Groups whose Chermak-Delgado lattice is a subgroup lattice of an abelian group*

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June 14, 2022

Abstract

The Chermak-Delgado lattice of a finite group $G$ is a self-dual sublattice of the subgroup lattice of $G$. In this paper, we prove that, for any finite abelian group $A$, there exists a finite group $G$ such that the Chermak-Delgado lattice of $G$ is a subgroup lattice of $A$.

Keywords Chermak-Delgado lattice; subgroup lattice; finite $p$-groups


1 Introduction

Suppose that $G$ is a finite group, and $H$ is a subgroup of $G$. The Chermak-Delgado measure of $H$ (in $G$) is denoted by $m_G(H)$, and defined as $m_G(H) = |H| \cdot |C_G(H)|$. The maximal Chermak-Delgado measure of $G$ is denoted by $m^*(G)$, and defined as

$$m^*(G) = \max \{m_G(H) \mid H \leq G\}.$$ 

Let

$$CD(G) = \{H \mid m_G(H) = m^*(G)\}.$$ 

Then the set $CD(G)$ forms a sublattice of $L(G)$ (the subgroup lattice of $G$), which is called the Chermak-Delgado lattice of $G$. It was first introduced by Chermak and Delgado [9], and revisited by Isaacs [12]. In the last years, there has been a growing interest in understanding this lattice (see e.g. [1-11], [13-17], [19-22]).

A Chermak-Delgado lattice is always self-dual. So the question arises: Which types of self-dual lattices can be Chermak-Delgado lattices of finite groups? In [5], it is proved that, for any integer $n$, a chain of length $n$ can be a Chermak-Delgado lattice of a finite $p$-group.

*This work was supported by NSFC (No. 11971280)
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A quasi-antichain is a lattice consisting of a maximum, a minimum, and the atoms of the lattice. The width of a quasi-antichain is the number of atoms. For a positive integer \( w \geq 3 \), a quasi-antichain of width \( w \) is denoted by \( \mathcal{M}_w \). In [6], it was proved that \( \mathcal{M}_w \) can be a Chermak-Delgado lattice of a finite group if and only if \( w = 1 + p^a \) for some positive integer \( a \) and some prime \( p \).

An \( m \)-diamond is a lattice with subgroups in the configuration of an \( m \)-dimensional cube. A mixed \( n \)-string is a lattice with \( n \) components, adjoined end-to-end so that the maximum of one component is identified with the minimum of the other component. The following theorem gives more self-dual lattices which can be Chermak-Delgado lattices of finite groups.

**Theorem 1.1.** ([4]) If \( \mathcal{L} \) is a Chermak-Delgado lattice of a finite \( p \)-group \( G \) such that both \( G/Z(G) \) and \( G' \) are elementary abelian, then so are \( \mathcal{L}^+ \) and \( \mathcal{L}^{++} \), where \( \mathcal{L}^+ \) is a mixed 3-string with center component isomorphic to \( \mathcal{L} \) and the remaining components being \( m \)-diamonds, \( \mathcal{L}^{++} \) is a mixed 3-string with center component isomorphic to \( \mathcal{L} \) and the remaining components being lattice isomorphic to \( \mathcal{M}_{p+1} \).

By [18, Theorem 8.1.4], \( \mathcal{L}(A) \) is always self-dual for any finite abelian group \( A \). If \( A \) is a cyclic \( p \)-group, then \( \mathcal{L}(A) \) is chain, and hence can be a Chermak-Delgado lattice of a finite \( p \)-group. In [2], it is proved that, if \( A \) is an elementary abelian \( p \)-group, then \( \mathcal{L}(A) \) can be a Chermak-Delgado lattice of a finite \( p \)-group. In this paper, we prove that, for any finite abelian group \( A \), \( \mathcal{L}(A) \) can be a Chermak-Delgado lattice of a finite group. The main results are:

**Theorem 1.2.** For any finite abelian \( p \)-group \( A \), there exists a finite \( p \)-group \( G \) such that \( \mathcal{CD}(G) \) is isomorphic to \( \mathcal{L}(A) \).

