LIMIT THEOREMS FOR A DIFFUSION PROCESS WITH A ONE-SIDED BROWNIAN POTENTIAL

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Abstract

We consider a diffusion process $X(t)$ with a one-sided Brownian potential starting from the origin. The limiting behavior of the process as time goes to infinity is studied. For each $t > 0$, the sample space describing the random potential is divided into two parts, $\tilde{A}_t$ and $\tilde{B}_t$, both having probability $\frac{1}{2}$, in such a way that our diffusion process $X(t)$ exhibits quite different limiting behavior depending on whether it is conditioned on $\tilde{A}_t$ or on $\tilde{B}_t$ ($t \to \infty$). The asymptotic behavior of the maximum process of $X(t)$ is also investigated. Our results improve those of Kawazu, Suzuki, and Tanaka (2001).

Keywords: Random environment; diffusion process; occupation time

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1. Introduction

In [4] a diffusion process $\{X(t), t \geq 0\}$ with a one-sided Brownian potential was studied, and it was shown that the limit distribution of $t^{-1/2}X(t)$ as $t \to \infty$ exists and is given by

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx + \frac{1}{2} \delta_0(dx),$$

the support being $[0, \infty)$. The long-time behavior of $X(t)$ is diffusive (in the sense that a limit distribution exists under the Brownian scaling) with probability $\frac{1}{2}$ and subdiffusive with the remaining probability $\frac{1}{2}$.

In this paper we treat the same model and give much more precise statements. In fact we prove, among other things, that $(\log t)^{-2}X(t)$ has a limit distribution with probability $\frac{1}{2}$; for the precise meaning of this, see Theorem 1.5.

Let us describe our model, following [4]. We denote by $\mathbb{W}$ the space of continuous functions $w$ defined on $\mathbb{R}$ and vanishing identically on $[0, \infty)$. Let $P$ be the Wiener measure on $\mathbb{W}$, namely the probability measure on $\mathbb{W}$ such that $\{w(-x), x \geq 0, P\}$ is a Brownian motion with time parameter $x$. By $\Omega$ we denote the space of real-valued, continuous functions defined on $[0, \infty)$. For $\omega \in \Omega$, we write $X(t) \equiv X(t, \omega) \equiv \omega(t)$, the value of $\omega$ at $t$. For $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, we let $P^{x_0}_w$ be the probability measure on $\Omega$ such that $\{X(t), t \geq 0, P^{x_0}_w\}$ is a diffusion process with generator

$$L_w = \frac{1}{2} e^{w(x)} \frac{d}{dx} \left( e^{-w(x)} \frac{d}{dx} \right).$$

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starting from \( x_0 \). We define the probability measure \( \mathcal{P}^{x_0} \) on \( \mathbb{W} \times \Omega \) by
\[
\mathcal{P}^{x_0}(dw \, d\omega) = P(dw) P^{x_0}_w(d\omega).
\]
We regard \( \{X(t), t \geq 0, \mathcal{P}^{x_0}\} \) as a process defined on the probability space \( (\mathbb{W} \times \Omega, \mathcal{P}^{x_0}) \) and call it a diffusion process with a one-sided Brownian potential. Our aim is to clarify the limiting behavior of \( \{X(t), t \geq 0, \mathcal{P}^{0}\} \) as \( t \to \infty \).

For the case in which \( w(x) \) does not vanish identically for \( x \geq 0 \) or, more precisely, the case in which \( \{w(x), x \geq 0, P\} \) and \( \{w(-x), x \geq 0, P\} \) are independent Brownian motions, the corresponding diffusion process, \( \{X(t), t \geq 0, \mathcal{P}^{0}\} \), was introduced by Brox [1] and Schumacher [5] as a diffusion analogue of Sinai’s random walk [6]. In [1] and [5] it was proved that \( \{(\log t)^{-2} X(t), t \geq 0, \mathcal{P}^{0}\} \) has a nondegenerate limit distribution.

We begin by presenting the result of [4]. Let \( \mathcal{M} \) be the space of probability laws on \( \Omega \) and let \( \rho \) be the Prokhorov metric on \( \mathcal{M} \). Let \( \{X(t), t \geq 0, \mathcal{P}^{0}\} \) be a diffusion process with a one-sided Brownian potential. Set
\[
X_{\lambda}(t) = \lambda^{-1/2} X(\lambda t), \quad t \geq 0,
\]
for a constant \( \lambda > 0 \), and denote by \( P_{\lambda}(w) \in \mathcal{M} \) the probability law of the process \( \{X_{\lambda}(t), t \geq 0, \mathcal{P}^{0}_{\lambda}\} \). Also, denote by \( P_{\mathcal{N}} \in \mathcal{M} \) the probability law of the identically vanishing process, and by \( P_{\mathcal{R}} \in \mathcal{M} \) the probability law of the reflecting Brownian motion on \( [0, \infty) \) starting from 0.

**Theorem 1.1.** ([4]) For any \( \varepsilon \) such that \( 0 < \varepsilon < \rho(P_{\mathcal{N}}, P_{\mathcal{R}})/2 \),

\[
\lim_{\lambda \to \infty} \mathbb{P}[\rho(P_{\lambda}(w), P_{\mathcal{N}}) < \varepsilon] = \frac{1}{2}, \quad \lim_{\lambda \to \infty} \mathbb{P}[\rho(P_{\lambda}(w), P_{\mathcal{R}}) < \varepsilon] = \frac{1}{2}.
\]

Our present results (namely Theorems 1.2 and 1.3, stated below) imply Theorem 1.1. To state the theorems, we introduce some notation.

For \( w \in \mathbb{W} \) and \( x_0 \in \mathbb{R} \), the diffusion process \( \{X(t), t \geq 0, \mathcal{P}^{x_0}_w\} \) can be constructed from a Brownian motion via time change and scale change [3, p. 165]. See [4] for the explicit representation of this diffusion process. The scale function of the process is given by
\[
S(x) = \int_0^x e^{w(y)} \, dy, \quad x \in \mathbb{R}.
\]
If \( S(x) \to -\infty \) (as \( x \to -\infty \)), then the diffusion process \( \{X(t), t \geq 0, \mathcal{P}^{x_0}_w\} \) is recurrent and, hence, conservative. By restricting the whole space \( \mathbb{W} \) to the set of \( w \)’s satisfying \( S(x) \to -\infty \) (as \( x \to -\infty \)), which still has \( P \)-measure 1, we may assume that the diffusion process \( \{X(t), t \geq 0, \mathcal{P}^{x_0}_w\} \) is recurrent for any \( w \).

