J. Austral. Math. Soc. 23 (Series A) (1977), 129–146.

ON GROUPS GENERATED BY THREE-DIMENSIONAL SPECIAL UNITARY GROUPS II

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(Received 19 November 1974; revised 30 October 1975)

Introduction

We shall determine in this paper groups of types D_n, E_6, E_7 and E_8 generated by $SU(3, q)$'s, q odd, $q > 3$. These groups are defined in Phan (1975). [We shall refer to this paper as I]. Acquaintance with the results of I is assumed. The identification of groups of type D_4 is similar to that of $SU(n, q)$. We actually construct an isomorphism from the universal group of type D_4 onto $\mathrm{Spin}^+(8, q)$. This direct approach does not appear to be feasible for groups of type D_n with $n \geqq 5$. Fortunately Wong's recent result (1974) is applicable here. But his theorem requires that the characteristic of the field be odd; hence unlike the unitary case, we assume that q is odd and $q > 3$. Using Wong's theorem, we proceed to show by induction that groups of type D_n are homomorphic images of $\mathrm{Spin}^+(2n, q)$ or $\mathrm{Spin}^-(2n, q)$ according as n is even or n is odd.

We then use our result on groups of type D_n and the structure of these groups to show the existence of Steinberg's generators and relations in groups of types E_6, E_7 and E_8. It turns out that these are either Chevalley groups or their twisted analogues.

1. Groups of types D_n

Let U be a vector space of dimension m over a field K of odd characteristic and f a non degenerate symmetric bilinear form on U. The set of isometries forms the orthogonal group $O_m(K, f) = O(U)$. The subgroup of determinant 1 of the orthogonal group and the commutator subgroup $O(U)'$ are denoted by $SO(K, f) = SO(U)$ and $\Omega(U)$ respectively. When m is even and K is finite, there are two non equivalent symmetric bilinear forms giving rise to non isomorphic orthogonal groups. When K is finite of order q and the

https://doi.org/10.1017/S1446788700018140 Published online by Cambridge University Press

index of f is $m/2$, we also denote $\Omega(K, f)$ by $\Omega^+(m, q)$. In the case that the index of f is $m/2-1$ and $|K| = q$ we denote $\Omega_m(K, f)$ by $\Omega^-(m, q)$. The corresponding subgroups $\text{Spin}_m(K, f) = \text{Spin}(U)$ of the Clifford group are denoted by $\text{Spin}^+(m, q)$ and $\text{Spin}^-(m, q)$ respectively [Dieudonné (1955)].

We shall next show that $\Omega_m(K, f)$ can be embedded in the special unitary group of some hermitian space depending on m and f. Let V be a non degenerate hermitian space of dimension $2n \geqq 4$ over the finite field F of q^2 elements. We shall assume throughout this paper that *q is odd and $q > 3$*. We denote the hermitian form by $(\,,)$. Let $B = \{v_1, v_2, \cdots, v_{2n}\}$ be an orthonormal basis of V. Let L_i^* (resp. Γ_i^*), $1 \leqq i \leqq n-1$ denote the subgroup of $SU(V)$ whose restriction to the subspace $V_i = \{v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i+2}\}$ is represented by the matrices

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \bar{\alpha} & 0 & \bar{\beta} \\ -\bar{\beta} & 0 & \bar{\alpha} & 0 \\ 0 & -\beta & 0 & \alpha \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \bar{\alpha} & \bar{\beta} & 0 \\ 0 & -\beta & \alpha & 0 \\ -\bar{\beta} & 0 & 0 & \bar{\alpha} \end{pmatrix}$$

$\alpha, \beta \in F$, $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ $(\bar{x} = x^q)$ and L_i^* (resp. Γ_i^*) fixes elementwise the orthogonal complement $V_i^\perp$ of V_i. Let H_i^* (resp. K_i^*) denote the diagonal subgroup of L_i^* (resp. Γ_i^*). We note that L_i^*, Γ_i^* are isomorphic to $SU(2, q)$ and H_i^*, K_i^* are cyclic of order $q+1$, and generate an abelian subgroup of $SU(V)$.

Let V_0 be the subspace of V consisting of vectors whose column coordinate matrix has the form ${}^t(x_1, \bar{x}_1, x_2, \bar{x}_2, \cdots, x_n, \bar{x}_n)$. Clearly V_0 is n-dimensional over F. But we can also regard V_0 in the usual way as an $2n$-dimensional space over F_0, the subfield of q elements in F. We check that the hermitian form when restricted to V_0 induces a non degenerate symmetric bilinear form over F_0 and the subgroup $G^* = \langle L_i^*, \Gamma_i^* \mid 1 \leqq i \leqq n-1 \rangle$ is faithful on V_0. Moreover the elements of G^* are isometries of V_0 and hence $G^* \subseteq \Omega(V_0)$ as each L_i^*, Γ_i^* is a perfect group. We verify that V_i contains two dimensional totally degenerate subspaces (over F_0) e.g. $\{xv_{2i-1} + \bar{x}v_{2i} + \sigma x v_{2i+1} + \overline{\sigma x} v_{2i+2}\}$ where $x \in F$ and $\sigma\bar{\sigma} = -1$. It also contains an anisotropic space of dimension 2 e.g. $\{xv_{2i-1} + \bar{x}v_{2i}\}$. Therefore V_0 has index n when n is even and index $n-1$ when n is odd. We collect these facts in the following

LEMMA 1.1. *The space V_0 with the form $(\,,)\,|\,V_0$ is a non singular orthogonal space of dimension $2n$ over F_0 and index n or $n-1$ according as n is even or odd. The group $G^* = \langle L_i^*, \Gamma_i^* \mid 1 \leqq i \leqq n-1 \rangle$ is a subgroup of $\Omega(V_0)$.*

REMARK. The space V_0 is always regarded as over F_0 unless otherwise specified.

A simple computation produces the following

LEMMA 1.2. *Let* s_1, r_i *be elements of* Γ_1^*, L_i^* *respectively such that* $s_1(v_1) = v_4$, $s_1(v_4) = -v_1$; $r_i(v_{2i-1}) = v_{2i+1}$, $r_i(v_{2i+1}) = -v_{2i-1}$. *We define inductively* $s_{i+1} = r_i s_i r_{i+1} s_i^{-1} r_i^{-1}$, $1 \leqq i \leqq n-2$. *Then*

(i) $\Gamma_{i+1}^* = r_i s_i L_{i+1}^* s_i^{-1} r_i^{-1}$ *and hence* $G^* = \langle \Gamma_1^*, L_i^* \mid 1 \leqq i \leqq n-1 \rangle$;

(ii) $\langle L_i^*, L_{i-1}^* \rangle$, $\langle L_i^*, L_{i-1}^* \rangle$, $\langle L_i^*, \Gamma_{i-1}^* \rangle$ *and* $\langle L_i^*, \Gamma_{i+1}^* \rangle$ *are isomorphic to* $SU(3, q)$;

(iii) *Statement* (ii) *with* L_i^* *replaced by* Γ_i^*;

(iv) $[L_i^*, L_j^*] = [L_i^*, \Gamma_j^*] = [\Gamma_i^*, \Gamma_j^*] = [L_i^*, \Gamma_1^*] = 1$, $j \neq i-1$, i, $i+1$;

(v) $\langle L_i^*, H_{i-1}^* \rangle$, $\langle L_i^*, H_{i+1}^* \rangle$, $\langle L_i^*, K_{i-1}^* \rangle$, $\langle L_i^*, K_{i+1}^* \rangle$ *are isomorphic to* $GU(2, q)$;

(vi) *Statement* (v) *with* L_i^* *replaced by* Γ_i^*;

(vii) $H_i^* H_j^* = H_i^* \times H_j^*$; $H_i^* K_j^* = H_i^* \times K_j^*$; $K_i^* K_j^* = K_i^* \times K_j^*$, $i \neq j$.

