INFINITE FAMILIES OF CONGRUENCES MODULO 3 AND 9 FOR BIPARTITIONS WITH 3-CORES

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Abstract

Let $A_3(n)$ denote the number of bipartitions of $n$ with 3-cores. Recently, Lin ['Some results on bipartitions with 3-core', J. Number Theory 139 (2014), 44–52] established some congruences modulo 4, 5, 7 and 8 for $A_3(n)$. In this paper, we prove several infinite families of congruences modulo 3 and 9 for $A_3(n)$ by employing two identities due to Ramanujan.

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1. Introduction

A partition $\lambda$ of a positive integer $n$ is any nonincreasing sequence of positive integers whose sum is $n$. For a positive integer $t \geq 2$, a partition is said to be a $t$-core partition if its Ferrers graph does not contain a hook whose length is a multiple of $t$. For any nonnegative integer $n$, let $a_t(n)$ denote the number of $t$-core partitions of $n$. From [10], the generating function for $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1^t}.$$  

Here and throughout this paper, for any positive integer $k$, $f_k$ is defined by

$$f_k := \prod_{n=1}^{\infty} (1 - q^{kn}).$$

Numerous properties of $a_t(n)$ have been extensively studied (see, for example, [2, 6, 7, 9–12, 17]).

A bipartition $(\lambda, \mu)$ of $n$ is a pair of partitions $(\lambda, \mu)$ such that the sum of all of the parts is $n$. Arithmetic properties of the number of bipartitions of $n$ have been established (see, for example, [1, 5, 8, 16]). Numerous arithmetic properties have been...
proved for bipartitions with certain restrictions on each partition (see, for example, [4, 13–15, 18]). A bipartition with \( t \)-cores is a pair of partitions \((\lambda, \mu)\) such that \( \lambda \) and \( \mu \) are both \( t \)-cores. Let \( A_t(n) \) denote the number of bipartitions of \( n \) with \( t \)-cores. It is easy to see that the generating function of \( A_t(n) \) is

\[
\sum_{n=0}^{\infty} A_t(n)q^n = \frac{f_{2t}}{f_1}.
\] (1.1)

Very recently, Lin [14] discovered some congruences modulo 4, 5, 7 and 8. For example, he proved that for \( n \geq 0 \) and \( \alpha \geq 0 \),

\[
A_3\left(4^{\alpha+1}n + \frac{11 \times 4^\alpha - 2}{3}\right) \equiv 0 \pmod{4},
\]

and

\[
A_3\left(16^{\alpha+1}n + \frac{8 \times 16^\alpha - 2}{3}\right) \equiv 0 \pmod{5}.
\]

The aim of this paper is to prove several infinite families of congruences modulo 3 and 9 for \( A_3(n) \). Our main result can be stated as follows.

**Theorem 1.1.** For all \( \alpha \geq 0 \) and \( n \geq 0 \),

\[
A_3\left(64^\alpha n + \frac{2(64^\alpha - 1)}{3}\right) \equiv A_3(n) \pmod{3},
\] (1.2)

\[
A_3\left(64^{\alpha+1}n + \frac{2 \times 6^{\alpha+5} - 2}{3}\right) \equiv 0 \pmod{3},
\] (1.3)

\[
A_3\left(4^{9\alpha} n + \frac{2(4^{9\alpha} - 1)}{3}\right) \equiv A_3(n) \pmod{9},
\] (1.4)

\[
A_3\left(4^{9(\alpha+1)} n + \frac{2 \times 18^{\alpha+17} - 2}{3}\right) \equiv 0 \pmod{9}.
\] (1.5)

Thanks to (1.4), we can deduce the following corollary.

**Corollary 1.2.** For all integers \( \alpha \geq 0 \),

\[
A_3\left(r_i \times 4^{9\alpha} + \frac{2(4^{9\alpha} - 1)}{3}\right) \equiv i \pmod{9},
\]

where \( r_0 = 17, r_1 = 9, r_2 = 1, r_3 = 10, r_4 = 3, r_5 = 2, r_6 = 5, r_7 = 11 \) and \( r_8 = 4 \).