For \( \omega \in \Omega \), we write
\[
a(t) \equiv a(t, \omega) = \int_0^t 1_{(0, \infty)}(X(s)) \, ds, \quad t \geq 0, \tag{1.1}
\]
where \( 1_A \) denotes the indicator function of the (generic) set \( A \). Then, for any \( w \in \mathbb{W} \) and \( x_0 \in \mathbb{R} \), we have
\[
\mathbb{P}^{x_0}_w \left[ \lim_{t \to \infty} a(t) = \infty \right] = 1,
\]
since the diffusion process \(\{X(t), t \geq 0, P^0_w\}\) is recurrent. In what follows, we reduce \(\Omega\) so that it equals the set of \(\omega\)s satisfying \(a(t) \to \infty\) (as \(t \to \infty\)). For \(\lambda > 0\) and \(\omega \in \Omega\), we let

\[
a_\lambda(t) \equiv a_\lambda(t, \omega) = \int_0^t 1_{(0,\infty)}(X_\lambda(s)) \, ds, \quad t \geq 0.
\]

Since \(a_\lambda(t) = \lambda^{-1} a(\lambda t) \to \infty\) (as \(t \to \infty\)), we can define

\[
a_\lambda^{-1}(t) = \inf\{s > 0: a_\lambda(s) > t\}, \quad t \geq 0,
\]

the right-continuous inverse function of \(a_\lambda(t)\). We also let

\[
G_\lambda(t) = X_\lambda(a_\lambda^{-1}(t)), \quad t \geq 0.
\]

Then \(\{G_\lambda(t), t \geq 0, P^0_w\}\) is a reflecting Brownian motion on \([0, \infty)\) starting from 0.

For \(w \in \mathbb{W}\) and \(a \in \mathbb{R}\), we let

\[
\sigma(a) = \sigma(a, w) = \sup\{x < 0: w(x) = a\},
\]

and introduce two subsets \(A\) and \(B\) of \(\mathbb{W}\) as follows:

\[
A = \{w \in \mathbb{W}: \sigma(\frac{1}{2}) > \sigma(-\frac{1}{2})\},
\]

\[
B = \{w \in \mathbb{W}: \sigma(\frac{1}{2}) < \sigma(-\frac{1}{2})\}.
\]

Each of these subsets has \(P\)-measure \(\frac{1}{2}\). For \(w \in \mathbb{W}\) and \(\lambda > 0\), we define \(w_\lambda \in \mathbb{W}\) by

\[
w_\lambda(x) = \lambda^{-1} w(\lambda^2 x), \quad x \in \mathbb{R}.
\]

Then

\[
\{w_\lambda, P\} \overset{\text{d}}{=} \{w, P\}, \quad (1.2)
\]

where \(\overset{\text{d}}{=}\) denotes equality in distribution. For each \(\lambda > 0\), we also introduce two further subsets \(A_\lambda\) and \(B_\lambda\) of \(\mathbb{W}\) as follows:

\[
A_\lambda = \{w \in \mathbb{W}: w_\lambda \in A\},
\]

\[
B_\lambda = \{w \in \mathbb{W}: w_\lambda \in B\}.
\]

Each of these also has \(P\)-measure \(\frac{1}{2}\), by (1.2).

In the following theorems, \(P[\cdot \mid \cdot]\) denotes the conditional probability. We write \(\tilde{A}_\lambda = A_{\log \lambda}\) and \(\tilde{B}_\lambda = B_{\log \lambda}\).

**Theorem 1.2.** For any \(T > 0\) and \(\varepsilon > 0\),

\[
\lim_{\lambda \to \infty} P_w^0 \left\{ \sup_{0 \leq t \leq T} |X_\lambda(t) - G_\lambda(t)| < \varepsilon \right\} > 1 - \varepsilon \left| \tilde{A}_\lambda \right| = 1.
\]

For \(w \in \mathbb{W}\), we let

\[
\zeta = \zeta(w) = \sup\{x < 0: w(x) - \min_{x \leq y \leq 0} w(y) = 1\},
\]

\[
M = M(w) = \begin{cases} \sigma(\frac{1}{2}) & \text{if } w \in A, \\ \zeta(w) & \text{if } w \in B, \end{cases}
\]

\[
V = V(w) = \min_{x \leq M} w(x).
\]

We also define \(b \equiv b(w)\) in \((M, 0)\) by \(w(b) = V\). Note that \(b\) is determined uniquely by \(w\) (\(P\)-almost surely).
Theorem 1.3. For any $\varepsilon > 0$,
\[
\lim_{t \to \infty} P[P^0_0((\log t)^{-2}X(t) - b(w)\log t) < \varepsilon ] > 1 - \varepsilon \mid \mathcal{B}_t] = 1.
\]

To state the result on the maximum process of $X(t)$, we let
\[ H(w) = \max_{M \leq x \leq 0} w(x). \]

Note that $H(w) = \frac{1}{2}$ if $w \in A$ and $0 < H(w) < \frac{1}{2}$ if $w \in B$.

Theorem 1.4. For any $\varepsilon > 0$,
\[
\lim_{t \to \infty} \mathcal{P}_0^0 \left\{ \frac{\log \max_{0 \leq s \leq t} X(s)}{\log t} - H(w) > \varepsilon \right\} = 0.
\]

Our present results, together with those of [4], immediately imply the following theorem.

Theorem 1.5. In each of the following instances, the distribution of $\tilde{X}_t$ under $P^0_0$ tends to a limit distribution as $t \to \infty$, as described.

- $\tilde{X}_t = t^{-1/2}X(t)$: limit distribution is $\mu_1(dx) = \frac{1}{2} \sqrt{2/\pi} e^{-x^2/2} dx + \frac{1}{2} \delta_0(dx)$; support is $[0, \infty)$.
- $\tilde{X}_t = (\log t)^{-2}X(t)$: limit distribution is $\mu_{II}(dx) = P\{b \in dx \cap B\} + \frac{1}{2} \delta_\infty(dx)$; support is $(-\infty, 0) \cup \{\infty\}$.
- $\tilde{X}_t = t^{-1/2} \max_{0 \leq s \leq t} X(s)$: limit distribution is $\mu_{III}(dx) = \frac{1}{2} P\{\max_{0 \leq s \leq 1} X(s) \in dx\} + \frac{1}{2} \delta_0(dx)$; support is $[0, \infty)$.
- $\tilde{X}_t = \log(\max_{0 \leq s \leq t} X(s))/\log t$: limit distribution is $\mu_{IV}(dx) = P\{H \in dx\}$; support is $(0, \frac{1}{2}]$.
- $\tilde{X}_t = (\log t)^{-2} \min_{0 \leq s \leq t} X(s)$: limit distribution is $\mu_V(dx) = P\{M \in dx\}$; support is $(-\infty, 0)$.

Moreover, the Laplace transforms of the distributions of $b$, $H$, and $M$ appearing in the definitions of $\mu_{II}$, $\mu_{IV}$, and $\mu_{V}$ are as follows. For $\xi > 0$,
\[
E[e^{\xi b}, B] = \frac{\sinh(\sqrt{2\xi}/2)}{\sqrt{2\xi} \cosh(\sqrt{2\xi})},
\]
\[
E[e^{\xi H}, A] = \frac{1}{2} e^{\xi/2},
\]
\[
E[e^{\xi H}, B] = \int_0^{1/2} e^{\xi x} dx,
\]
\[
E[e^{\xi M}, A] = \frac{\sinh(\sqrt{2\xi}/2)}{\sinh(\sqrt{2\xi})},
\]
\[
E[e^{\xi M}, B] = \frac{\sinh(\sqrt{2\xi}/2)}{(\sinh(\sqrt{2\xi}) \cosh(\sqrt{2\xi})).
\]

Here $E[\cdot, A]$ denotes the expectation with respect to $P$ on the set $A$. 