LEMMA 1.3. *Let* $\tilde{L}_i$, $\tilde{\Gamma}_i$ *be the commutator subgroup of the inverse images of* L_i^*, Γ_i^* *in* Spin(V_0) *respectively and* $\tilde{H}_i$, $\tilde{K}_i$ *the intersection of* $\tilde{L}_i$, $\tilde{\Gamma}_i$ *with the inverse images of* H_i^*, K_i^* *in* Spin(V_0) *respectively. Set* $G = \langle \tilde{L}_i, \tilde{\Gamma}_i \mid 1 \leqq i \leqq n-1 \rangle$. *Let* n_i, p_i *be representatives of inverse images of* r_i, s_i *in* Spin(V_0) *respectively. Then* (i)–(vii) *remain valid with* L_i^*, Γ_i^*, H_i^*, K_i^*, r_i, s_i *replaced by* $\tilde{L}_i$, $\tilde{\Gamma}_i$, $\tilde{H}_i$, $\tilde{K}_i$, n_i, p_i *respectively. Moreover* $\tilde{L}_i \cong L_i^* \cong \Gamma_i^* \cong \tilde{\Gamma}_i$, $\tilde{H}_i \cong H_i^* \cong K_i^* \cong \tilde{K}_i$ *and* $G/\langle z \rangle$ *is isomorphic to* G^* *where* z *is the product of the involutions in* $\tilde{L}_1$ *and* $\tilde{\Gamma}_1$.

PROOF. First we note that Spin(V_0) is a non splitting central extension of a subgroup of order 2 by $\Omega(V_0)$. Since both $SU(2, q)$ and $SU(3, q)$ have trivial Schur multipliers (except $SU(2, 9)$, whose Schur multiplier has order 3), it follows the inverse image in Spin(V_0) of a subgroup in $\Omega(V_0)$ isomorphic to $SU(2, q)$ or $SU(3, q)$ is a direct product [Griess (1972)]. The assertions are now clear.

COROLLARY 1.4. *The groups* $\tilde{G}$ *and* G^* *are groups of type* D_n *generated by* $SU(3, q)$'s.

LEMMA 1.5. $G^* = \Omega(V_0)$ *and* $\tilde{G} =$ Spin(V_0).

PROOF. We shall prove the lemma by induction on n. The cases $n = 2$ and 3 are clear by I. Assume then $n > 3$. Let $U_1 = \langle v_i \mid 1 \leqq i \leqq 2n-2 \rangle \cap V_0$; $U_2 = \langle v_i \mid 3 \leqq i \leqq 2n-2 \rangle \cap V_0$; $U_3 = \langle v_i \mid 3 \leqq i \leqq 2n \rangle \cap V_0$ and $U_0 = \langle v_1, v_2 \rangle \cap V_0$. We shall regard $\Omega(U_i)$ as a subgroup of $\Omega(V_0)$ in a natural way.

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Let $g \in \Omega(V_0)$. The projection of $g(U_0)$ into U_3 is a subspace of dimension at most two. As U_2 has index at least 2 it contains all possible symmetric bilinear spaces of dimension $\leqq 2$. By Witt's theorem, we can choose suitable elements $a \in \Omega(U_3)$ and $b \in \Omega(U_1)$ such that $ag(U_0) \subseteq U_1$ and $(bag)(U_0) \subseteq U_0$. Since $H_1^*\langle r_1 s_1\rangle|_{U_0} \cong O(U_0)$, we can assume $bag|_{U_0} =$ identity. It follows that $bag \in \Omega(U_3)$ and therefore $g \in \Omega(U_3)\Omega(U_1)\Omega(U_3)$. The result now follows by induction.

REMARK. It was Wong (1974) who first identified the group G^*.

Since we are assuming that q is odd, we can give a weaker definition of a group of type X generated by $SU(3, q)$'s. That is the set of subgroups L_i satisfies the following

(a) $G = \langle L_i \mid i \in X\rangle$;

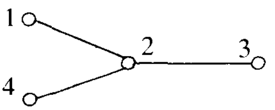
(b) $[L_i, L_j] = 1$ if $\{i, j\}$ is not an edge;

(c) $\langle L_i, L_j\rangle \cong SU(3, q)$ if $\{i, j\}$ is an edge;

(d) $[Z(L_i), Z(L_j)] = 1$ for all i, j in X.

Because there is only one class of four groups in $SU(3, q)$, it follows immediately that there exists cyclic subgroup H_i of order $q + 1$ such that $H_iH_j = H_i \times H_j$ and $\langle L_i, H_j\rangle \cong \langle L_j, H_i\rangle \cong GU(2, q)$ if $\{i, j\}$ is an edge.

We shall now investigate universal group G of type D_4. Clearly universal groups of types D_2 and D_3 are $SU(2, q) \times SU(2, q)$ and $SU(4, q)$ respectively by I. Let the graph of G be



By (1.5) of I we have

$$\langle L_1, L_2, L_3\rangle \cong \langle L_1, L_2, L_4\rangle \cong \langle L_4, L_2, L_3\rangle \cong SU(4, q).$$

Let U be a non degenerate hermitian space over F with orthonormal basis $\{u_1, u_2, u_3, u_4\}$. We may then regard $SU(U)$ as generated by the subgroups

$$A = \begin{pmatrix} \alpha & \beta & & \\ -\bar\beta & \bar\alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & & & \\ & \alpha & \beta & \\ & -\bar\beta & \bar\alpha & \\ & & & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \alpha & \beta \\ & & -\bar\beta & \bar\alpha \end{pmatrix}$$

α, $\beta \in F$ and $\alpha\bar\alpha + \beta\bar\beta = 1$. Thus we may identify L_1, L_2, L_3 with A, B, C,

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respectively and H_1, H_2, H_3 with the diagonal subgroups of A, B, C respectively. We have similar identification in the other two cases.