**Theorem 1.3.** For any finite abelian group \( A \), there exists a finite group \( G \) such that \( \mathcal{CD}(G) \) is isomorphic to \( \mathcal{L}(A) \).

### 2 Preliminary

We gather next some basic properties of the Chermak-Delgado lattice, which will be used often throughout the paper without further reference.

**Theorem 2.1.** ([9]) Suppose that \( G \) is a finite group and \( H, K \in \mathcal{CD}(G) \).

1. \( \langle H, K \rangle = HK \). Hence a Chermak-Delgado lattice is modular.
2. \( C_G(H \cap K) = C_G(H)C_G(K) \).
3. \( C_G(H) \in \mathcal{CD}(G) \) and \( C_G(C_G(H)) = H \). Hence a Chermak-Delgado lattice is self-dual.
4. Let \( M \) be the maximal member of \( \mathcal{CD}(G) \). Then \( M \) is characteristic in \( G \) and \( \mathcal{CD}(M) = \mathcal{CD}(G) \).
(5) The minimal member of \( \text{CD}(G) \) is characteristic, abelian, and contains \( \mathbb{Z}(G) \).

We also need the following lemmas:

**Theorem 2.2.** ([7, Theorem 2.9]) For any finite groups \( G \) and \( H \), \( \text{CD}(G \times H) = \text{CD}(G) \times \text{CD}(H) \).

**Lemma 2.3.** [2, Lemma 3.3] Suppose that \( G \) is finite group, \( H \leq G \) such that \( G = H \cdot C_G(H) \). If \( H \in \text{CD}(H) \), then \( H \) is contained in the unique maximal member of \( \text{CD}(G) \).

**Lemma 2.4.** [20, Lemma 5] Let \( G \) be a finite \( p \)-group. Then \( \text{CD}(G) = [G/\mathbb{Z}(G)] \) if and only if the interval \([G/\mathbb{Z}(G)]\) of \( \mathcal{L}(G) \) is modular and \( G' \) is cyclic.

In this section, we prove that, for any finite abelian group \( A \), \( \mathcal{L}(A \times A) \) can be a Chermak-Delgado lattice of a finite group. Although this result can be deduced from our main theorem, the proof is independent and short.

**Lemma 2.5.** Let \( A \) be a finite abelian \( p \)-group. Then there exists a finite \( p \)-group \( G \) such that \( \text{CD}(G) \) is isomorphic to \( \mathcal{L}(A \times A) \).

**Proof** Assume that the type of \( A \) is \((p^{e_1}, p^{e_2}, \ldots, p^{e_m})\), where \( e_1 \geq e_2 \geq \cdots \geq e_m \).

Let \( G \) be the group generated by \( 2m \) elements \( x_1, \ldots, x_m, y_1, \ldots, y_m \) subject to the defining relations:

\[
[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ if } i \neq j,
\]

\[
x_i^{p^{e_i}} = y_i^{p^{e_i}} = z^{p^{e_1}} = 1, \quad [x_i, y_i] = z^{p^{e_1-e_i}}, \quad [z, x_i] = [z, y_i] = 1 \text{ for } 1 \leq i \leq m.
\]

Let \( P_i = \langle x_i, y_i, z \rangle \). Then \( Z(P_i) = \langle z \rangle \). Thus \( G \) is also the central product of \( P_i \). It is easy to see that \( G' = Z(G) = \langle z \rangle \) and \( G/\mathbb{Z}(G) \cong A \times A \). By Lemma 2.4, \( \text{CD}(G) \) is just the interval \([G/\mathbb{Z}(G)]\). Hence \( \text{CD}(G) \cong \mathcal{L}(G/\mathbb{Z}(G)) \cong \mathcal{L}(A \times A) \).

**Theorem 2.6.** For any finite abelian group \( A \), there exists a finite group \( G \) such that \( \text{CD}(G) \) is isomorphic to \( \mathcal{L}(A \times A) \).