Let $\eta(t, x)$ be the local time at $x$ of the Brox–Schumacher diffusion process. Hu and Shi [2] showed that, for any $x \in \mathbb{R}$,

$$\frac{\log \eta(t, x)}{\log t} \overset{d}{\to} U \wedge  \hat{U}, \quad t \to \infty,$$

where ‘$\overset{d}{\to}$’ denotes convergence in distribution and $U$ and $\hat{U}$ are independent random variables uniformly distributed in $(0, 1)$.

For our diffusion process $\{X(t), t \geq 0, \mathcal{F}^0\}$ with a one-sided Brownian potential, Z. Shi (private communication (2000)) informed us that the same method based on the second Ray–Knight theorem as in [2] can be used to show that

$$\frac{\log a(t)}{\log t} \overset{d}{\to} 1 \wedge (2U), \quad t \to \infty,$$

where $U$ is a random variable uniformly distributed in $(0, 1)$.

In Section 3 we investigate the asymptotic behavior of the occupation time $a(t)$ as $t \to \infty$.

### 2. Preliminaries

For $\lambda > 0$, $w \in \mathcal{W}$, and $x_0 \in \mathbb{R}$, let $P_{x_0}^{\lambda w}$ be the probability measure on $\Omega$ such that $\{X(t), t \geq 0, P_{x_0}^{\lambda w}\}$ is a diffusion process with generator

$$L_{\lambda w} = \frac{1}{2} e^{\lambda w(x)} \frac{d}{dx} \left( e^{-\lambda w(x)} \frac{d}{dx} \right),$$

starting from $x_0$. Denote by $E_{x_0}^{\lambda w}$ the expectation with respect to $P_{x_0}^{\lambda w}$. The following lemma was proved in [1].

**Lemma 2.1.** ([1].) For any $\lambda > 0$ and $w \in \mathcal{W}$,

$$\{X(t), t \geq 0, P_{x_0}^{\lambda w}\} \overset{d}{=} \{\lambda^{-2} X(\lambda^4 t), t \geq 0, P_{w}^{0}\}.$$

In preparation for the proofs of Theorem 1.2 and Theorem 1.3, we present the following theorems.

**Theorem 2.1.** Let $\mu \equiv \mu(\lambda) = \lambda^{1/4} \log \lambda$. Then, for any $T > 0$ and $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} \mathbb{P}_{x_0}^{\lambda w} \left[ \sup_{0 \leq t \leq T} |X(t) - G(t)| < \varepsilon \bigg| \mu \lambda \right] = 1,$$

where $G(t) = X(a^{-1}(t))$ and $a^{-1}(t) = \inf\{s > 0 : a(s) > t\}$, the right-continuous inverse function of $a(t)$ defined in (1.1).

**Theorem 2.2.** Let $r$ be a real-valued function of $\lambda > 0$ such that $r(\lambda) \to 1$ (as $\lambda \to \infty$). Then there exists a subset $B^\theta$ of $B$, with $\mathbb{P}[B \setminus B^\theta] = 0$, such that, for any $w \in B^\theta$ and $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} \mathbb{P}_{x_0}^{\lambda w} \left[ |X(e^{\lambda r(\lambda)}) - b(w)| < \varepsilon \right] = 1.$$

We remark that to prove Theorem 2.2 it is enough to show the following proposition.
Proposition 2.1. There exists a subset $B^\#$ of $B$, with $P(\{B \setminus B^\#\}) = 0$, such that for any $w \in B^\#$ the following holds: there exists a $\delta > 0$ such that, for any $r_1$ and $r_2$ satisfying $1 - \delta < r_1 < r_2 < 1 + \delta$ and any $\varepsilon > 0$,

$$\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P^{0}_\omega(|X(\lambda^2 r) - b(w)| < \varepsilon) = 1. \quad (2.1)$$

In Section 4 we prove Theorem 2.1 and Theorem 1.2, in Section 5 we prove Proposition 2.1 and Theorem 1.3, and in Section 6 we prove Theorem 1.4.

3. Asymptotic behavior of the occupation time $a(t)$ as $t \to \infty$

In this section we investigate the limiting behavior of $\{t^{-1} a(t), t \geq 0, R^0\}$ as $t \to \infty$. To do so, we need two lemmas. The first, Lemma 3.1, which is needed for the proof of Theorem 2.1, will be proved in Section 4.

Lemma 3.1. There exists a subset $A^\#$ of $A$, with $P(\{A \setminus A^\#\}) = 0$, such that, for any $w \in A^\#$ and $T > 0$,

$$\lim_{\lambda \to \infty} E^{0}_\omega \left[ \frac{1}{T} e^{\lambda} \int_0^T 1_{(0, \infty)}(X(s)) \, ds \right] = 1. \quad (3.1)$$

The second lemma, Lemma 3.2, can be obtained from Proposition 2.1.

Lemma 3.2. There exists a subset $B^\#$ of $B$, with $P(\{B \setminus B^\#\}) = 0$, such that, for any $w \in B^\#$,

$$\lim_{\lambda \to \infty} E^{0}_\omega \left[ \frac{1}{e^{\lambda}} \int_0^{e^\lambda} 1_{(-\infty, 0)}(X(s)) \, ds \right] = 1.$$

The main result in this section is as follows.

Theorem 3.1. For any $\varepsilon > 0$,

$$\lim_{t \to \infty} P \left\{ P^{0}_w \left[ \frac{1}{t} \int_0^t 1_{(0, \infty)}(X(s)) \, ds > 1 - \varepsilon \right] > 1 - \varepsilon \right\} \mathbb{A}_t = 1, \quad (3.2)$$

$$\lim_{t \to \infty} P \left\{ P^{0}_w \left[ \frac{1}{t} \int_0^t 1_{(0, \infty)}(X(s)) \, ds < \varepsilon \right] > 1 - \varepsilon \right\} \mathbb{B}_t = 1. \quad (3.3)$$

Proof. We prove (3.2); we can prove (3.3) in the same way by using Lemma 3.2. By Lemma 3.1 we have, for $w \in A^\#$,

$$\lim_{t \to \infty} E^{0}_w \left[ \frac{1}{t} \int_0^t 1_{(0, \infty)}(X(s)) \, ds \right] = 1.$$

Here $A^\#$ is a subset of $A$ with $P(\{A \setminus A^\#\}) = 0$. Therefore, it follows that

$$\lim_{t \to \infty} P \left\{ P^{0}_w \left[ \frac{1}{t} \int_0^t 1_{(0, \infty)}(X(s)) \, ds > 1 - \varepsilon \right] > 1 - \varepsilon \right\} \mathbb{A}_t = 1$$

for any $\varepsilon > 0$. Moreover, (1.2) and Lemma 2.1 imply that

$$\lim_{t \to \infty} P \left\{ P^{0}_w \left[ \frac{1}{t} \int_0^t 1_{(0, \infty)}((\log t)^4 s) \, ds > 1 - \varepsilon \right] > 1 - \varepsilon \right\} \mathbb{A}_{\log t} = 1. \quad (3.4)$$
Changing the variable of the integral in (3.4) yields
\[
\lim_{u \to \infty} \mathbb{P} \left\{ \mathbb{P}_0 \left[ \frac{1}{u} \int_0^u 1_{(0,\infty)}(X(s)) \, ds > 1 - \varepsilon \right] > 1 - \varepsilon \left| A_{\log t} \right. \right\} = 1, \tag{3.5}
\]
where \( t \equiv t(u) \) is determined by \( u = t(\log t)^4 \).