LEMMA 1.6. *Let g and g' be in $SU(U)$. Then one of the following holds*

(i) $g \in CBABAC$;

(ii) *there exists* $c \in C$ *such that* $gc \in ACBCBABA$ *and* $c^{-1}g' \in BABCBABA$;

(iii) *g has the form*

$$\begin{pmatrix} 1 & 0 & \alpha & \sigma\alpha \\ 0 & 1 & \lambda\alpha & \lambda\sigma\alpha \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}^{ab}$$

for a suitable $a \in A$ and a diagonal element $b \in B$. ($\times$ denotes an unspecified entry in the matrix).

PROOF. If $g(u) \in \langle u_3, u_4\rangle$ for some $u \in \langle u_3, u_4\rangle$ of unit length, then there exist c_1, c_2 in C such that $c_2(u_4) = u$ and $c_1gc_2(u_4) = u_4$. Then $g \in CBABAC$ as the stabilizer of u_4 in $SU(U)$ is $\langle A, B\rangle = BABA$. Therefore we may assume $g(u_i) \notin \langle u_3, u_4\rangle$, $i = 3, 4$. We now choose an element

$$c = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & x & y \\ & & -\bar{y} & \bar{x} \end{pmatrix}, \quad x\bar{x} + y\bar{y} = 1$$

with $y \neq 0$ and $x = \zeta\bar{y}$. Let pr be the projection map into $\langle u_1, u_2\rangle$. Suppose

$$\operatorname{pr} g(u_3) = \alpha u_1 + \beta u_2$$

$$\operatorname{pr} g(u_4) = \gamma u_1 + \delta u_2.$$

Then $\operatorname{pr} gc\,(u_4) = y\{(\alpha + \gamma\bar{\zeta})u_1 + (\beta + \delta\bar{\zeta})u_2\}$ which has length

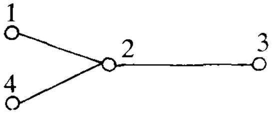
$$(1) \qquad L = y\bar{y}\{\alpha\bar{\alpha} + \beta\bar{\beta} + (\alpha\bar{\gamma} + \beta\bar{\delta})\zeta + (\bar{\alpha}\gamma + \bar{\beta}\delta)\bar{\zeta} + (\gamma\bar{\gamma} + \delta\bar{\delta})\zeta\bar{\zeta}\}.$$

For (ii) to hold, we must be able to choose ζ such that it does not satisfy the equations $1 + \zeta\bar{\zeta} = 0$; $L = 0$ and a non trivial polynomial in ζ of degree at most $q + 1$ which expresses the length of the projection of $c^{-1}g'(u_4)$ into $\langle u_1, u_2, u_3\rangle$. (See [I; 1.7] for details). Clearly such ζ exists if $L = 0$ is non trivial and if $q^2 - 3(q + 1) > 0$ that is $q > 4$. Thus it remains to consider the case when L is identically zero i.e.

$$\alpha\bar{\alpha} + \beta\bar{\beta} = \gamma\bar{\gamma} + \delta\bar{\delta} = \alpha\bar{\gamma} + \beta\bar{\delta} = 0. \tag{2}$$

We may assume none of α, β, γ, δ is zero; otherwise we are back to the situation of (i). Thus $\beta = \lambda\alpha \neq 0$ where $\lambda\bar{\lambda} = -1$. Equation (2) shows that $\alpha u_1 + \beta u_2$ and $\gamma u_1 + \delta u_2$ are non zero isotropic vectors orthogonal to each other. Because $\langle u_1, u_2\rangle$ is not totally degenerate, it follows that $\gamma = \sigma\alpha$ and $\delta = \sigma\beta$ for some $\sigma \in \dot{F}$. If $\sigma\bar{\sigma} = -1$, then there exists $c' \in C$ such that $\mathrm{pr}\, gc'(u_4) = 0$ and we are again in (i). So $\sigma\bar{\sigma} = -1$. It follows then the projections of $g^{-1}(u_1)$ and $g^{-1}(u_2)$ into $\langle u_1, u_2\rangle$ are orthogonal vectors of unit length. It is now clear that (iii) follows. This completes the proof.

LEMMA 1.7. *Let G be a universal group of type D_4 generated by $SU(3, q)$'s with the following graph*



Then $G = (NL_3)^3 N$ *where* $N = \langle L_1, L_2, L_4\rangle$.

PROOF. We have already remarked that

$$\langle L_1, L_2, L_4\rangle \cong \langle L_1, L_2, L_3\rangle \cong \langle L_4, L_2, L_3\rangle \cong SU(4, q).$$

Let $M = L_2L_1L_2L_4L_2L_1L_2$. Then each element g in G belongs to $N(L_3M)^m N$ for some integer $m > 0$ since we have the following identities

$$\langle L_1, L_2\rangle = L_1L_2L_1L_2 = L_2L_1L_2L_1 \tag{1}$$

and

$$\begin{aligned} N &= L_4\langle L_1, L_2\rangle L_4\langle L_1, L_2\rangle \\ &= \langle L_1, L_2\rangle L_4\langle L_1, L_2\rangle L_4 \end{aligned} \tag{2}$$

by (1.7) of I. We also need the identity

$$\begin{aligned} N &= \langle L_1, L_2\rangle\langle L_2, L_4\rangle\langle L_1, L_2\rangle \\ &= \langle L_2, L_4\rangle\langle L_1, L_2\rangle\langle L_2, L_4\rangle. \end{aligned} \tag{3}$$

Let $Y = NL_3ML_3ML_3N$. We want to show $Y = G$. It suffices to prove that an element

$$x = c_1m_1c_2m_2c_3m_3c_4$$

belongs to Y where $c_i \in L_3$, $m_i \in M$. First we may assume that $c_i \notin H_3$, otherwise we are done. Let $m_i = b_{4i-3}a_{2i-1}b_{4i-2}d_ib_{4i-1}a_{2i}b_{4i}$ where $a_j \in L_1$, $b_j \in L_2$ and $d_j \in L_4$.

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In the remaining proof we shall use the letters a, b, c, d to denote arbitrary elements of L_1, L_2, L_3, L_4 respectively. Since we shall be interested in the factorization of G only, we use the same letter in an equation to denote possible different elements. We use $y \equiv z$ to denote $NyN = NzN$. On many occasions, we need to introduce suitably chosen fixed elements in L_3. These will be denoted by c^*, $\tilde{c}$, $\bar{c}$ etc. We look at different forms of the element x.

(i) We may suppose $b_1a_1b_2d_1b_3a_2b_4$ satisfies either (i) or (ii) of (1.6).