### 2. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we present a proof of Theorem 1.1. We first present the following lemma.
Lemma 2.1. Let $a$ and $b$ be two integers. If
\[
\sum_{n=0}^{\infty} c(n)q^n \equiv af_1^{16} + bq f_2^{24} f_8^{-1} \quad (\text{mod } 9),
\]
then
\[
\sum_{n=0}^{\infty} c(4n + 2)q^n \equiv (5a + 8b)f_1^{16} + 4aq f_2^{24} f_8^{-1} \quad (\text{mod } 9).
\]

Proof. The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt’s book [3, page 40]:
\[
f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q f_2^2 f_8^4 f_4^{14}
\]
and
\[
\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^4 f_8^8} + 4q f_2^2 f_8^4 f_4^{10}
\]
Substituting (2.3) and (2.4) into (2.1),
\[
\sum_{n=0}^{\infty} c(n)q^n \equiv a\left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q f_2^2 f_8^4 f_4^{14}\right)^2 + b q f_2^{24}\left(\frac{f_4^{14}}{f_2^4 f_8^8} + 4q f_2^2 f_8^4 f_4^{10}\right)
\]
\[
\equiv a f_4^{10} f_2^2 f_8^4 + (b - 7a)q f_4^{28} f_2^8 f_8^4 + (6a + 8b)q^2 f_4^{16}
\]
\[
+ (7b - 4a)q^3 f_2^4 f_4^4 f_8^8 + 4aq^4 f_2^8 f_8^4 f_4^{16} \quad (\text{mod } 9).
\]
Extracting the terms with even powers of $q$ on both sides of (2.5), then replacing $q^2$ by $q$,
\[
\sum_{n=0}^{\infty} c(2n)q^n \equiv a f_4^{20} f_2^2 f_8^{16} + (6a + 8b)q f_2^{16} + 4aq^2 f_2 f_4^{16} f_8^{-4} \quad (\text{mod } 9).
\]
Substituting (2.3) and (2.4) into (2.6),
\[
\sum_{n=0}^{\infty} c(2n)q^n \equiv a f_4^{20} f_2^{16} f_4^{14} f_8^4 + 4q^2 f_2^4 f_4^{14} f_8^{-2} + (6a + 8b)q f_2^{16}
\]
\[
+ 4aq^2 f_2^{16} f_4^{10} f_2^4 f_8^4 f_4^{14} f_8^{-2}
\]
\[
\equiv a f_4^{12} f_2^{12} f_8^{12} + f_2^8 f_4^{16} f_8^{20} + 7aq^{2} f_2^{20} f_4^{16} f_8^{8}
\]
\[
+ 4aq^{2} f_2^{36} f_4^{12} f_8^{8} + 4aq^{3} f_2^8 f_4^{24} f_8^4 + aq^{4} f_2^{12} f_4^{12} f_8^{8} \quad (\text{mod } 9).
\]
Congruence (2.2) follows from (2.7). This completes the proof. □
We are now ready to prove Theorem 1.1 by using Lemma 2.1.

**Proof of Theorem 1.1.** Setting \( t = 3 \) in (1.1),

\[
\sum_{n=0}^{\infty} A_3(n) q^n = \frac{f_3^6}{f_1^2}.
\]  
(2.8)

By the binomial theorem, it is easy to check that

\[
f_3^6 \equiv f_1^{18} \pmod{9}.
\]  
(2.9)

Combining (2.8) and (2.9),

\[
\sum_{n=0}^{\infty} A_3(n) q^n \equiv f_1^{16} \pmod{9}.
\]  
(2.10)

