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A further twist to helicity

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In fluid dynamics, helicity measures the correlation between velocity and its curl, vorticity, over a spatial volume. Under 'ideal' conditions (vanishing viscosity and either homogeneneous density or when pressure may be regarded as a function of density alone), helicity is a topological invariant closely related to the knottedness of vortex lines (Moffatt 1969 *J. Fluid Mech.* **35** (1), 117–129). Helicity is conserved following a material volume for compact vorticity distributions, i.e. when the vorticity field is tangent to the surface of the volume. There is a related helicity invariant in ideal magnetohydrodynamics involving the correlation between the magnetic potential and its curl, the magnetic field. Helicity is a fragile invariant in the sense that relaxing any one of the ideal conditions results in non-conservation. Unlike energy and enstrophy (mean-square vorticity), helicity is not positive (or sign) definite. Viscous diffusion can create both positive and negative helicity when vortex lines reconnect, something which is topologically forbidden in an ideal fluid where vortex lines move as material curves. Moreover, variable density or more generally compressibility destroys conservation and weakens the association between helicity and vortex-line topology. Furthermore, in compressible flows, the velocity field is not entirely determined from the vorticity field. A recent paper by Boutros & Gibbon (2025) *J. Fluid Mech.* in this journal explains how one can extend the definition of helicity to control and limit the non-conservation of helicity. This offers a promising way forward in using helicity to characterise flow properties in computational studies of high Reynolds number flows.

Key words: topological fluid dynamics

1. Context

Vorticity plays a fundamental role in the evolution of many fluid flows, particularly when the effects of compressibility and viscosity may be neglected. Under these conditions, the velocity field $\boldsymbol{u}$ is uniquely determined by the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ throughout the domain via the Biot–Savart integral ('uncurling' the definition of vorticity), together with a potential flow when impermeable boundaries are present (the latter also depends only on $\boldsymbol{\omega}$ throughout the domain). In an unbounded three-dimensional domain, $\mathbb{R}^3$, we have explicitly

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}(\boldsymbol{x}', t) \times (\boldsymbol{x}' - \boldsymbol{x})\ \mathrm{d}V'}{|\boldsymbol{x}' - \boldsymbol{x}|^3}, \tag{1.1}$$

showing that the velocity depends on vorticity in a non-local manner. However, the evolution of vorticity not only depends on velocity but also, in general, on pressure P and density ρ

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \frac{\nabla \rho \times \nabla P}{\rho^2} . \tag{1.2}$$

The last term is commonly referred to as 'baroclinic torque', and contributes to the local rate of change of vorticity when isosurfaces of pressure and density are misaligned. In atmospheric dynamics, this term is responsible for the development of an onshore sea breeze when the water temperature is cooler than the land temperature. When (isotropic) viscosity is present, there is an additional term, $\nu \nabla^2 \boldsymbol{\omega}$, on the right-hand side of (1.2), where ν is the kinematic viscosity coefficient.

In general, even in the absence of diffusion of any form, the baroclinic torque in (1.2) implies that vorticity does not remain compact (localised in space) even if it was initially so. For an incompressible flow, the density ρ remains constant on fluid particles yet may change locally by advection

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0 . \tag{1.3}$$

Pressure itself is determined from a Poisson equation arising from taking the divergence of Euler's momentum equation, and depends non-locally on both $\boldsymbol{u}$ and ρ. The key point is that the baroclinic torque does not generally vanish. Hence, spatial variations in density may generate vorticity throughout space, especially when gravity is present and the flow is stratified (then density variations give rise to internal waves or gravity waves). The exception is when both density and vorticity are compact in the same region of space. In this case, they remain compact and the region moves as a material volume, even though vortex lines are not material curves in this case.

While vorticity flux is not conserved under the action of the baroclinic torque in (1.2), there is a scalar quantity q called 'potential vorticity' that is. One may show directly that $q = \boldsymbol{\omega} \cdot \nabla \rho$ obeys the same material conservation equation, (1.3), satisfied by density. Conservation of q is a direct consequence of the fact that circulation $\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}$ around any closed material curve C lying on a surface of constant density is invariant. This follows from manipulating the momentum equation, using the fact that $\rho =$ constant on C. Stokes' theorem shows that circulation may also be interpreted as the vorticity flux through a surface S bounded by C

$$\Gamma = \iint_S \boldsymbol{\omega} \cdot \boldsymbol{n}\ \mathrm{d}S , \tag{1.4}$$

where $\boldsymbol{n}$ is a unit normal directed out of S, i.e. parallel to $\nabla \rho$. Shrinking C to a point then shows that the potential vorticity q is conserved pointwise. (One uses the fact that the volume $\mathrm{d}V$ between two constant density surfaces ρ and $\rho + \mathrm{d}\rho$ is conserved due to incompressibility, together with $\mathrm{d}V = \mathrm{d}S\ \mathrm{d}n$, $\boldsymbol{n} = \nabla \rho / |\nabla \rho|$ and $\mathrm{d}\rho = |\nabla \rho|\ \mathrm{d}n$; combining these relations gives $\Gamma = q\ \mathrm{d}V / \mathrm{d}\rho$ and hence q is conserved because Γ

and $\mathrm{d}V$ are, while $\mathrm{d}\rho$ is just a constant.) In fact, potential vorticity is conserved even for compressible but adiabatic flows, and has proven to be an important quantity for interpreting many atmospheric and oceanic phenomena; see McIntyre (2015) for a comprehensive historical review.