Let us prove that
\[
\lim_{u \to \infty} \mathbb{P} \{ A_{\log t(u)} \ominus A_{\log u} \} = 0, \tag{3.6}
\]
where \( A_1 \ominus A_2 = (A_1 \cap A_2^c) \cup (A_1^c \cap A_2) \) for sets \( A_1 \) and \( A_2 \). We note that \( r = \log t(u) / \log u \to 1 \) as \( u \to \infty \).

Using the fact that \( w_{\lambda_1, \lambda_2} = (w_{\lambda_1})_{\lambda_2} \) for \( \lambda_1, \lambda_2 > 0 \), we have
\[
\mathbb{P} \{ A_{\log t(u)} \ominus A_{\log u} \} = \mathbb{E} \left[ \left| 1_{A(w_{\text{r}})} - 1_{A(w)} \right| \right], \tag{3.8}
\]
where \( \mathbb{E} \) denotes the expectation with respect to \( \mathbb{P} \). By (1.2), the right-hand side of (3.8) is equal to
\[
\mathbb{E} \left[ \left| 1_{A(w_{\text{r}})} - 1_{A(w)} \right| \right],
\]
which converges to 0 as \( u \to \infty \), due to (3.7). This proves (3.6). From (3.6) it follows that \( \mathbb{P} \{ \cdots \left| A_{\log t(u)} \right\} \to 1 \) (as \( u \to \infty \)) is equivalent to \( \mathbb{P} \{ \cdots \left| A_{\log u} \right\} \to 1 \) (as \( u \to \infty \)). Hence, by (3.5), we obtain (3.2).

**Corollary 3.1.** The probability distribution of \( t^{-1} \int_0^t 1_{(0,\infty)}(X(s)) \, ds \) under \( \mathbb{P}_0 \) converges to \( \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \) as \( t \to \infty \).

\[\text{4. Proofs of Theorem 2.1 and Theorem 1.2}\]

In this section we prove Theorem 2.1 and Theorem 1.2. First we introduce a lemma from [4].

For \( \omega \in \Omega \), let
\[
\tau (a) \equiv \tau (a, \omega) = \inf \{ t > 0 : X(t) = a \}, \quad a \in \mathbb{R}.
\]

**Lemma 4.1.** ([4]) Let \( w \in \mathcal{W} \) and \( a < 0 \). Assume that \( w(a) > w(x) \) for all \( x > a \). Then, for any \( \varepsilon > 0 \),
\[
\lim_{\lambda \to \infty} \mathbb{P}_w^0 \{ e^{\lambda (J - \varepsilon)} < \tau (a) < e^{\lambda (J + \varepsilon)} \} = 1,
\]
where \( J = \max \{ J_0, 2w(a) \} \) and \( J_0 = w(a) - \min \{ w(x) : x \geq a \} \).

Next we prove Lemma 3.1.

**Proof of Lemma 3.1.** Let \( w \in A \). In this case we note that \( V > -\frac{1}{2} \). First we choose \( r_0 \) and \( r_1 \) satisfying \( -V < r_0 < r_1 < \frac{1}{2} \). We then see that
\[
\lim_{\lambda \to \infty} \mathbb{P}_w^0 (e^{\lambda r_0}) < \tau (\sigma (r_1))) = \lim_{\lambda \to \infty} \frac{\int_0^{\sigma (r_1)} e^{\lambda w(x)} \, dx}{\int_{\sigma (r_1)} e^{\lambda w(x)} \, dx + e^{\lambda r_0}} = 1, \tag{4.1}
\]
since \( \lim_{\lambda \to \infty} \lambda^{-1} \log \int_{\sigma(r)}^{0} e^{\lambda w(x)} \, dx = r_1 > r_0 \). Moreover, an application of Lemma 4.1 with \( a = \sigma(r_1) \) gives

\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ e^{\lambda (2r_1 - \varepsilon)} < \tau(\sigma(r_1)) < e^{\lambda (2r_1 + \varepsilon)} \} = 1
\]  (4.2)

for any \( \varepsilon > 0 \). Combining (4.1) and (4.2) yields

\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau(e^{r_0}) < e^{\lambda \theta} \} = 1 \quad \text{for some } \theta \in (0, 1).
\]  (4.3)

Next we choose a \( \rho > 1/2 \) satisfying

\[
\min\{w(x) : \sigma(\rho) \leq x \leq \sigma(\rho/2)\} > V.
\]  (4.4)

(Note that the set of \( w \in A \) for which there is no \( \rho > 1/2 \) satisfying (4.4) is \( P \)-negligible.) By applying Lemma 4.1 with \( a = \sigma(\rho) \), we have

\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ e^{\lambda (2\rho - \varepsilon)} < \tau(\sigma(\rho)) < e^{\lambda (2\rho + \varepsilon)} \} = 1
\]  (4.5)

for any \( \varepsilon > 0 \). Therefore, for any \( T > 0 \),

\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau(\sigma(\rho)) > Te^{\lambda} \} = 1.
\]  (4.6)

Now we define \( m_{\lambda w} \), a probability measure on \( I_{\lambda} = [\sigma(\rho), \infty) \), by

\[
m_{\lambda w}(E) = \frac{\int_{E\cap[\sigma(\rho),0]} e^{-\lambda w(x)} \, dx + \int_{E\cap(0,e^{r_0})} e^{\rho x} \, dx}{\int_{\sigma(\rho)}^{0} e^{-\lambda w(x)} \, dx + e^{\rho x}}
\]

for any Borel set \( E \) in \( I_{\lambda} \). This is the invariant probability measure for the reflecting \( \mathcal{L}_{\lambda w} \)-diffusion process on \( I_{\lambda} \). Note that

\[
\lim_{\lambda \to \infty} m_{\lambda w}((0, e^{r_0})) = \lim_{\lambda \to \infty} \frac{e^{r_0}}{\int_{\sigma(\rho)}^{0} e^{-\lambda w(x)} \, dx + e^{\rho x}} = 1
\]  (4.7)

since \( \lim_{\lambda \to \infty} \lambda^{-1} \log \int_{\sigma(\rho)}^{0} e^{-\lambda w(x)} \, dx = -V < r_0 \).