**Proof** Let \( A = A_1 \times \cdots \times A_n \), where \( A_i \) is the Sylow \( p_i \)-subgroup of \( A \). By Lemma 2.5, there exist finite group \( P_i \) such that \( \text{CD}(P_i) \) is isomorphic to \( \mathcal{L}(A_i \times A_i) \). Let \( G = P_1 \times \cdots \times P_n \). By Theorem 2.2,

\[
\text{CD}(G) = \text{CD}(P_1) \times \cdots \times \text{CD}(P_n)
\]

\[
\cong \mathcal{L}(A_1 \times A_1) \times \cdots \times \mathcal{L}(A_n \times A_n)
\]

\[
= \mathcal{L}(A \times A)
\]

\( \square \)
3 The groups $G(p, e)$

For any prime $p$ and an integer $e \geq 1$, we use $G(p, e)$ to denote the finite $p$-group generated by 3 elements $x, y, w$ subject to the following defining relations:

- $\langle x, y \rangle = z_1, \langle y, w \rangle = z_2, \langle w, x \rangle = z_3$,
- $x^{p^e} = y^{p^e} = w^{p^e} = z_1^{p^e} = z_2^{p^e} = z_3^{p^e} = 1$,
- $[z_i, x] = [z_i, y] = [z_i, w] = 1$ for all $i = 1, 2, 3$.

In this section, we prove that the Chermak-Delgado lattice of $G(p, e)$ is isomorphic to a subgroup lattice of a cyclic group of order $p^{e+1}$. This group will be used to construct an example in the proof of Theorem 1.2. Let $G = G(p, e)$. Then it is easy to check the following results:

- $d(G) = 3$, $\exp(G) = p^e$, $Z(G) = G' = \langle z_1, z_2, z_3 \rangle$,
- $|Z(G)| = p^{3e}$, $|G/Z(G)| = p^{3e}$, $m_G(G) = m_G(Z(G)) = p^{9e}$.

Lemma 3.1. Assume that $G = G(p, e)$ and $Z(G) < H < G$.

1. If $H/Z(G)$ is cyclic, then $m_G(H) < m_G(G)$.
2. If $H/Z(G)$ is not cyclic, then $m_G(H) \leq m_G(G)$, where "=" holds if and only if the type of $H/Z(G)$ is $(p^{e_1}, p^{e_2}, p^{e_3})$ for some $1 \leq e_1 < e$.

Proof (1) Let $H = \langle h, Z(G) \rangle$ and $H/Z(G)$ be of order $p^{e_1}$. Then we may let

$h = x^{k_1p^{e_1}}y^{k_2p^{e_1}}w^{k_3p^{e_1}}$,

where $p \nmid k_i$ for some $i$. Without loss of generality, we may assume that $p \nmid k_1$. Replacing $x$ with $x^{k_1}y^{k_2}w^{k_3}$, we have $h = x^{p^{e_1}}$. It is easy to check that $C_G(H) = \langle x, y^{p^{e_1}}, w^{p^{e_1}} \rangle Z(G)$. Since $|C_G(H)/Z(G)| = p^{3e-2e_1}$,

$|H/Z(G)| \cdot |C_G(H)/Z(G)| = p^{3e-e_1} < p^{3e} = |G/Z(G)|$.

Hence $m_G(H) = |H| \cdot |C_G(H)| < |G| \cdot |Z(G)| = m_G(G)$.

(2) Let $H = \langle h_1, h_2, h_3 \rangle Z(G)$ and $H/Z(G)$ be of type $(p^{e_1}, p^{e_2}, p^{e_3})$, where $e_1 \geq e_2 \geq e_3 \geq 0$. Since $H/Z(G)$ is not cyclic, $e_2 \geq 1$. By a similar argument as (1), we may assume that $h_1 = x^{p^{e_1}}$. We may let

$h_2 = x^{k_1p^{e_1}}y^{k_2p^{e_1}}w^{k_3p^{e_1}}$,

where $p \nmid k_i$ for some $2 \leq i \leq 3$. Without loss of generality, we may assume that $p \nmid k_2$. Replacing $y$ with $x^{k_1}y^{k_2}w^{k_3}$, we have $h_2 = y^{p^{e_2}}$. It is easy to check that

$C_G(H) = C_G(h_1) \cap C_G(h_2) = \langle y^{p^{e_2}}, y^{p^{e_1}}, w^{p^{e_1}} \rangle Z(G)$.