Setting \( a = 1, b = 0 \) in (2.1) and using Lemma 2.1 and (2.10), we see that

\[
\sum_{n=0}^{\infty} A_3(4n+2) q^n \equiv 5 f_1^{16} + 4q \frac{f_2^{24}}{f_1^8} \pmod{9}.
\]  
(2.11)

If we apply Lemma 2.1 repeatedly, starting from (2.11),

\[
\sum_{n=0}^{\infty} A_3(16n+10) q^n \equiv 3 f_1^{16} + 2q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.12)

\[
\sum_{n=0}^{\infty} A_3(64n+42) q^n \equiv 4 f_1^{16} + 3q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.13)

\[
\sum_{n=0}^{\infty} A_3(256n+170) q^n \equiv 8 f_1^{16} + 7q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.14)

\[
\sum_{n=0}^{\infty} A_3(1024n+682) q^n \equiv 6 f_1^{16} + 5q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.15)

\[
\sum_{n=0}^{\infty} A_3(4096n+2730) q^n \equiv 7 f_1^{16} + 6q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.16)

\[
\sum_{n=0}^{\infty} A_3(16384n+10922) q^n \equiv 2 f_1^{16} + q \frac{f_2^{24}}{f_1^8} \pmod{9},
\]  
(2.17)

\[
\sum_{n=0}^{\infty} A_3(65536n+43690) q^n \equiv 8q \frac{f_2^{24}}{f_1^8} \pmod{9}
\]  
(2.18)

and

\[
\sum_{n=0}^{\infty} A_3(262144n+174762) q^n \equiv f_1^{16} \pmod{9}.
\]  
(2.19)
In view of (2.10) and (2.13), we see that for \( n \geq 0, \)
\[
A_3(64n + 42) \equiv A_3(n) \pmod{3}. \quad (2.20)
\]
Congruence (1.2) follows from (2.20) and mathematical induction.

By (2.12), we see that
\[
\sum_{n=0}^{\infty} A_3(16n + 10)q^n \equiv 2q\frac{f_{12}^{24}}{f_{8}^{4}} \pmod{3}. \quad (2.21)
\]
Substituting (2.4) into (2.21),
\[
\sum_{n=0}^{\infty} A_3(16n + 10)q^n \equiv 2q\frac{f_{28}^{4}}{f_{8}^{2}} + q^2 f_{4}^{16} + 2q^3 f_{4}^{4} f_{8}^{8} \pmod{3},
\]
which implies that for \( n \geq 0, \)
\[
A_3(64n + 10) \equiv 0 \pmod{3}. \quad (2.22)
\]
Replacing \( n \) by \( 64n + 10 \) in (1.2) and employing (2.22), we arrive at (1.3).

It follows from (2.10) and (2.19) that for \( n \geq 0, \)
\[
A_3(262144n + 174762) \equiv A_3(n) \pmod{9}. \quad (2.23)
\]
Congruence (1.4) follows from (2.23) and mathematical induction.

Substituting (2.4) into (2.18),
\[
\sum_{n=0}^{\infty} A_3(65536n + 43690)q^n \equiv 8q\frac{f_{28}^{4}}{f_{8}^{2}} + q^2 f_{4}^{16} + 2q^3 f_{4}^{4} f_{8}^{8} \pmod{9},
\]
which implies that for \( n \geq 0, \)
\[
A_3(262144n + 43690) \equiv 0 \pmod{9}.
\]
Replacing \( n \) by \( 262144n + 43690 \) in (1.4), we get (1.5).

To conclude this paper, we give a proof of Corollary 1.2.

**Proof of Corollary 1.2.** Setting \( n = r_i \) in (1.4) and then employing the facts
\( A_3(1) = 2, \ A_3(2) = 5, \ A_3(3) = 4, \ A_3(4) = 8, \ A_3(5) = 6, \ A_3(9) = 10, \ A_3(10) = 21, \)
\( A_3(11) = 16, A_3(17) = 18, \) we can deduce Corollary 1.2. This completes the proof. \( \square \)
References


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