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Some elementary inequalities in function theory

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1. Let $0 < r < 1$ and let $f(z)$ be regular¹ for $|z| \leq 1$.

Then from Cauchy's integral

$$(1) \quad f(r) = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \frac{d\zeta}{\zeta - r} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta} d\theta}{e^{i\theta} - r}$$

we have the inequality

$$(2) \quad (1 - r) |f(r)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

This inequality is clearly not the best possible, for the factor $e^{i\theta} - r$ is varying in modulus and we took its minimum modulus. But Cauchy's integral is not the only integral with the value $f(r)$. We have only to replace $1/(\zeta - r)$ by a function regular for $|z| \leq 1$ except for a simple pole at $\zeta = r$ to obtain another representation of $f(r)$. For example

$$(3) \quad (1 - r^2) f(r) = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \frac{1 - r\zeta}{\zeta - r} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - re^{i\theta}}{1 - re^{-i\theta}} d\theta,$$

from which we deduce the inequality

$$(4) \quad (1 - r^2) |f(r)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

¹ It is sufficient in what follows if $f(z)$ is regular for $|z| < 1$ and if $\int_0^{2\pi} |f(re^{i\theta})| d\theta$ is bounded for $0 \leq r < 1$, but the simpler case adequately illustrates our arguments.

This result is best possible and equality is attained by the function $f(z) = 1/(1 - rz)^2$, for with this function

$$(5) \quad (1 - r^2)f(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$

and the integrand is always positive.

If we apply the inequality to $f(ze^{ia})$ we have

$$(6) \quad (1 - r^2)|f(re^{ia})| \leq \frac{1}{2\pi} \int_a^{2\pi+a} |f(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta,$$

or for $|z| < 1$

$$(7) \quad (1 - |z|^2)|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

2. We may expect to find similar inequalities involving $f'(z)$ by using a suitable auxiliary function with a double pole. The function

$$(8) \quad \left(\frac{1 - r\zeta}{\zeta - r}\right)^2 = \frac{(1 - r^2)^2 - 2r(1 - r^2)(\zeta - r) + r^2(\zeta - r)^2}{(\zeta - r)^2}$$

is of constant modulus unity when $|z| = 1$; and hence by calculating the residue of the integrand we find

$$(9) \quad (1 - r^2)^2 f'(r) - 2r(1 - r^2)f(r) = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \left(\frac{1 - r\zeta}{\zeta - r}\right)^2 d\zeta,$$

and there follows the inequality

$$(10) \quad |(1 - r^2)^2 f'(r) - 2r(1 - r^2)f(r)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

It would of course be of more interest to calculate the best possible inequality of this type for $f'(z)$ itself, and this is quite an easy deduction. If we apply the last inequality to $f(z)h(z)$ and choose $h(z)$ so that $|h(e^{i\theta})| = 1$, there results

$$(11) \quad |(1 - r^2)^2 f'(r)h(r) + (1 - r^2)f(r)\{(1 - r^2)h'(r) - 2rh(r)\}| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

This will give us the required inequality if

$$(12) \quad (1 - r^2)h'(r) - 2rh(r) = 0.$$

The condition $|h(e^{i\theta})| = 1$ will be satisfied if

$$(13) \quad h(z) = \frac{z - y}{1 - zy} \quad (-1 < y < 1).$$

For this function

$$(14) \quad \frac{h'(r)}{h(r)} = \frac{1}{r-y} + \frac{y}{1-ry} = \frac{1-y^2}{(r-y)(1-ry)},$$

so we must choose y to satisfy the quadratic equation

$$(15) \quad (1-r^2)(1-y^2) = 2r(r-y)(1-ry).$$

The roots of this equation are $y = r \pm (1-r^2)/\sqrt{1+r^2}$. The root $y = r - (1-r^2)/\sqrt{1+r^2}$ will always lie in the range $-1 < y < 1$ if r lies in $0 < r < 1$. With this value of y we find $h(r) = 1/[r + \sqrt{1+r^2}]$, and so obtain the inequality

$$(16) \quad \frac{(1-r^2)^2}{r + \sqrt{1+r^2}} |f'(r)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

This inequality is best possible and equality can be attained. The equation from which it was deduced can be written in the form

$$(17) \quad \frac{(1-r^2)^2}{r + \sqrt{1+r^2}} f'(r) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-re^{i\theta}}{e^{i\theta}-r} \right)^2 \left(\frac{e^{i\theta}-y}{1-ye^{i\theta}} \right) e^{i\theta} d\theta \\ = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \left(\frac{1-re^{i\theta}}{1-re^{-i\theta}} \right)^2 \left(\frac{1-ye^{-i\theta}}{1-ye^{i\theta}} \right) d\theta$$

and the integrand will be positive if, for example,

$$(18) \quad f(z) = \frac{(1-yz)^2}{(1-rz)^4}, \quad y = r - (1-r^2)/\sqrt{1+r^2}.$$

Hence for this function our inequality is an equality.

The restriction to real z is clearly unnecessary as before, so we can state that for $|z| < 1$

$$(19) \quad \frac{(1-|z|^2)^2}{|z| + \sqrt{1+|z|^2}} |f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

($z = 0$ obviously need not be excluded.) Similar results could be obtained for higher derivatives but would involve greater algebraic difficulties.

Added in proof.—The inequality (7) was proved less simply by Egervary, *Math. Annalen*, 99 (1928), 542-561, and then by Landau, *Math. Zeits.*, 29 (1929), 461. The present authors have in preparation a general theory of these and similar problems.

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