By the comparison theorem for one-dimensional diffusion processes, for \( \lambda > 0 \) we can construct diffusion processes \( \{Y_{\lambda}(t), t \geq 0\} \) and \( \{Z_{\lambda}(t), t \geq 0\} \), on a probability space \( (\Omega, \tilde{\mathbb{P}}) \), with the following properties:

\[
\{Y_{\lambda}(t), t \geq 0\} \quad \text{is a reflecting } \mathcal{L}_{\lambda w} \text{-diffusion process on } \{\sigma(\rho), \infty\} \text{ starting from } e^{r_0}. \quad \text{(4.7)}
\]

\[
\{Z_{\lambda}(t), t \geq 0\} \quad \text{is a reflecting } \mathcal{L}_{\lambda w} \text{-diffusion process on } I_{\lambda} \text{ with initial distribution } m_{\lambda w}.
\]

\[
\tilde{\mathbb{P}} \{Y_{\lambda}(t) \geq Z_{\lambda}(t) \text{ for all } t \geq 0\} = 1. \quad \text{(4.8)}
\]

Since \( \{Z_{\lambda}(t), t \geq 0\} \) is a stationary process with invariant probability measure \( m_{\lambda w} \), by (4.6) we have

\[
\lim_{\lambda \to \infty} \tilde{\mathbb{E}} \left[ \frac{1}{Te^{\lambda}} \int_0^{Te^{\lambda}} I_{(0,\infty)}(Z_{\lambda}(s)) \, ds \right] = 1 \quad \text{(4.9)}
\]
A diffusion process with a one-sided Brownian potential

for any \( T > 0 \). Here \( \bar{E} \) denotes the expectation with respect to \( \bar{P} \). Moreover, (4.8) and (4.9) imply that

\[
\lim_{\lambda \to \infty} \bar{E} \left[ \frac{1}{T e^\lambda} \int_0^T e^\lambda I_{(0, \infty)}(Y_\lambda(s)) \, ds \right] = 1. \tag{4.10}
\]

Let us prove (3.1). Using the strong Markov property of \( \{X(t), t \geq 0, P_w\} \), for any \( T > 0 \) we obtain

\[
\begin{align*}
E^0_{\mu_w} \left[ \frac{1}{T e^\lambda} \int_0^T e^\lambda I_{(0, \infty)}(X(s)) \, ds \right] &
\geq E^0_{\mu_w} \left[ \frac{1}{T e^\lambda} \int_{\tau(e^{\theta \lambda})}^T e^\lambda I_{(0, \infty)}(X(s)) \, ds, \tau(e^{\theta \lambda}) < e^{\lambda \theta} \right] \\
&\geq E^0_{\mu_w} \left[ \frac{1}{T e^\lambda} \int_{0}^{e^{\lambda \theta}} e^\lambda I_{(0, \infty)}(X(s)) \, ds \right] P^0_{\lambda w} \{ \tau(e^{\theta \lambda}) < e^{\lambda \theta} \}.
\end{align*}
\]

(4.11)

Owing to (4.3), (4.5), (4.7), and (4.10), we see that the right-hand side of (4.11) converges to \( 1 \) as \( \lambda \to \infty \). Hence, we obtain (3.1).

We now present three lemmas in preparation for the proof of Theorem 2.1.

**Lemma 4.2.** Let \( \mu \equiv \mu(\lambda) = \lambda^{1/4} \log \lambda \). Then, for any \( T > 0 \) and \( \varepsilon > 0 \),

\[
\lim_{\lambda \to \infty} P \left[ \mu_{\mu_w} \left( \sup_{0 \leq t \leq T} |t - a(t)| < \varepsilon \right) \geq 1 - \varepsilon, A_{\log \lambda} \right] = 1.
\]

**Proof.** Using Chebyshev’s inequality, we have

\[
P \left[ \mu_{\mu_w} \left( \sup_{0 \leq t \leq T} |t - a(t)| \geq \varepsilon, A_{\log \lambda} \right) \right] = P \left[ \mu_{\mu_w} \{ T - a(T) \geq \varepsilon, A_{\log \lambda} \} \right]
\leq P \left[ \frac{1}{\varepsilon} E^0_{\mu_w} \{ T - a(T) \} \geq \varepsilon, A_{\log \lambda} \right]. \tag{4.12}
\]

By Lemma 2.1, we have

\[
\begin{align*}
E^0_{\mu_w} \{ a(T) \} &= E^0_{\mu_w} \left[ \int_0^T I_{(0, \infty)}(X(s)) \, ds \right] \\
&= E^0_w \left[ \int_0^T I_{(0, \infty)}(X_{\lambda(\log \lambda)^4}(s)) \, ds \right] \\
&= E^0_w \left[ \frac{1}{\lambda} \int_0^{T_{\lambda}} I_{(0, \infty)}(X_{\log \lambda}(s)) \, ds \right] \\
&= E^0_{\log \lambda \mu_{\log \lambda}} \left[ \frac{1}{\lambda} \int_0^{T_{\lambda}} I_{(0, \infty)}(X(s)) \, ds \right] \\
&= E^0_{\log \lambda \mu_{\log \lambda}} \left[ \frac{1}{\lambda} a(T_{\lambda}) \right].
\end{align*}
\]

Therefore, the right-hand side of (4.12) is equal to

\[
P \left[ \frac{1}{\varepsilon} E^0_{\log \lambda \mu_{\log \lambda}} \{ T - \frac{1}{\lambda} a(T_{\lambda}) \} \right] \geq \varepsilon, A_{\log \lambda} = P \left[ \frac{1}{\varepsilon} E^0_{\log \lambda \mu_{\log \lambda}} \{ T - \frac{1}{\lambda} a(T_{\lambda}) \} \right] \geq \varepsilon, A.
\]
This probability converges to 0 as \( \lambda \to \infty \), since
\[
\lim_{\lambda \to \infty} E_0^{\log \lambda} \left[ \frac{1}{T \lambda} a(T \lambda) \right] = 1 \quad \text{for } w \in A^\#.
\]
by Lemma 3.1. Here \( A^\# \) is a subset of \( A \) with \( P(A \setminus A^\#) = 0 \). Hence, we obtain Lemma 4.2.

**Lemma 4.3.** Let \( \mu \equiv \mu(\lambda) = \lambda^{1/4} \log \lambda \). Then, for any \( T > 0 \) and \( \varepsilon > 0 \),
\[
\lim_{\lambda \to \infty} P \left[ \sup_{0 \leq t \leq T} |G(t) - G(a(t))| < \varepsilon \right] > 1 - \varepsilon, \quad A_{\log \lambda} = 1.
\]

**Proof.** We note that, for any \( w \in \mathbb{W} \) and \( \lambda > 0 \), \( \{G(t), t \geq 0, P_0^{\mu w \mu}\} \) is a reflecting Brownian motion on \([0, \infty)\) starting from 0. Since, with probability 1, Brownian sample paths are locally Hölder continuous with exponent \( \gamma \), for every \( \gamma \in (0, \frac{1}{2}) \), we obtain Lemma 4.3 from Lemma 4.2.

The following lemma plays an important role in the proof of Theorem 2.1.

**Lemma 4.4.** Let \( f \) be a real-valued, continuous function of \( t \geq 0 \) with \( f(0) = 0 \), and let \( \alpha(t) = \int_0^t 1_{(0, \infty)}(f(s)) \, ds \).