Suppose not. We may identify $\langle L_1, L_2, L_3\rangle$ (resp. $\langle L_1, L_2, L_4\rangle$) with $SU(U)$ so that L is identified with A, L_2 with B and L_3 (resp. L_4) with C. After suitable changes in the c_i, m_i of $c_2m_2c_3m_3c_4$ using (1) and (2), we may suppose

$$z = \begin{pmatrix} 1 & 0 & \rho & \sigma\rho \\ 0 & 1 & \lambda\rho & \lambda\sigma\rho \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

for some $\rho, \sigma, \lambda \in \dot{F}$ such that $\lambda\bar{\lambda} = \sigma\bar{\sigma} = -1$. Let

$$c_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \eta & \tau \\ & & -\bar{\tau} & \bar{\eta} \end{pmatrix} \quad \text{and} \quad c^* = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & x & y \\ & & -\bar{y} & \bar{x} \end{pmatrix}$$

where $\eta\bar{\eta} + \tau\bar{\tau} = 1 = x\bar{x} + y\bar{y}$ and $\zeta y = \bar{x} \neq 0$. By (1.7) of Phan (1976), there exists suitable ζ such that

$$(c^*)^{-1}b_7a_4b_8c_3b_9a_5b_{10} \quad \text{and} \quad c_2b_5a_3b_6c^*$$

belong to $\langle L_1, L_2\rangle L_3\langle L_1, L_2\rangle$ provided $q^2 - 3(q+1) > 0$. Suppose

$$c_2b_5a_3b_6c^* = ecf.$$

where $e, f \in \langle L_1, L_2\rangle$ and $c \in L_3$. Assume that

$$(b_5a_3b_6)(u_3) = \alpha u_1 + \beta u_2 + \gamma u_3$$

$$(e)(u_3) = \delta u_1 + \varepsilon u_2 + \chi u_3$$

and

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$$c = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & x' & y' \\ & & -\bar{y}' & \bar{x}' \end{pmatrix}$$

We note that $y' \neq 0$ as $c_2 \notin H_3$. Thus $\delta = (y/y')\alpha$; $\varepsilon = (y/y')\beta$ and $\chi = (y/y')(\eta\gamma + \tau\zeta)$. If ze satisfies (i) or (ii) of (1.6), then we are done. Otherwise we have

$$(\delta + \rho\chi)/\sigma\rho = (\varepsilon + \lambda\rho\chi)/\sigma\lambda\rho$$

and $(\delta + \rho\chi)(\overline{\delta + \rho\chi}) + \sigma\bar{\sigma}\rho\bar{\rho} = 0$. This implies that $\lambda\delta = \varepsilon$ and so $\chi\bar{\chi} = 1$. Then the second equation above becomes

$$\alpha\bar{\alpha} + \bar{\alpha}(\gamma\eta + \tau\zeta)\rho + \alpha(\overline{\gamma\eta + \tau\zeta})\bar{\rho} = 0. \tag{4}$$

If $\alpha = 0$, then $\beta = 0$ because $\lambda(y/y')\alpha = (y/y')\beta$. Therefore $b_5a_3b_6 \in \langle L_1, H_2 \rangle$ i.e. $b_5a_3b_6 = \bar{a}\bar{b}$ where $\bar{a} \in L_1$ and $\bar{b} \in H_2$. So

$$\begin{aligned} x &= c_1m_1c_2\bar{a}\bar{b}d_2b_7a_4b_8c_3m_3c_4 \\ &\equiv c_1mc_2'b_7a_4b_8c_3m_3c_4 \text{ for a suitable } m \text{ in } M \text{ and } c_2' \text{ in } L_3 \\ &\quad \text{by (1) and (2).} \\ &= (c\ bab)d(bab\ c\ bab\ c\ bab)d\ bab\ c \\ &= (c\ bab\ \tilde{c})d(\tilde{c})^{-1}(bab\ c\ bab\ c\ bab)d\ bab\ c \\ &= (bab\ c\ bab\ a)d(bab\ c\ baba)d\ bab\ c \quad \text{by (1.7) of I.} \end{aligned}$$

So $x \in Y$. Thus we may suppose $\alpha \neq 0$; that is, (4) is a non trivial equation in ζ of degree at most q. Now if $q^2 - 4q - 3 > 0$, there exist a suitable ζ not satisfying (4) and so this completes the proof of (i).

(ii) We may suppose $b_7a_4b_8c_3b_9a_5b_{10}$ satisfies either (i) or (ii) of 1.6. The proof is the same as in (i).

(iii) If $b_7a_4b_8c_3b_9a_5b_{10} \in L_3L_2L_1L_2L_1L_3$, then $x \in Y$.

We have

$$\begin{aligned} x &= c\ bab\ d\ bab\ c\ bab\ d(bab\ c\ bab)d\ bab\ c \\ &= c\ bab\ d\ bab\ c\ bab\ d(c\ baba\ c)d\ babc \\ &= (c\ bab\ c^*)d(c^{*-1}\ bab\ c\ babc)d\ bab\ a\ d\ cbabc \\ &\equiv (c\ bab\ a)d(bab\ c\ baba)\ dbaba\ d\ cbabc \quad \text{by (1.7) of I} \\ &= c\ bab\ a\ d\ bab\ c(baba\ d\ baba\ d)\ cbabc \\ &= cbab\ a\ d\ bab\ c(d\ bab\ d\ bab\ a)cbabc \\ &= c(bab\ a\ d\ bab\ d)c\ bab\ d\ bab\ a\ c\ bab\ c \end{aligned}$$

$$= d\ c\ bab\ d\ baba\ c\ bab\ d\ baba\ a\ babc \quad \text{by (1) and (2)}$$
$$\equiv c\ bab\ d(baba\ c\ bab\ \bar{c})d(\bar{c}^{-1}\ babacbabc)$$
$$= c\ bab\ d\ babc\ baba\ d\ bab\ c\ baba \quad \text{by (1.7) of I.}$$

So $x \in Y$

(iv) If $b_1a_1b_2d_1b_3a_2b_4 \in L_4L_2L_1L_2L_1L_4$, then $x \in Y$.
We have

$$\begin{aligned} x &= c(bab\ d\ bab)c\ bab\ d\ bab\ c\ bab\ d\ bab\ c \\ &= c(d\ baba\ d)c\ bab\ d\ bab\ c\ bab\ d\ bab\ c \\ &\equiv cbaba\ c\ dbabdbab\ c\ bab\ d\ bab\ c \\ &\equiv cbab\ c\ bab\ d\ bab\ c\ bab\ d\ bab\ c \quad \text{by (1) and (2)} \\ &\equiv (cbabc\ bab\ c^*)d(c^{*-1}\ babc\ bab)d\ babc \\ &\equiv bab\ c\ baba\ d\ bab\ c\ baba\ d\ bab\ c \\ &\in Y \end{aligned}$$

(v) If $b_1a_1b_2d_1b_3a_2b_4 \in L_1L_4L_2L_4L_2L_1L_2L_1L_4$, then $x \in Y$.
We have

$$\begin{aligned} x &= c(bab\ d\ bab)c\ bab\ d\ bab\ c\ babd\ bab\ c \\ &\equiv c\ bdbaba\ d\ c\ bab\ d\ babc\ babd\ babc \\ &\equiv cbd\ bab\ c\ bab\ d(bab\ c\ bab)d\ bab\ c \quad \text{by (1) and (2).} \end{aligned}$$