Helicity, a word first coined by Moffatt (1969) from the Greek word 'helix' ($\varepsilon\lambda\iota\xi$) meaning 'coiled' or 'twisted', was originally introduced into fluid dynamics for ideal fluids in which the baroclinic torque vanishes in the vorticity equation (Moreau 1961; Moffatt 2018). Then either $\rho =$ constant in space, or pressure is functionally related to density, $P = P(\rho)$, a situation referred to as a 'barotropic' fluid. The resulting vorticity equation is then formally identical to the equation governed by an infinitesimal segment of a material line, implying that vortex lines are themselves material. This means that the circulation, i.e. the flux of vorticity, through any material surface is invariant. Hence, a vortex tube, whose surface consists of vortex lines lying tangent to it, has constant circulation along its length as it moves and deforms. Stretching of the tube necessarily shrinks its cross-sectional area to preserve volume, thereby intensifying the vorticity magnitude.

Helicity was originally defined by

$$H = \iiint_V \boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\omega} \ \mathrm{d}V \ , \tag{1.5}$$

where V can be any volume, but most naturally it is the material volume occupied by one or more vortex tubes. In an ideal fluid, an initially compact vorticity distribution stays that way, so it is natural to consider a material volume containing the entire vorticity distribution at all times (in any case, the integrand vanishes outside this volume). Then Moreau (1961) showed that H is invariant in an ideal fluid, a result which extends to ideal magnetohydrodynamics upon replacing $\boldsymbol{u}$ by the magnetic potential $\boldsymbol{A}$, and $\boldsymbol{\omega}$ by the magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ (Woltjer 1958; Moffatt 2018). See also the collection of works in Ricca (2001).

Moffatt (1969) noticed that H is directly related to the topology of vortex lines. To see this, we rewrite H using the Biot–Savart solution for $\boldsymbol{u}$

$$H = \frac{1}{4\pi} \iiint_V \iiint_{V'} \frac{[\boldsymbol{\omega}(\boldsymbol{x}', t) \times (\boldsymbol{x}' - \boldsymbol{x})] \boldsymbol{\cdot} \boldsymbol{\omega}(\boldsymbol{x}, t)}{|\boldsymbol{x}' - \boldsymbol{x}|^3} \ \mathrm{d}V' \ \mathrm{d}V \ . \tag{1.6}$$

Each vortex line within V is a material curve. Consider two such curves C and C', and let $\boldsymbol{r}$ and $\boldsymbol{r}'$ denote points on these curves. Remarkably, Gauss showed that the integral

$$L = \frac{1}{4\pi} \oint_C \oint_{C'} \frac{[\,\mathrm{d}\boldsymbol{r}' \times (\boldsymbol{r}' - \boldsymbol{r})] \boldsymbol{\cdot}\ \mathrm{d}\boldsymbol{r}}{|\boldsymbol{r}' - \boldsymbol{r}|^3} = \frac{1}{4\pi} \oint_C \oint_{C'} \frac{[\,\mathrm{d}\boldsymbol{r} \times\ \mathrm{d}\boldsymbol{r}'] \boldsymbol{\cdot} (\boldsymbol{r}' - \boldsymbol{r})}{|\boldsymbol{r}' - \boldsymbol{r}|^3}, \tag{1.7}$$

gives the number of times the two curves link, the so-called 'linking number'. (Note that C and C' can be the same curve.) Now consider a single closed vortex tube. Along each vortex line within the tube, the vorticity may be written in the form $\boldsymbol{\omega} = |\boldsymbol{\omega}|\ \mathrm{d}\boldsymbol{r}/\ \mathrm{d}n$, where $\mathrm{d}n$ is the differential arclength. Moreover, in the circulation integral (1.4), we may orientate the surface S locally so that the normal $\boldsymbol{n} = \boldsymbol{\omega}/|\boldsymbol{\omega}| = \ \mathrm{d}\boldsymbol{r}/\ \mathrm{d}n$. The helicity integral (1.6) thereby reduces to the product of the linking number L and the circulation $\varGamma$ squared: $H = \varGamma^2 L$. This follows because the flux of vorticity through all cross sections of the tube is $\varGamma$. For a set of vortex tubes, having circulations $\varGamma_i$ (for $i = 1,\ 2\ \ldots$), the result generalises to $H = \sum_i \sum_j \varGamma_i \varGamma_j L_{ij}$, where L_{ij} is the linking number between tube i and tube j. The upshot is that H directly measures the topology of the vorticity distribution.