Take \( T > 0 \) and assume that there exists a \( T_1 > T \) such that \( \alpha(T_1) > T \). Define the right-continuous inverse function of \( \alpha(t) \) by
\[
\alpha^{-1}(t) = \inf \{ s > 0 : \alpha(s) > t \}, \quad 0 \leq t \leq T,
\]
and let \( g(t) = f(\alpha^{-1}(t)), \quad 0 \leq t \leq T \).

Then
\[
|g(t) - f(t)| = |g(t) - g(\alpha(t)))| - \min\{f(t), 0\}, \quad 0 \leq t \leq T. \quad (4.13)
\]

**Proof.** First assume that \( \alpha^{-1}(\alpha(t)) = t \). In this case we notice that \( f(t) \geq 0 \) and
\[
|g(t) - f(t)| = |g(t) - f(\alpha^{-1}(\alpha(t)))| = |g(t) - g(\alpha(t))|,
\]
which establishes (4.13).

Next assume that \( \alpha^{-1}(\alpha(t)) \neq t \). In this case \( \alpha^{-1}(\alpha(t)) > t \). Moreover, \( f \leq 0 \) on the interval \([t, \alpha^{-1}(\alpha(t))]) \) and \( f(\alpha^{-1}(\alpha(t))) = 0 \), i.e. \( g(\alpha(t)) = 0 \). Noting that \( g(t) \geq 0 \), we obtain
\[
|g(t) - f(t)| = |g(t)| + |f(t)| = |g(t) - g(\alpha(t))| - \min\{f(t), 0\}.
\]
This completes the proof of the lemma.

**Proof of Theorem 2.1.** By virtue of Lemma 4.4, we have
\[
P \left[ \sup_{0 \leq t \leq T} |G(t) - X(t)| \geq \varepsilon \right] \geq \varepsilon, \quad A_{\log \lambda}
\]
\[
\leq P \left[ \sup_{0 \leq t \leq T} |G(t) - G(a(t))| \geq \frac{\varepsilon}{2}, \quad A_{\log \lambda} \right]
\]
\[
+ P \left[ \inf_{0 \leq t \leq T} X(t) \leq -\frac{\varepsilon}{2}, \quad A_{\log \lambda} \right]. \quad (4.14)
\]
The first term on the right-hand side of (4.14) converges to 0 as \( \lambda \to \infty \), due to Lemma 4.3. The second term is bounded above by
\[
P \left\{ \inf_{0 \leq t \leq T} X(t) \leq -\frac{\varepsilon}{2} \geq \frac{\varepsilon}{2} \right\} = P \left\{ \inf_{0 \leq t \leq T} X(t) \leq -\frac{\varepsilon}{2} \geq \frac{\varepsilon}{2} \right\},
\]
which converges to 0 as \( \lambda \to \infty \), by [4, Lemma 4.2]. This completes the proof of Theorem 2.1.

**Proof of Theorem 1.2.** By combining Theorem 2.1 and Lemma 2.1, we have
\[
\lim_{\lambda \to \infty} P \left\{ P_0 \{ \sup_{0 \leq t \leq T} |X_{\mu^t}(t) - G_{\mu^t}(t)| < \varepsilon \} > 1 - \varepsilon \right\} = 1
\]
or, equivalently,
\[
\lim_{\nu \to \infty} P \left\{ P_0 \{ \sup_{0 \leq t \leq T} |X_{\nu}(t) - G_{\nu}(t)| < \varepsilon \} > 1 - \varepsilon \right\} = 1,
\]
where \( \lambda \equiv \lambda(\nu) \) is determined by \( \nu = \lambda(\log \lambda)^4 \). We obtain Theorem 1.2 by the same argument as in the proof of Theorem 3.1.

**5. Proofs of Proposition 2.1 and Theorem 1.3**

In this section we prove Proposition 2.1 and Theorem 1.3. We begin by introducing a lemma due to Brox [1]. Let \( w \in W \) and \( \alpha < m < \beta < 0 \). We call a triple of negative numbers \( \Delta = (\alpha, m, \beta) \) a valley of \( w \) if the following conditions are satisfied.

(i) \( w(\alpha) > w(x) > w(m) \) for all \( x \in (\alpha, m) \) and \( w(\beta) > w(x) > w(m) \) for all \( x \in (m, \beta) \).

(ii) \( w(\alpha) - w(m) > H_{\beta,m} \) and \( w(\beta) - w(m) > H_{\alpha,m} \).

For a valley \( \Delta = (\alpha, m, \beta) \), we call \( D(\Delta) \) the depth of \( \Delta \) and \( A(\Delta) = H_{\beta,m} \wedge H_{\alpha,m} \) the inner directed ascent of \( \Delta \). A valley \( \Delta = (\alpha, m, \beta) \) is said to contain \( x_0 \) if \( \alpha < x_0 < \beta \).

**Lemma 5.1.** ([1]) Let \( w \in W \) and let \( \Delta = (\alpha, m, \beta) \) be a valley of \( w \) containing \( x_0 \). Then, for any \( r_1 \) and \( r_2 \) satisfying \( A(\Delta) < r_1 < r_2 < D(\Delta) \) and any \( \varepsilon > 0 \),
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P^0 \{ |X_{e^{r\lambda}}(t) - m| < \varepsilon \} = 1.
\]

Let us now prove Proposition 2.1.

**Proof of Proposition 2.1.** Let \( w \in B \). In this case \( M = \zeta \) and \( V < -\frac{1}{2} \). Let \( V' = \max_{x \geq b} w(x) \) and define \( b'(w) \) in \((b, 0)\) by \( w(b') = V' \). Note that \( b' \) is determined uniquely by \( w \) (P-almost surely).

First we consider the case \( V' > V > 1 \). Let \( c = c(w) = \sup\{x < b': w(x) = 0\} \).
and define \( \tilde{w} \in \mathcal{W} \) by
\[
\tilde{w}(x) = \begin{cases} 
  w(x) & \text{for } x \geq c, \\
  -x + c & \text{for } x < c.
\end{cases}
\]

We can choose a \( c' < c \) satisfying
\[
\tilde{w}(b') < \tilde{w}(c') < \frac{1}{2},
\]
\[
\tilde{J} := \left\{ \tilde{w}(c') - \min_{c' \leq x \leq 0} \tilde{w}(x) \right\} \lor 2\tilde{w}(c') < 1.
\]

An application of Lemma 4.1 with \( a = c' \) yields
\[
\lim_{\lambda \to \infty} P_{\lambda, \tilde{w}}^0 \{ \tau(c') < e^{\tilde{J} + \epsilon} \} = 1 \quad (5.1)
\]
for any \( \epsilon > 0 \). Since
\[
P_{\lambda, \tilde{w}}^0 \{ \tau(c') < e^{\tilde{J} + \epsilon} \} \leq P_{\lambda, w}^0 \{ \tau(c) < e^{\tilde{J} + \epsilon} \} = P_{\lambda, w}^0 \{ \tau(c) < e^{\tilde{J} + \epsilon} \},
\]
(5.1) implies that
\[
\lim_{\lambda \to \infty} P_{\lambda, w}^0 \{ \tau(c) < e^{\tilde{J} + \epsilon} \} = 1 \quad (5.2)
\]
for some \( \theta_0 \in (0, 1) \).