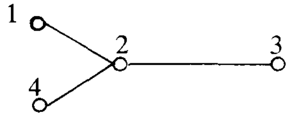
If the bracketed term above belongs to $L_3L_2L_1L_2L_1L_3$, then we are done by (iii). In view of (ii), we may suppose it satisfies (ii) of (1.6). Therefore

$$\begin{aligned} x &\equiv c\ b\ d\ bab\ c\ bab\ d\ (a\ cbc\ baba)d\ babc \\ &\equiv c\ b\ d\ (bab\ c\ babac)\ dbcbaba\ d\ bab\ c \\ &\equiv cbd(cbc\ aba\ bcbc)\ dbcbaba\ d\ babc \quad \text{by (3)} \\ &\equiv (cbd\ cbc)aba(bcbc\ dbcb\ )aba\ d\ babc \quad \text{by (3)} \\ &\equiv cbc\ (dbdb\ aba\ dbd)\ cbc(dbdb\ aba\ dbab)c \\ &\equiv cbc(aba\ dbd\ baba)cbc(aba\ dbd\ bab)c \quad \text{by (3)} \\ &\equiv cbc\ aba\ dbd(baba\ cbc\ bab)d(bab\ c) \quad \text{by (1), (2)} \\ &\equiv cbc\ aba\ dbd(baba\ cbcbab\ \bar{c})d(\bar{c}^{-1}\ babc) \\ &\equiv cbc\ aba\ dbd\ bab\ c\ baba\ d\ bab\ c \\ &\equiv (cbc\ bab)d(bab\ c\ bab)dbab\ c \quad \text{by (1), (2)} \\ &\equiv c\ bab\ d\ bab\ c\ bab\ d\ bab\ c \quad \text{by (1.7) of I} \\ &\in Y \end{aligned}$$

In view of (i), (iv), (v) and (1.6), the proof is now complete.

THEOREM 1.8. *A universal group* G *of type* D_4 *generated by* $SU(3,q)$*'s is isomorphic to* $\mathrm{Spin}(V_0)$ *where* $\dim V_0 = 8$.

PROOF. Let the graph of G be as follows



By (1.4) and (1.4) of I, we have homomorphisms

$$G \xrightarrow{\phi} \operatorname{Spin}(V_0) \xrightarrow{\chi} \Omega(V_0)$$

where $\dim V_0 = 8$ such that $\theta(L_i) = L_i^*$ $i = 1, 2, 3$ and $\theta(L_4) = \Gamma_1^*$ where $\theta = \chi\phi$. We observe that $z_1 z_4 \in \ker\theta$ where $z_i \in Z(L_i)^{\#}$. Suppose $g \in \ker\theta$. We consider the following possibilities.

(i) $g = n_1 c n_2$ where $n_i \in N = \langle L_1, L_2, L_4\rangle$ and $c \in L_3$. Then $cn \in \ker\theta$ where $n = n_2 n_1$. Hence $\theta(c)(v_8) = \theta(n)^{-1}(v_8) = v_8$. Therefore $c = 1$ as $\theta|_{L_3}$ is an isomorphism and the stabilizer of v_8 in L_3^* is 1. Therefore $g \in N$. On the other hand $\theta|_N$ has kernel $\langle z_1 z_4\rangle$. Hence $g \in \langle z_1 z_4\rangle$.

(ii) $g = n_1 c_1 n_2 c_2 n_3$ where $n_i \in N$ and $c_i \in L_3$. Again we have $c_1 n_2 c_2 n \in \ker\theta$ where $n = n_3 n_1$. We may suppose $c_1, c_2 \notin H_3$, otherwise we are back to (i). Let $x = \theta(c_1)\theta(n_2)\theta(c_2) = \theta(n)^{-1}$. By comparing the images of v_8 (resp. v_7) for both expressions of x, we see that $\theta(n_2)$ fixes $\langle v_6\rangle$ (resp. $\langle v_5\rangle$). It follows that $n_2 \in L_1 L_4 H_2$. Therefore $n_2 c_2 \in L_3 N$ and so we are in (i).

(iii) $g = n_1 c_1 n_2 c_2 n_3 c_3 n_4$ where $n_i \in N$, $c_i \in L_3$. We may suppose none of c_i belongs to H_3. The proof of (v) in (1.7) shows that either (i) or (ii) of (1.6) or g has the form in (ii) above. Hence we may assume that $n_2 = b_1 d b_2 a b_3$ where $a \in L_1$, $b_i \in L_2$ and $d \in L_4$. If one of a, b_1, b_3, d belongs to $H_1 H_2 H_4$, then we may reduce the form of g to case (ii) above using 1.7 of I and bearing in mind the relation $L_3 L_2 L_4 L_2 L_3 \subseteq L_2 L_4 L_3 L_2 L_3 L_4 L_2$ ((2.2) of I. Let $x = \theta(c_1 b_1 d b_2 a b_3 c_2) = \theta(n_3 c_3 n_4 n_1)^{-1}$. Using the second expression for x, we see that the projection of $x(v_8)$ into $\langle v_7\rangle$ is 0. On the other hand, because none of c_1, c_2, b_1, b_3, a, d belongs to $H_1 H_2 H_3 H_4$, the projection of $\theta(c_2)(v_8)$ [resp. $\theta(b_3 c_2)(v_8)$; $\theta(a b_3 c_2)(v_8)$; $\theta(d b_2 a b_3 c_2)(v_8)$; $\theta(b_1 d b_2 a b_3 c_2)(v_8)$; $\theta(c_1 b_1 d b_2 a b_3 c_2)(v_8)$] into $\langle v_8\rangle$ [resp. $\langle v_4\rangle$; $\langle v_2\rangle$; $\langle v_3\rangle$; $\langle v_5\rangle$; $\langle v_7\rangle$] is a non zero vector. This is a contradiction. Thus we have shown that $\ker\theta = \langle z_1 z_4\rangle$ and ϕ is an isomorphism. This completes the proof.

THEOREM 1.9. *Let G be a universal group of type D_n generated by $SU(3, q)$'s, $n \geqq 2$. Then G is isomorphic to* $\operatorname{Spin}(V_0)$ *where* $\dim V_0 = 2n$.

PROOF. The result holds for $2 \leqq n \leqq 4$ as remarked earlier and by (1.7). We may suppose $n \geqq 5$. Let the graph of G be as follows

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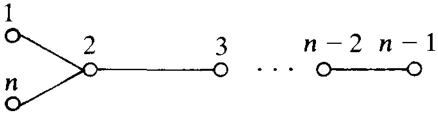


By (1.4), we have homomorphisms

$$G \xrightarrow{\psi} \mathrm{Spin}\,(V_0) \xrightarrow{\phi} \Omega(V_0).$$

Let $\theta = \phi\psi$ and $Z = \langle z_1 z_n \rangle$ where $z_i \in Z(L_i)^{\#}$. We shall prove by induction on n that $\ker\theta = Z$. Thus we may suppose that $\langle L_n, L_i \mid 1 \leqq i \leqq n-2 \rangle \cong \mathrm{Spin}^{\varepsilon}(2(n-1), q)$ where $\varepsilon = +$ if $n-1$ is even and $\varepsilon = -$ if $n-1$ is odd and $P = \langle L_n, L_i \mid 1 \leqq i \leqq 3 \rangle \cong \mathrm{Spin}^{+}(8, q)$.