2. A new twist

Recognising the importance of helicity as a measure of flow topology, Boutros & Gibbon (2025) have sought to extend the concept to non-ideal conditions, when the baroclinic torque term in (1.2) contributes to the vorticity evolution, and more generally for compressible flows. In the latter situation, the term $\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})$ in (1.2) must be replaced by $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}$, valid when $\nabla\cdot\boldsymbol{u}\neq 0$. Boutros & Gibbon (2025) extend the definition of helicity to

$$H=\iiint_V \rho\boldsymbol{u}\cdot\rho\boldsymbol{\omega}\ \mathrm{d}V=\iiint_V \boldsymbol{p}\cdot(\nabla\times\boldsymbol{p})\ \mathrm{d}V\ , \tag{2.1}$$

where $\boldsymbol{p}=\rho\boldsymbol{u}$ is the momentum vector. This form permits a topological interpretation, to the extent that $\boldsymbol{p}$ can be obtained from $\nabla\times\boldsymbol{p}$ by 'uncurling'; then one arrives at an expression for H like that given in (1.6), only now $\nabla\times\boldsymbol{p}$ takes the place of $\boldsymbol{\omega}$. This requires $\nabla\cdot\boldsymbol{p}=0$, known as the 'anelastic approximation' in atmospheric sciences. Otherwise, H is not strictly topological.

Although there appears to be no form of H that is invariant in non-ideal flows, they argue that this particular form is distinguished. First of all, $\mathrm{d}H/\mathrm{d}t$ can be written in a way which does not involve pressure, and does not rely on the specification of an equation of state. Therefore, their results apply to any thermodynamic formulation of the compressible system of equations. Second, they provide a bound on $|\mathrm{d}H/\mathrm{d}t|$. For incompressible flows (with arbitrary density), they prove that

$$|\mathrm{d}H/\mathrm{d}t|<2|q|_\infty E\ , \tag{2.2}$$

where $E=\frac{1}{2}\iiint_V \rho|\boldsymbol{u}|^2\ \mathrm{d}V$ is the kinetic energy, an invariant in the absence of gravity, and $|q|_\infty$ is the maximum value of $|q|$, also invariant since the potential vorticity q is materially conserved. In effect, (2.2) limits how fast changes in flow topology can occur. They also define a minimum 'topological length scale' λ_H through

$$\lambda_H^{-1}=\left[\langle|\mathrm{d}H/\mathrm{d}t|\rangle^2/(\bar{\rho}E^3)\right]^{1/7}\leqslant\left[4|q|_\infty^2/(\bar{\rho}E)\right]^{1/7}\ , \tag{2.3}$$

using (2.2); here the angled brackets denote a time average while $\bar{\rho}$ is the (constant) average density over the domain (assumed periodic). One may view λ_H^{-1} as a maximum 'helicity wavenumber'. Remarkably, they show that it is bounded from above simply by a constant – the right-hand side of (2.3).

In the compressible case, the bound on $|\mathrm{d}H/\mathrm{d}t|$ is less tight, and there is an additional term on the right-hand side of (2.2) equal to $2|\rho\boldsymbol{u}\cdot\rho\boldsymbol{\omega}|_\infty$ multiplied by the volume integral of $|\nabla\cdot\boldsymbol{u}|$. None of these terms are constant or have simple upper bounds – even the potential vorticity q is no longer materially conserved. Hence, the bound derived in this case is most likely to be useful in the perturbative regime of weakly compressible flows.

To increase the practical applicability of helicity for interpreting flow topology, two issues deserve attention. First, in atmospheric and oceanic flows, gravity plays a crucial role, as does the background rotation of the planet. Gravity gives rise to buoyancy forces, which contribute to the baroclinic torque in the vorticity equation. Rotation can be incorporated by working with the absolute vorticity (including that associated with the planetary rotation), since then the vorticity equation is formally unchanged. However, at intermediate and large scales, which are energetically dominant, the planetary contribution to the absolute vorticity dominates, and the associated absolute velocity field is orthogonal to it, giving no contribution to helicity. A second issue that deserves further attention is the role of viscosity and thermal diffusion. Viscosity breaks topology, allowing vortex-line reconnection. This is the dominant dissipative process in turbulent flows, insofar as vorticity is the paramount dynamical field. However, a recent study by Scheeler *et al.* (2014) indicates that helicity is remarkably well preserved through reconnection events. Knotted vortex lines may uncoil under the action of viscosity, but in the process new helical structures are produced that largely compensate the change in total helicity. Understanding the apparent robustness of helicity from a mathematical point of view remains a significant challenge.

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