On the other hand, we see that \( \Delta = (\zeta, b, b') \) is a valley of \( w \) of depth 1 containing \( c \). Thus, there exists a negative number \( \alpha \) such that \( \alpha < \zeta \) and \( \Delta' = (\alpha, b, b') \) is a valley of \( w \) containing \( c \) with \( A(\Delta') < 1 < D(\Delta') \). (The set of \( w \in B \) for which there is no \( \alpha \) satisfying this condition is \( \mathbb{P} \)-negligible [1].) Therefore, by Lemma 5.1, there exists a \( \delta_0 > 0 \) such that, for any \( r_1 \) and \( r_2 \) satisfying \( 1 - \delta_0 < r_1 < r_2 < 1 + \delta_0 \) and any \( \epsilon > 0 \),
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P_{\lambda, w}^c \{ |X(e^{\lambda r}) - b| < \epsilon \} = 1 \quad (5.3)
\]
Using the strong Markov property of \( \{ X(t), t \geq 0, \mathbb{P} \cdot \} \) and (5.2) and (5.3), we obtain (2.1) for any \( \delta \in (0, \delta_0 \wedge (1 - \delta_0)) \) in the case \( V' - V > 1 \).

Next we let \( V' - V < 1 \). In this case we note that \( 0 < V' < w(\zeta) < \frac{1}{2} \). Thus, we can choose a \( \rho' \) satisfying \( V' < \rho' < w(\zeta) \), and note that \( \sigma(\rho') < b \). Applying Lemma 4.1 with \( a = \sigma(\rho') \) yields
\[
\lim_{\lambda \to \infty} P_{\lambda, w}^0 \{ e^{\lambda(\rho' - V - \epsilon)} < \tau(\sigma(\rho')) < e^{\lambda(\rho' - V + \epsilon)} \} = 1 \quad (5.6)
\]
for any \( \epsilon > 0 \). Since \( \rho' - V < 1 \), it follows that
\[
\lim_{\lambda \to \infty} P_{\lambda, w}^0 \{ \tau(\sigma(\rho')) < e^{\lambda \theta_1} \} = 1 \quad \text{for some } \theta_1 \in (0, 1). \quad (5.4)
\]

Also, we can choose a \( \rho \) satisfying
\[
w(\zeta) < \rho < \frac{1}{2}, \quad \min_{\zeta \leq x \leq 0} [w(x) : \sigma(\rho) \leq x] > V.
\]
(5.5)
(Note that the set of \( w \in B \) for which there is no \( \rho \) satisfying (5.5) is \( \mathbb{P} \)-negligible.) An application of Lemma 4.1 with \( a = \sigma(\rho) \) yields
\[
\lim_{\lambda \to \infty} P_{\lambda, w}^0 \{ e^{\lambda(\rho - V - \epsilon)} < \tau(\sigma(\rho)) < e^{\lambda(\rho - V + \epsilon)} \} = 1 \quad (5.6)
\]
for any $\varepsilon > 0$. Let $\tau_\lambda = \tau(\sigma(\rho)) \wedge \tau(\rho^{1/2})$; we then observe that
\[
\lim_{\lambda \to \infty} P^0_{\lambda,w}(\tau(\sigma(\rho)) < \tau(\rho^{1/2})) = \lim_{\lambda \to \infty} \frac{e^{\lambda/2}}{\int_{\sigma(\rho)}^{0} e^{\lambda w(x)} \, dx + e^{\lambda/2}} = 1, \tag{5.7}
\]
because $\lim_{\lambda \to \infty} \lambda^{-1} \log \int_{\sigma(\rho)}^{0} e^{\lambda w(x)} \, dx = \rho < \frac{1}{2}$. By (5.6) and (5.7), for any $\varepsilon > 0$ we have
\[
\lim_{\lambda \to \infty} P^0_{\lambda,w}(e^{\lambda(\rho-V-\varepsilon)} < \tau_\lambda < e^{\lambda(\rho-V+\varepsilon)}) = 1.
\]
Since $\rho - V > 1$, for any small $\delta_1 > 0$ we may consider the process $\{X(t), 0 \leq t \leq e^{\lambda(1+\delta_1)}, P^0_{\lambda,w}\}$ to be a reflecting $\mathcal{L}_{\lambda,w}$-diffusion process on $I'_\lambda = [\sigma(\rho), e^{\lambda/2}]$. We define $m'_{\lambda,w}$, a probability measure on $I'_\lambda$, by
\[
m'_{\lambda,w}(E) = \frac{\int_{E \cap [\sigma(\rho),0]} e^{-\lambda w(x)} \, dx + \int_{E \cap (0,e^{\lambda/2})} \, dx}{\int_{\sigma(\rho)}^{0} e^{-\lambda w(x)} \, dx + e^{\lambda/2}}
\]
for any Borel set $E$ in $I'_\lambda$. This is the invariant probability measure for the reflecting $\mathcal{L}_{\lambda,w}$-diffusion process on $I'_\lambda$. Notice that, for any $\varepsilon > 0$ satisfying $[b-\varepsilon, b+\varepsilon] \subset [\sigma(\rho), 0]$, 
\[
\lim_{\lambda \to \infty} m'_{\lambda,w}((b-\varepsilon, b+\varepsilon)) = \lim_{\lambda \to \infty} \frac{\int_{b-\varepsilon}^{b+\varepsilon} e^{-\lambda w(x)} \, dx}{\int_{\sigma(\rho)}^{0} e^{-\lambda w(x)} \, dx + e^{\lambda/2}} = 1, \tag{5.8}
\]
since
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \int_{b-\varepsilon}^{b+\varepsilon} e^{-\lambda w(x)} \, dx = -V > \frac{1}{2},
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \int_{(b-\varepsilon,b+\varepsilon) \cap [\sigma(\rho),0]} \, dx < -V.
\]
Recall that $\sigma(\rho') < b < 0$. In the following, $\varepsilon > 0$ is chosen to be small enough that $\sigma(\rho') < b-\varepsilon$ and $b+\varepsilon < 0$. Let $\{X^R_\lambda(t), t \geq 0\}$ be a reflecting $\mathcal{L}_{\lambda,w}$-diffusion process on $I'_\lambda$ with initial distribution $m'_{\lambda,w}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is a stationary process. From (5.8), it follows that
\[
\lim_{\lambda \to \infty} \tilde{\mathbb{P}}[b - \varepsilon < X^R_\lambda(0) < b + \varepsilon] = 1 \tag{5.9}
\]
and that, for any $r_1$ and $r_2$ satisfying $0 < r_1 < r_2$,
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} \tilde{\mathbb{P}}[b - \varepsilon < X^R_\lambda(e^{\lambda r}) < b + \varepsilon] = 1. \tag{5.10}
\]
By (5.9), (5.10), and the comparison theorem for one-dimensional diffusion processes, we deduce that
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P^0_{\lambda,w}[X(e^{\lambda r}) < b + \varepsilon] = 1, \tag{5.11}
\]
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P^0_{\lambda,w}[X(e^{\lambda r}) > b - \varepsilon] = 1, \tag{5.12}
\]
for any $r_1$ and $r_2$ satisfying $0 < r_1 < r_2 < 1 + \delta_1$. 