Published online by Cambridge University Press

https://doi.org/10.4153/S0008439522000418
In this section, we require the following notation and straightforward results for a finite $e$-group $G$.

For any prime $p$, let $m = [G/Z(G)] = p^{e_1}$, and $|G| = p^{e_2 - 2e_1}$.

\[ |H/Z(G)| \cdot |C_G(H)/Z(G)| = p^{3e_1 + e_2 - e_1} \leq p^{3e_1} = |G/Z(G)|, \]

where “$\Rightarrow$” holds if and only if $e_3 = e_1$. Hence $m_G(H) = |H| \cdot |C_G(H)| \leq |G| \cdot |Z(G)| = m_G(G)$, where “$\Rightarrow$” holds if and only if $e_1 = e_2 = e_3$.

**Theorem 3.2.** Let $G = G(p, e)$. Then $G \in CD(G)$ and $CD(G)$ is a chain of length $e$. Moreover, $H \in CD(G)$ if and only if $H = \langle x^{p^{e_1}-1}, y^{p^{e_1}-1}, w^{p^{e_1}-1} \rangle Z(G)$ for some $0 \leq e_1 \leq e$.

**Proof** By Lemma 3.1, $m^*(G) = m_G(G) = p^{9e}$, and $H \in CD(G)$ if and only if the type of $H/Z(G)$ is $(p^{e_1}, p^{e_1}, p^{e_1})$ for some $0 \leq e_1 \leq e$. Hence all elements of $CD(G)$ are $\langle x^{p^{e_1}-1}, y^{p^{e_1}-1}, w^{p^{e_1}-1} \rangle Z(G)$ where $0 \leq e_1 \leq e$.

### 4 The proof of main results

For any prime $p$ and an abelian $p$-group $A$ with type $(p^{e_1}, p^{e_2}, \ldots, p^{e_m})$, where $e_1 \geq e_2 \geq \cdots \geq e_m$, we use $G_A$ to denote the finite $p$-group generated by $3m$ elements $x_1, \ldots, x_m, y_1, \ldots, y_m, w_1, \ldots, w_m$ subject to the following defining relations:

- $x_i^{p^{e_i}} = y_i^{p^{e_i}} = w_i^{p^{e_i}} = z_i^{p^{e_1}} = 1$ for $1 \leq i \leq m$,
- $[x_i, x_j] = [y_i, y_j] = [w_i, w_j] = [x_i, y_j] = [y_i, w_j] = [w_i, x_j] = 1$ if $i \neq j$,
- $[x_i, y_i] = z_i^{p^{e_1}-e_i}$, $[y_i, w_i] = z_i^{p^{e_1}-e_i}$, $[w_i, x_i] = z_i^{p^{e_1}-e_i}$ for $1 \leq i \leq m$,
- $[z_i, x_i] = [z_j, y_i] = [z_j, w_i] = 1$ for $1 \leq i \leq m$ and $j = 1, 2, 3$.

In this section, we require the following notation and straightforward results for a finite $p$-group $G = G_A$.

- $Z(G) = G' = \langle z_1, z_2, z_3 \rangle$ is of order $p^{3e_1}$.
- Let $P_i = \langle x_i, y_i, w_i \rangle$ for $1 \leq i \leq m$. Then $P_i \cong G(p, e_i)$, $|P_i Z(G)/Z(G)| = p^{3e_i}$ and $G$ is the central product $P_1 * P_2 * \cdots * P_m$.
- Let $X = \langle x_1, x_2, \ldots, x_m \rangle$, $Y = \langle y_1, y_2, \ldots, y_m \rangle$ and $W = \langle w_1, w_2, \ldots, w_m \rangle$. Then $X \cong Y \cong W \cong A$.
- Let $n = e_1 + e_2 + \cdots + e_m$. Then $|A| = p^n$, $|G/Z(G)| = p^{3n}$, $|G| = p^{3n+3e_1}$ and $m_G(G) = p^{3n+6e_1}$.
- Let $\alpha, \beta, \gamma$ be isomorphisms from $A$ to $X, Y, W$ respectively such that $x_i^{\alpha^{-1}} = y_i^{\beta^{-1}} = w_i^{\gamma^{-1}}$ for all $1 \leq i \leq m$.
- For $a \in A$, let $a^x = \langle a^\alpha, a^\beta, a^\gamma \rangle Z(G)$.
- For $B \leq A$, let $B^x = \langle B^\alpha, B^\beta, B^\gamma \rangle Z(G) = \prod_{b \in B} b^x$. 5
The proof of Theorem 1.2

