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Some elementary inequalities in function theory

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1. Let $0 < r < 1$ and let $f(z)$ be regular[1] for $|z| \leqq 1$.
Then from Cauchy's integral

$$(1) \qquad f(r) = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)\,\frac{d\zeta}{\zeta - r} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\frac{e^{i\theta}\,d\theta}{e^{i\theta} - r}$$

we have the inequality

$$(2) \qquad (1 - r)\,|f(r)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta.$$

This inequality is clearly not the best possible, for the factor $e^{i\theta} - r$ is varying in modulus and we took its minimum modulus. But Cauchy's integral is not the only integral with the value $f(r)$. We have only to replace $1/(\zeta - r)$ by a function regular for $|z| \leqq 1$ except for a simple pole at $\zeta = r$ to obtain another representation of $f(r)$. For example

$$(3) \quad (1 - r^2)\,f(r) = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)\,\frac{1 - r\zeta}{\zeta - r}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\frac{1 - re^{i\theta}}{1 - re^{-i\theta}}\,d\theta,$$

from which we deduce the inequality

$$(4) \qquad (1 - r^2)\,|f(r)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta.$$

[1] It is sufficient in what follows if $f(z)$ is regular for $|z| < 1$ and if $\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$ is bounded for $0 \leqq r < 1$, but the simpler case adequately illustrates our arguments.

This result is best possible and equality is attained by the function $f(z) = 1/(1 - rz)^2$, for with this function

$$(5) \qquad (1 - r^2) f(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$

and the integrand is always positive.

If we apply the inequality to $f(ze^{i\alpha})$ we have

$$(6) \qquad (1 - r^2) |f(re^{i\alpha})| \leqq \frac{1}{2\pi} \int_\alpha^{2\pi + \alpha} |f(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta,$$

or for $|z| < 1$

$$(7) \qquad (1 - |z|^2) |f(z)| \leqq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta.$$

2. We may expect to find similar inequalities involving $f'(z)$ by using a suitable auxiliary function with a double pole. The function

$$(8) \qquad \left(\frac{1 - r\zeta}{\zeta - r}\right)^2 = \frac{(1 - r^2)^2 - 2r(1 - r^2)(\zeta - r) + r^2 (\zeta - r)^2}{(\zeta - r)^2}$$

is of constant modulus unity when $|z| = 1$; and hence by calculating the residue of the integrand we find

$$(9) \qquad (1 - r^2)^2 f'(r) - 2r(1 - r^2) f(r) = \frac{1}{2\pi i} \int_{|\zeta| = 1} f(\zeta) \left(\frac{1 - r\zeta}{\zeta - r}\right)^2 d\zeta,$$

and there follows the inequality

$$(10) \qquad |(1 - r^2)^2 f'(r) - 2r(1 - r^2) f(r)| \leqq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta.$$

It would of course be of more interest to calculate the best possible inequality of this type for $f'(z)$ itself, and this is quite an easy deduction.. If we apply the last inequality to $f(z).h(z)$ and choose $h(z)$ so that $|h(e^{i\theta})| = 1$, there results

$$(11) \qquad |(1 - r^2)^2 f'(r) h(r) + (1 - r^2) f(r) \{(1 - r^2) h'(r) - 2rh(r)\}| \leqq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta.$$

This will give us the required inequality if

$$(12) \qquad (1 - r^2) h'(r) - 2rh(r) = 0.$$

The condition $|h(e^{i\theta})| = 1$ will be satisfied if

$$(13) \qquad h(z) = \frac{z - y}{1 - zy} \qquad (-1 < y < 1).$$

For this function

$$\frac{h'(r)}{h(r)} = \frac{1}{r-y} + \frac{y}{1-ry} = \frac{1-y^2}{(r-y)(1-ry)}, \tag{14}$$

so we must choose y to satisfy the quadratic equation

$$(1-r^2)(1-y^2) = 2r(r-y)(1-ry). \tag{15}$$

The roots of this equation are $y = r \pm (1-r^2)/\sqrt{(1+r^2)}$. The root $y = r - (1-r^2)/\sqrt{(1+r^2)}$ will always lie in the range $-1 < y < 1$ if r lies in $0 < r < 1$. With this value of y we find $h(r) = 1/[r + \sqrt{(1+r^2)}]$, and so obtain the inequality

$$\frac{(1-r^2)^2}{r+\sqrt{(1+r^2)}} |f'(r)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\, d\theta. \tag{16}$$

This inequality is best possible and equality can be attained. The equation from which it was deduced can be written in the form

$$\begin{aligned} \frac{(1-r^2)^2}{r+\sqrt{(1+r^2)}} f'(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-re^{i\theta}}{e^{i\theta}-r}\right)^2 \left(\frac{e^{i\theta}-y}{1-ye^{i\theta}}\right) e^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-re^{i\theta}}{1-re^{-i\theta}}\right)^2 \left(\frac{1-ye^{-i\theta}}{1-ye^{i\theta}}\right) d\theta \end{aligned} \tag{17}$$

and the integrand will be positive if, for example,

$$f(z) = \frac{(1-yz)^2}{(1-rz)^4}, \qquad y = r - (1-r^2)/\sqrt{(1+r^2)}. \tag{18}$$

Hence for this function our inequality is an equality.

The restriction to real z is clearly unnecessary as before, so we can state that for $|z| < 1$

$$\frac{(1-|z|^2)^2}{|z| + \sqrt{(1+|z|^2)}} |f'(z)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\, d\theta. \tag{19}$$

($z = 0$ obviously need not be excluded.) Similar results could be obtained for higher derivatives but would involve greater algebraic difficulties.

Added in proof.—The inequality (7) was proved less simply by Egervary, *Math. Annalen*, **99** (1928), 542-561, and then by Landau, *Math. Zeits.*, **29** (1929), 461. The present authors have in preparation a general theory of these and similar problems.

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