Now, by (5.4), we notice that
\[
\lim_{\lambda \to \infty} P^0_{\lambda w} \{ \tau (b - \epsilon) < e^{\lambda \theta_1} \} = 1 \quad \text{for some } \theta_1 \in (0, 1).
\] (5.13)

Choose any \( \delta \in (0, \delta_1 \wedge (1 - \theta_1)) \). Then, by the strong Markov property of \( \{ X(t), t \geq 0, P_{\lambda w} \} \), (5.11), and (5.13), for any \( r_1 \) and \( r_2 \) satisfying \( 1 - \delta < r_1 < r_2 < 1 + \delta \) we obtain
\[
\lim_{\lambda \to \infty} \inf_{r \in [r_1, r_2]} P^0_{\lambda w} \{ X(e^{\lambda r}) < b + \epsilon \} = 1.
\] (5.14)

Combining (5.12) and (5.14) yields (2.1) for \( V' - V < 1 \). This completes the proof of Proposition 2.1.

**Proof of Theorem 1.3.** Using Lemma 2.1 and (1.2), we have
\[
P \{ P^0_{\lambda w} \{ |\lambda^{-2} X(e^{\lambda r}) - b(w_\lambda)| < \epsilon \} > 1 - \epsilon, B \}
= P \{ P^0_{\lambda w} \{ |X(e^{\lambda r}) - b(w)| < \epsilon \} > 1 - \epsilon, B \},
\] (5.15)
where \( r(\lambda) = \frac{1}{2} - \frac{1}{\lambda} \log \lambda \). The right-hand side of (5.15) converges to \( \frac{1}{2} \) as \( \lambda \to \infty \), by virtue of Theorem 2.2, which is derived from Proposition 2.1 as we remarked above. We hence obtain Theorem 1.3.

### 6. Proof of Theorem 1.4

We first present a lemma in preparation for the proof of Theorem 1.4.

**Lemma 6.1.** Let \( r \) be a real-valued function of \( \lambda > 0 \) such that \( r(\lambda) \to 1 \) (as \( \lambda \to \infty \)). Then, for almost all \( w \in W \) (with respect to \( P \)) and any \( \epsilon > 0 \),
\[
\lim_{\lambda \to \infty} P^0_{\lambda w} \{ e^{\lambda (H - \epsilon)} \leq \max_{0 \leq s \leq e^{\lambda r(\lambda)}} X(s) \leq e^{\lambda (H + \epsilon)} \} = 1.
\]

**Proof.** We prove that, for almost all \( w \in W \),
\[
\lim_{\lambda \to \infty} P^0_{\lambda w} \{ \tau(e^{\lambda (H - \epsilon)}) < e^{\lambda r(\lambda)} < \tau(e^{\lambda (H + \epsilon)}) \} = 1,
\] (6.1)
which clearly implies the lemma. Let \( w \in W \) and, for any \( \epsilon \) such that \( 0 < \epsilon < H(w) \), let
\[
M' = \begin{cases} 
\sigma(\frac{1}{2} - \epsilon/2) & \text{if } w \in A, \\
\sup \left\{ x < \sigma(-\frac{1}{2}) : w(x) = \min_{x \leq y \leq \sigma(-\frac{1}{2})} w(y) = 1 - \epsilon/2 \right\} & \text{if } w \in B.
\end{cases}
\]
Then we see that
\[
\lim_{\lambda \to \infty} P^0_{\lambda w} \{ \tau(e^{\lambda (H - \epsilon)}) < \tau(M') \} = \lim_{\lambda \to \infty} \frac{\int_{M'} e^{\lambda w(x)} \, dx}{\int_{M'} e^{\lambda w(x)} \, dx + e^{\lambda (H - \epsilon)}} = 1,
\] (6.2)
since
\[
\lim_{\lambda \to \infty} \lambda^{-1} \log \int_{M'} e^{\lambda w(x)} \, dx = \max_{M' \leq x \leq 0} w(x) \geq H - \frac{\epsilon}{2} > H - \epsilon.
\]
Moreover, by applying Lemma 4.1 with $a = M'$, we have
\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau (M') < e^{\lambda r(\lambda)} \} = 1.
\tag{6.3}
\]
Combining (6.2) and (6.3) yields
\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau (e^{\lambda(H-\varepsilon)}) < e^{\lambda r(\lambda)} \} = 1.
\tag{6.4}
\]
Next, for any $\varepsilon > 0$ we let
\[
M'' = \begin{cases} 
\sigma \left( \frac{1}{2} + \varepsilon / 2 \right) & \text{if } w \in A, \\
\sup \{ x < \sigma (-1/2) : w(x) - \min_{x \leq y \leq \sigma(-1/2)} w(y) = 1 + \varepsilon / 2 \} & \text{if } w \in B.
\end{cases}
\]
Then we have
\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau (e^{\lambda(H+\varepsilon)}) > \tau (M'') \} = \lim_{\lambda \to \infty} \frac{e^{\lambda(H+\varepsilon)}}{\int_{M''} e^{\lambda w(x)} \, dx + e^{\lambda(H+\varepsilon)}} = 1,
\tag{6.5}
\]
since
\[
\lim_{\lambda \to \infty} \lambda^{-1} \log \int_{M''} e^{\lambda w(x)} \, dx = \max_{M'' \leq x \leq 0} w(x) \leq H + \frac{\varepsilon}{2} < H + \varepsilon.
\]
Moreover, an application of Lemma 4.1 with $a = M''$ yields
\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ \tau (M'') > e^{\lambda r(\lambda)} \} = 1.
\tag{6.6}
\]
By (6.5) and (6.6), we obtain
\[
\lim_{\lambda \to \infty} P_{\lambda w}^0 \{ e^{\lambda r(\lambda)} < \tau (e^{\lambda(H+\varepsilon)}) \} = 1,
\]
which, combined with (6.4), proves (6.1). The proof of Lemma 6.1 is thus complete.

**Proof of Theorem 1.4.** By Lemma 2.1 and (1.2), we have
\[
\int_{\mathbb{W}} P(dw) P_{\lambda w}^0 \left\{ \log \max_{0 \leq s \leq 1} X(s) - H(w) > \varepsilon \right\} = \int_{\mathbb{W}} P(dw) P_{\lambda w}^0 \left\{ \left| 2 \log \lambda + \log \max_{0 \leq s \leq e^{\lambda r(\lambda)}} X(s) - H(w) \right| > \varepsilon \right\},
\tag{6.7}
\]
where $r(\lambda) = 1 - 4\lambda^{-1} \log \lambda$. The right-hand side of (6.7) converges to 0 as $\lambda \to \infty$, by Lemma 6.1. We hence obtain Theorem 1.4.

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