Let n_i be an element of L_i which inverts H_i. Set $\Gamma_1 = L_n$; $p_1 = n_n$ and $K_1 = H_n$. We define inductively $\Gamma_{i+1} = n_i p_i L_{i+1} p_i^{-1} n_i^{-1}$ and $K_{i+1} = n_i p_i H_{i+1} p_i^{-1} n_i^{-1}$. From the isomorphism of $\langle L_1, L_2, \Gamma_1 \rangle$ onto $SU(4, q)$ we obtain that $[L_2, \Gamma_2] = 1$; $n_2 p_2 L_1 p_2^{-1} n_2^{-1} = \Gamma_1$ and $n_2 p_2 \Gamma_1 p_2^{-1} n_2^{-1} = L_1$. Since $n_1 p_1 \langle L_2, L_3 \rangle p_1^{-1} n_1^{-1} = \langle \Gamma_2, L_3 \rangle$ and $n_1 p_1 \langle H_2, H_3 \rangle p_1^{-1} n_1^{-1} = \langle K_2, H_3 \rangle$, we easily verify that $M = \langle \Gamma_2, L_i \mid 2 \leqq i \leqq n-1 \rangle$ is a universal group of type D_{n-1} (universal because $z_1 z_n \neq 1$ in P and $z_2 n_1 p_1 z_2 p_1^{-1} n_1^{-1} = z_1 z_n$). Similarly $N = \langle \Gamma_4, L_i \mid 4 \leqq i \leqq n-1 \rangle$ is a universal group of type D_{n-3}. We note finally from the isomorphism of $\langle L_2, L_3, \Gamma_2 \rangle$ onto $SU(4, q)$, $n_3 p_3$ interchanges L_2, Γ_2 by conjugation. Thus $\langle L_1, L_2, \Gamma_1 \rangle$ commutes elementwise with Γ_4 as $[L_1, L_3] = n_2 p_2 [\Gamma_1, L_3] p_2^{-1} n_2^{-1} = 1$; $[\Gamma_1, \Gamma_3] = 1$ and $\langle L_1, L_2, \Gamma_1 \rangle = \langle L_1, \Gamma_2, \Gamma_1 \rangle$.

Let $V_1 = \{xv_1 + \bar{x}v_2\}$; $V_2 = \{xv_3 + \bar{x}v_4 + yv_5 + \bar{y}v_6\}$; $V_3 = \{xv_7 + \bar{x}v_8\}$ and $V_4 = (V_1 + V_2 + V_3)^{\perp}$, $x, y \in F$. We regard $\Omega(U)$ naturally as a subgroup of $\Omega(V_0)$ however U is a subspace of V_0. We verify that $\Omega(V_1 + V_2) = \theta(\langle L_1, \Gamma_1, L_2 \rangle)$; $\Omega(V_3 + V_4) = \theta(N)$; $\Omega(V_1 + V_2 + V_3) = \theta(P)$ and $\Omega(V_2 + V_3 + V_4) = \theta(M)$ and $[\Omega(V_1 + V_2), \Omega(V_3 + V_4)] = 1$. We now apply Wong's Theorem 3A [Wong (1974)] and get that $G/Z \cong \Omega(V_0)$. Therefore $G \cong \mathrm{Spin}\,(V_0)$. The proof is now complete.

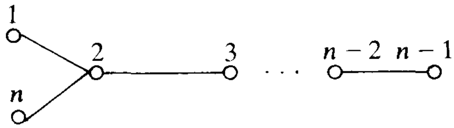
2. Groups of types E_6, E_7, E_8

The success in identifying groups of types E_6, E_7 and E_8 depends on the fact that groups of type D_n when n is even are also Chevalley groups of type D_n, and therefore they have Steinberg's generators and relations. For convenience in discussing the proof we introduce the following terminology.

DEFINITION. *Let M_1, M_2 be subgroups of a group X with $M_1 \cong SL(2, q)$ (q odd, $q > 3$) such that $\langle M_1, M_2 \rangle \cong SL(3, q)$ (respectively $SU(3, q)$) and $[Z(M_1), Z(M_2)] = 1$. We say M_1 is joined to M_2 linearly (resp. unitarily) in X. We remark that it follows there exist cyclic subgroups $H_i \subseteq M_i$ of order $q-1$*

(*resp.* $q+1$) *such that* $\langle H_1, H_2\rangle = H_1 \times H_2$ *and* $M_1H_2 \cong H_1M_2 \cong GL(2,q)$ (*respectively* $GU(2,q)$).

Next we need to know the structure of a universal group Y of type D_n in some details. We introduce the following notation. Let the graph of X be as follows



If L_i is joined to L_j unitarily, we can choose generators h_i, h_j of H_i, H_j; elements n_i, n_j in L_i, L_j which invert h_i, h_j respectively such that

$$n_j h_i n_j^{-1} = h_i h_j = n_i h_j n_i^{-1}.$$

Set $\Gamma_1 = L_n$; $p_1 = n_n$; $k_1 = h_n$ and define inductively $\Gamma_{i+1} = xL_{i+1}x^{-1}$; $p_{i+1} = xn_{i+1}x^{-1}$; $k_{i+1} = xh_{i+1}x^{-1}$; where $x = n_ip_i$. Let $\langle y_i\rangle = Z(L_i)$ and $\langle z_i\rangle = Z(\Gamma_i)$.

LEMMA 2.1. *The following hold for the group X defined above.*

(i) $[L_i, L_j] = [L_i, \Gamma_j] = [\Gamma_i, \Gamma_j] = [L_i, \Gamma_i] = 1$ if $j \neq i-1, i, i+1$

(ii) L_i *is joined to* Γ_{i-1}, L_{i-1}, Γ_{i+1}, L_{i+1} *unitarily*;

(iii) Γ_i *is joined to* Γ_{i-1}, L_{i-1}, Γ_{i+1}, L_{i+1} *unitarily*;

(iv) $p_ih_{i+1}p_i^{-1} = k_ik_{i+1} = n_{i+1}k_in_{i+1}^{-1}$;

(v) $p_ih_{i-1}p_i^{-1} = h_{i-1}k_i^{-1} = n_{i-1}k_i^{-1}n_{i-1}^{-1}$;

(vi) $k_{i+1} = h_ih_{i+1}k_i$;

(vii) $y_iz_i = y_1z_1$;

(viii) $n_{i+1}p_{i+1}L_ip_{i+1}^{-1}n_{i+1}^{-1} = \Gamma_i$.

PROOF. Identifying Y with Spin (V_0) where dim $V_0 = 2n$, we verify all the assertions easily.