Assume the type of $A$ is $(p^{e_1}, p^{e_2}, \ldots, p^{e_m})$, where $e_1 \geq e_2 \geq \cdots \geq e_m$. Let $G = G_A$. We will prove $CD$ in the unique maximal member of $(\mathcal{L}(A))$ in 6 steps.

(1) $G \in CD(G)$ and $m^*(G) = p^{3n+6\epsilon_1}$.

By Theorem 3.2, $P_i \in CD(P_i)$. Since $G = P_iC_G(P_i)$, by Lemma 2.3, $P_i$ is contained in the unique maximal member of $CD(G)$. Hence $G$ is the unique maximal member of $CD(G)$ and $m^*(G) = m_G(G) = p^{3n+6\epsilon_1}$.

(2) For any $a \in A$, there exists a subgroup $C_a$ of $A$ such that $C_X(a^\beta) = C_X(a^\gamma) = (C_a)^\beta$, $C_Y(a^\alpha) = C_Y(a^\gamma) = (C_a)^\beta$ and $C_W(a^\alpha) = C_W(a^\beta) = (C_a)^\gamma$.

Notice that for $x \in X$, $[x, a^\beta] = 1$ if and only if $[x, a^\gamma] = 1$. We have $C_X(a^\beta) = C_X(a^\gamma)$. Let $C_a = (C_X(a^\beta))^{a^{-1}}$. Then $C_X(a^\beta) = C_X(a^\gamma) = (C_a)^a$. Notice that for $c \in A$, $[c^\beta, a^\gamma] = 1$ if and only if $[c^\beta, a^\gamma] = 1$. We have $c \in C_a \iff c^\beta \in C_X(a^\gamma) \iff c^\beta \in C_Y(a^\gamma)$.

It follows that $C_Y(a^\gamma) = (C_a)^\beta$. By the symmetry, the conclusions hold.

(3) $C_G(a^\varphi) = (C_a)^\varphi$ and $a^\varphi \in CD(G)$.

Suppose $a$ is of order $p^t$. Then $|a^\varphi / Z(G)| = p^{3t}$. Since $[a^\alpha, G] \leq \langle z_p^{p^{e_1} - t}, z_p^{p^{e_2} - t} \rangle$, the length of the conjugacy class of $a^\alpha$ does not exceed $p^{2t}$. Hence $|C_G(a^\alpha)| \geq p^{3n+3e_1-2t}$ and $|C_G(a^\alpha)/Z(G)| \geq p^{3n-2t}$. Notice that

\[ C_G(a^\alpha)/Z(G) = XZ(G)/Z(G) \times C_Y(a^\alpha)Z(G)/Z(G) \times C_W(a^\alpha)Z(G)/Z(G), \]

\[ |XZ(G)/Z(G)| = |X| = p^n, \text{ and by (2),} \]

\[ |C_a| = |C_Y(a^\alpha)| = |C_W(a^\alpha)| = |C_Y(a^\alpha)Z(G)/Z(G)| = |C_W(a^\alpha)Z(G)/Z(G)|. \]

We have $|C_a| \geq p^{n-t}$. Hence $|(C_a)^\varphi / Z(G)| \geq p^{3n-3t}$. By (2), $(C_a)^\varphi \leq C_G(a^\varphi)$. Hence

\[ |a^\varphi / Z(G)| \cdot |C_G(a^\alpha)/Z(G)| \geq |a^\varphi / Z(G)| \cdot |(C_a)^\varphi / Z(G)| \geq p^{3n} = |G/Z(G)|. \]

It follows that

\[ m_G(a^\varphi) = |a^\varphi| \cdot |C_G(a^\alpha)| \geq |G| \cdot |Z(G)| = m^*(G). \]

Thus “$\Rightarrow$” holds, $C_G(a^\varphi) = (C_a)^\varphi$ and $a^\varphi \in CD(G)$.

