), we obtain

$$\lim_{\lambda\to\infty} \mathrm{P}^0_{\lambda w}\{\mathrm{e}^{\lambda r(\lambda)} < \tau(\mathrm{e}^{\lambda(H+\varepsilon)})\} = 1,$$

which, combined with (6.4), proves (6.1). The proof of Lemma 6.1 is thus complete.

Proof of Theorem 1.4. By Lemma 2.1 and (1.2), we have

$$\begin{aligned} &\int_{\mathbb{W}} \mathrm{P}(\mathrm{d}w)\,\mathrm{P}^0_w\left\{\left|\frac{\log\max_{0\le s\le \mathrm{e}^\lambda} X(s)}{\lambda} - H(w_\lambda)\right| > \varepsilon\right\} \\ &\quad = \int_{\mathbb{W}} \mathrm{P}(\mathrm{d}w)\,\mathrm{P}^0_{\lambda w}\left\{\left|\frac{2\log\lambda + \log\max_{0\le s\le \mathrm{e}^{\lambda r(\lambda)}} X(s)}{\lambda} - H(w)\right| > \varepsilon\right\}, \end{aligned} \tag{6.7}$$

where $r(\lambda) = 1 - 4\lambda^{-1}\log\lambda$. The right-hand side of (6.7) converges to 0 as $\lambda \to \infty$, by Lemma 6.1. We hence obtain Theorem 1.4.

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