DEFINITION. *We call Γ_i the dual of L_i in Y. We note that the dual of L_i in Y is unique if $n \geqq 5$; when $n = 4$, there are three subgroups L_3, Γ_3, Γ_1 which can be dual of L_1. In* (2.2)–(2.5), *we shall use the notation just introduced without further comment.*

LEMMA 2.2. *Let Y be the universal group of type D_n defined above. Then $C_Y(y_1)$ contains a perfect group C of index* 2 *which is the central product of $L_1 \times \Gamma_1$ and $\langle L_i, \Gamma_i \mid 3 \leqq i \leqq n-1\rangle$ and $Z(C) = \langle y_1, z_1\rangle$.*

PROOF. The assertions are straightforward [Iwahori (1970)]. (Note that we are assuming q odd, $q > 3$.)

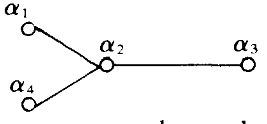
COROLLARY 2.3. *Let x be an involution in Y conjugate to y_1 in Y. Then there exists a unique subgroup L_x in Y such that $Z(L_x) = \langle x \rangle$.*

LEMMA 2.4. *Let Y be the universal group of type D_4, generated by $SU(3)$'s*

(i) *Suppose $q \equiv 1 \pmod 4$. Every involution x in $C_Y(y_1)' - Z(C_Y(y_1))$ is conjugate to $n_1p_1n_3p_3$ and the unique subgroup L_x of (2.3) is joined to L_1, Γ_1, L_3, Γ_3 linearly.*

(ii) *Suppose $q \equiv -1 \pmod 4$. Every involution x in $C_Y(y_1) - C_Y(y_1)'$ is conjugate to $h_2n_1p_1n_3p_3$ and the unique subgroup L_x of (2.3) is joined to L_1, Γ_1, L_3, Γ_3 linearly.*

PROOF. Since $Y \cong \mathrm{Spin}^+(8, q)$, we can regard Y as a universal Chevalley group of type D_4 with the following Dynkin diagram.



Corresponding to each root α, we have the one parameter unipotent subgroup $U_\alpha = \{x_\alpha(t) \mid t \in F_0\}$; $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$; $h_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)n_\alpha^{-1}$ and $X_\alpha = \langle U_\alpha, U_{-\alpha}\rangle$. Here we can assume $y_1 = h_{\alpha_1}(-1)$. Then $C_Y(y_1) = C\langle h_{\alpha_2}(\lambda)\rangle$ where $C = X_{\alpha_0}X_{\alpha_1}X_{\alpha_3}X_{\alpha_4}$, $\langle \lambda \rangle = \dot{F}$ and α_0 the longest root. We have $h_{\alpha_2}(\lambda)^2 \in C$ and $\langle X_{\alpha_i}, h_{\alpha_2}(\lambda)\rangle \cong GL(2, q)$, $i = 0, 1, 3, 4$; $[X_{\alpha_i}, X_{\alpha_j}] = 1$ $i \neq 2 \neq j$ and $i \neq j$.

Suppose $q \equiv 1 \pmod 4$. Then $h_{\alpha_2}(-1) \in C$ and $h_{\alpha_2}(-1) \notin Z(C) = \langle h_{\alpha_1}(-1), h_{\alpha_3}(-1), h_{\alpha_4}(-1)\rangle$. If x is an involution in $C - Z(C)$, it necessarily must have the form $x_0x_1x_3x_4$ where $x_i \in X_{\alpha_i}$ and $0(x_i) = 4$. Since $SL(2, q)$ contains just one class of elements of order 4, all involutions in $C - Z(C)$ are conjugate to $h_{\alpha_2}(-1)$. The assertions of (i) are now clear.

Next suppose $q \equiv -1 \pmod 4$. Let x be an involution in $C_Y(y_1) - C_Y(y_1)'$. Then x has the form $x = h_{\alpha_2}(-1)yz$ when $y \in X_{\alpha_1}$ and $z \in X_{\alpha_0}X_{\alpha_3}X_{\alpha_4}$. As $X_{\alpha_1}\langle h_{\alpha_2}(\lambda)\rangle \cong GL(2, q)$, $h_{\alpha_2}(-1)y$ is an involution conjugate in Y_{α_1} to $h_{\alpha_2}(-1)$, we may assume $y = 1$. A similar argument proves that $h_{\alpha_2}(-1)yz$ is conjugate to $h_{\alpha_2}(-1)$ in $C_Y(y_1)$. This completes the proof.

LEMMA. *Let Y be the universal group of type D_6 generated by $SU(3)$'s*

(i) *There are two classes of non central involutions with representatives y_1 and y_3y_5 and $C_Y(y_1) = C_Y(y_3y_5)$;*

(ii) *Suppose x, y are commuting involutions conjugate to y_1 in Y such that xy is not conjugate to y_1 in Y. Let L_x and L_y be the unique subgroups L_x and L_y of (2.3) with $Z(L_x) = \langle x \rangle$, $Z(L_y) = \langle y \rangle$. Then $[L_x, L_y] = 1$.*

PROOF. (i) The details can be easily computed [Iwahori (1970)].

To prove (ii), we introduce the usual Chevalley notation as in (2.4) since $Y \cong \mathrm{Spin}^+(12,q)$. We can assume $y_1 = h_{\alpha_1}(-1)$ and also $x = y_1$. Then $L_x = L_1 = X_{\alpha_1}$. If $y \in C_Y(y_1)'$, then $y = x_1x_2$ where $x_1 \in L_1$ and $x_2 \in \Gamma_1 \times \langle L_i, \Gamma_i \mid 3 \leqq i \leqq 5\rangle$ by (2.2). Suppose first $0(x_1) = 0(x_2) = 4$. From the structure of $SU(2,q)$, we get $xy = y_1x_1x_2 \underset{Y}{\sim} x_1x_2 = y$, a contradiction to our hypothesis. So we may suppose $0(x_1) \leqq 2$ i.e. $x_1 = 1$ or $x_1 = y_1$. We compute that the classes of involutions in $\Gamma_1 \times \langle L_i, \Gamma_i \mid 3 \leqq i \leqq 5\rangle$ have representatives $z_1, z_1y_3y_5, z_1y_3z_5, z_1y_3, y_3y_5, y_3z_5, y_5z_5, y_3$. Of these only those with representatives z_1, y_3 satisfy our requirement. In these cases we have $L_y = \Gamma_1$ or L_3 and so $[L_x, L_y] = 1$.

