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A universal semigroup

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J.H. Michael recently proved that there exists a metric semigroup U such that every compact metric semigroup with property P is topologically isomorphic to a subsemigroup of U ; where a semigroup S has property P if and only if for each x, y in S , $x \neq y$, there is a z in S such that $xz \neq yz$ or $zx \neq zy$.

A stronger result is proved here more simply. It is shown that for any set A of metric semigroups there exists a metric semigroup U such that each S in A is topologically isomorphic to a subsemigroup of U . In particular this is the case when A is the class of all separable metric semigroups, or more particularly the class of all compact metric semigroups.

THEOREM. *Let A be any set of metric semigroups. Then there exists a metric semigroup U such that for each S in A there is a mapping of S into U which is both a semigroup isomorphism and a homeomorphism.*

Proof. Let $A = \{S_i : i \in I\}$. Without loss of generality, assume $\sup_{x,y \in S_i} [d_i(x, y)] = 1$, for each i in I , where d_i is the metric in S_i . Adjoin an identity e_i to each S_i to give a semigroup T_i . Define a metric d_i^* on T_i as follows:

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$$d_i^*(x, y) = \begin{cases} d(x, y), & \text{if } x \text{ and } y \in S_i ; \\ 1, & \text{if } x = e_i \text{ and } y \in S_i ; \\ 1, & \text{if } x \in S_i \text{ and } y = e_i ; \\ 0, & \text{if } x = y = e_i . \end{cases}$$

It can be verified easily that d_i^* is a metric and furthermore this metric is compatible with the semigroup structure; that is each T_i is a metric semigroup.

Let U be the direct sum of $\{T_i : i \in I\}$. (See [1].) Define a metric d on U as follows:

Let x and y be any distinct elements of U. Then $x = \{x_i\}$ and $y = \{y_i\}$, where all but a finite number of x_i and y_i are equal to e_i. Put $d(x, y)$ equal to the sum of the non-zero $d_i^*(x_i, y_i)$, $i \in I$.

It is a routine matter to check that d is a metric and U with this metric is a metric semigroup. Finally, note that U does have the required properties.

COROLLARY. *Let A be the class of all metric separable semigroups or the class of all compact metric semigroups. Then there exists a metric semigroup U such that for each S in U, there is a mapping of S into U which is both a semigroup isomorphism and a homeomorphism.*

Proof. We only have to verify that A is a set. This follows immediately from the theorem of Urysohn ([2], p. 125) which states that every separable metric space is homeomorphic to a subset of the Hilbert cube.

References

[1] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Vol. II (Math. Surveys 7 (II), Amer. Math. Soc., Providence, Rhode Island, 1967).

[2] John L. Kelley, *General topology* (Van Nostrand, Toronto, New York, London, 1955).

[3] J.H. Michael, "A universal semigroup", *J. Austral. Math. Soc.* 11 (1970), 216-220.

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