(4) For any $B \leq A$, $B^\varphi \in CD(G)$ and there exists a subgroup $C_B$ of $A$ such that $C_G(B^\varphi) = (C_B)^\varphi$. Moreover, $|B| \cdot |C_B| = p^n$.

Let $C_B = \bigcap_{b \in B} C_b$. Since $B^\varphi = \prod_{b \in B} b^\varphi$, $B^\varphi \in CD(G)$ and

\[ C_G(B^\varphi) = \bigcap_{b \in B} C_G(b^\varphi) = \bigcap_{b \in B} (C_b)^\varphi = (C_B)^\varphi. \]
Since \(|B^e/Z(G)| = |B|^3| \) and \(|(C_B)^e/Z(G)| = |C_B|^3| \), we have
\[|B|^3 \cdot |C_B|^3 = |B^e/Z(G)| \cdot |(C_B)^e/Z(G)| = |G/Z(G)| = p^{3n}.\]

Hence \(|B| \cdot |C_B| = p^n| \).

(5) If \(K \in CD(G)\), then there exists a subgroup \(B\) of \(A\) such that \(K = B^e\).

Let \(H = C_G(K)\). Then \(H \in CD(G)\) and \(K = C_G(H)\). Let
\[B_1 = \{a \in A \mid \text{there exist } y \in Y, w \in W \text{ and } z \in Z(G) \text{ such that } a^nywz \in H\},\]
\[B_2 = \{a \in A \mid \text{there exist } x \in X, w \in W \text{ and } z \in Z(G) \text{ such that } xa^3wz \in H\},\]
\[B_3 = \{a \in A \mid \text{there exist } x \in X, y \in Y \text{ and } z \in Z(G) \text{ such that } xya^7z \in H\}.

Then \(B_1, B_2\) and \(B_3\) are subgroups of \(A\) and \(|H/Z(G)| \leq |B_1| \cdot |B_2| \cdot |B_3|\). By (2),
\[C_X(H) \leq C_X(B_2^e) = (C_B)^a.\]

Hence \(|C_X(H)| \leq |C_B|\). Similarly, \(|C_Y(H)| \leq |C_B|\) and \(|C_W(H)| \leq |C_B|\). It follows that
\[|H/Z(G)| \cdot |K/Z(G)| \leq |B_1| \cdot |B_2| \cdot |B_3| \cdot |C_B^a| \cdot |C_B^b| \cdot |C_B^c| = p^{3n} = |G/Z(G)|.\]

Since \(H \in CD(G)\), \(=\) holds. Hence
\[K = C_G(H) = \langle (C_B^a), (C_B^b), (C_B^c) \rangle Z(G)\]

and
\[C_X(H) = (C_B^a), C_Y(H) = (C_B^b) \text{ and } C_W(H) = (C_B^c).\]

By the symmetry, we also have
\[C_X(H) = (C_B^a), C_Y(H) = (C_B^b) \text{ and } C_W(H) = (C_B^c).\]

It follows that \(C_B^a = C_B^b = C_B^c\). Let \(B = C_B^a\). Then \(K = C_G(H) = B^e\).

(6) \(CD(G)\) is isomorphic to \(L(A)\).

It is a direct result of (4) and (5).

\[\square\]

The proof of Theorem 1.3

Let \(A = A_1 \times \cdots \times A_n\), where \(A_i\) is the Sylow \(p_i\)-subgroup of \(A\). By Theorem 1.2, there exist finite groups \(P_i\) such that \(CD(P_i)\) is isomorphic to \(L(A_i)\). Let \(G = P_1 \times \cdots \times P_n\).

By Theorem 2.2,
\[CD(G) = CD(P_1) \times \cdots \times CD(P_n) \cong L(A_1) \times \cdots \times L(A_n) = L(A)\]

\[\square\]

Acknowledgments I cordially thank the referee for detail reading and helpful comments, which helped me to improve the whole paper considerably.

7
References