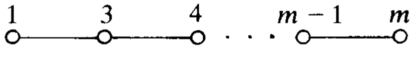
Now suppose $y \in C_Y(y_1) - C_Y(y_1)'$. Suppose $q \equiv 1 \pmod 4$. Then y must have the form $y = y_2x_1x_2$ where $x_1 \in L_1$ and $x_2 \in \Gamma_1 \times \langle L_i, \Gamma_i \mid 3 \leqq i \leqq 5\rangle$. From the fact that $\langle L_i, h_2\rangle \cong GU(2,q)$, we find that y_2x_1 is an involution conjugate in L_1 to y_2. Therefore we may assume $x_1 = 1$. Then $xy = y_1(y_2x_2) = n_1(y_2x_2)n_1^{-1} \underset{Y}{\sim} y$ in contradiction to our assumption. The case $q \equiv -1 \pmod 4$ is proved similarly regarding Y as a Chevalley group and the fact $\langle L_1, h_{\alpha_2}(\lambda)\rangle \cong GL(2,q)$ where $\langle \lambda \rangle = \dot{F}$. This completes the proof.

In the proofs of (2.6)–(2.8) we shall encounter certain subgroups which are homomorphic images $\bar{Y}$ of $Y = SU(m,q)$ for some integer $m > 0$. For convenience, we introduce the following uniform notation for elements and subsets of $\bar{Y}$. Let U be a non degenerate hermitian space of dimension m on which Y acts naturally. We choose an orthonormal basis $\{u_1, u_3, u_4, \cdots, u_{m+1}\}$. Let L^*_{ij} be the subgroup of Y which leaves $\langle u_i, u_j\rangle^\perp$ pointwise fixed. If $i < j$, let h^*_{ij}, n^*_{ij} be the elements of L^*_{ij} such that $h^*_{ij}(u_i) = \sigma u_i$, $h^*_{ij}(u_j) = (\sigma^{-1})u_j$, $n^*_{ij}(u_i) = u_j$ and $n^*_{ij}(u_j) = -u_i$ where $\langle \sigma \rangle$ is the subgroup of order $q+1$ in $\dot{F}$. The images of L^*_{ij}, h^*_{ij}, n^*_{ij} in $\bar{Y}'$ will be denoted by L_{ij}, h_{ij}, n_{ij} respectively.

Suppose that G is a group of certain type generated by $SU(3,q)$'s and L_i, L_j are subgroups such that L_i is joined to L_j unitarily. We can choose generators h_i, h_j of H_i, H_j, n_i, n_j of L_i, L_j respectively such that n_i, n_j inverts h_i, h_j and

$$n_ih_jn_i^{-1} = h_ih_j = n_jh_in_j^{-1}.$$

In particular, if G has a subgroup of type A_{m-1} isomorphic to $\bar{Y}$ and with subgraph



We may then identify h_{13}, $h_{i,i+1}$; n_{13}, $n_{i,i+1}$; $L_{i,i+1}$ with h_1, h_i; n_1, n_i; L_1, L_i respectively.

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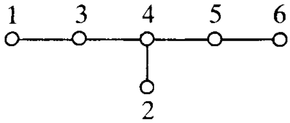
THEOREM 2.6. *Let G be a universal group of type E_6 generated by $SU(3,q)$'s. Then G is isomorphic to the twisted analogue of the universal Chevalley group of type E_6 over F_0.*

PROOF. Let the graph of G be as follows



In view of (1.4) and (2.3) of I, the subgroup generated by L_i, $i = 1,3,4,5,6$ is isomorphic to $SU(6,q)$ and so we can use the notation just introduced. Let $\theta = n_{14}n_{57}$. Then $[\theta, L_2] = 1$ as $n_{14} \in \langle L_1, L_3\rangle$ and $n_{57} \in \langle L_5, L_6\rangle$. The element θ interchanges the elements of the sets $\{L_1, L_3\}$; $\{L_5, L_6\}$ and $\{L_4, L_{17}\}$ by conjugation. We compute that $\theta n_{45}\theta^{-1} = n_{17}$ and L_{17} is joined to L_2 unitarily as L_4 is. Since L_4, L_5, L_6 commute elementwise with L_{17}, it follows $N = \langle L_{17}, L_i \mid 2 \leqq i \leqq 5\rangle \cong \mathrm{Spin}^-(10,q)$ as $(h_3h_5)^{q+1/2} \neq 1$ by (1.9). So $N \cong \mathrm{Spin}(V_0)$ where V_0 is the symmetric bilinear space introduced in §1. Set $V_i = \{xv_{2i-1} + \bar{x}v_{2i} + yv_{2i+1} + \bar{y}v_{2i+2}\}$, $x, y \in F$, $i = 1,2,3,4$. We may assume $L_3 \times L_5 = \mathrm{Spin}(V_1)$; $L_4 \subseteq \mathrm{Spin}(V_2)$; $L_2 \subseteq \mathrm{Spin}(V_3)$ and $L_{17} \subseteq \mathrm{Spin}(V_4)$. Since $n_{34}n_{56}L_4n_{56}^{-1}n_{34}^{-1} = L_{36}$, L_{36} is the dual of L_4 in N. So the subgroup $N_0 = \langle L_{17}, L_2, L_4, L_{36}\rangle \cong \mathrm{Spin}^+(8,q)$. Let L_0 be the dual of L_{17} in N. By (2.1), $[L_0, L_i] = 1$ $i = 3,4,5$. Since $\theta N_0\theta^{-1} = N_0$, it follows $\theta L_0\theta^{-1} = L_0$ by (2.2). Therefore $[L_0, L_1] = 1 = [L_0, L_6]$. In particular $[\theta, L_0] = 1$.

Let $z = n_{45}n_{17}n_{36}n_0$ where $n_0 = n_2n_{45}n_{36}n_2n_{36}^{-1}n_{45}^{-1}$ when $q \equiv 1 \pmod 4$ and $z = h_2n_{45}n_{17}n_{36}n_0$ when $q \equiv -1 \pmod 4$. By (2.4), z is an involution and there exists unique subgroup Γ_2 with $Z(\Gamma_2) = \langle z\rangle$ and Γ_2 is joined to L_{17}, L_0, L_{36}, L_4 linearly. Moreover $N_0 = \langle L_{17}, L_{45}, L_{36}, L_0, \Gamma_2\rangle$. Furthermore $L_{45} \times L_{36} = \mathrm{Spin}(V_2)$; $L_{17} \times L_0 = \mathrm{Spin}(V_4)$. Since $\langle \Gamma_2, L_{17}\rangle$ and $\langle \Gamma_2, L_{45}\rangle$ are isomorphic to $SL(3,q)$, it follows that there exist hyperbolic planes P_1, P_2 in V_2, V_4 respectively such that $\Gamma_2 \subseteq \mathrm{Spin}(P_1 + P_2)$. Let P_3 be the orthogonal complement (a hyperbolic plane) of P_2 in V_4. Thus $Q = (P_1 + P_2 + P_3)^\perp$ is a symmetric bilinear space of dimension 4 and index 1. We note that $Q \cap V_2$ is a hyperbolic plane in V_2 orthogonal to P_1. Let $S_2 = \mathrm{Spin}(Q)$. Then $S_2 \cong SL(2,q^2)$ [Dieudonné (1955)]; $\langle S_2, L_4\rangle = \langle L_i \mid 3 \leqq i \leqq 5\rangle$ and $[S_2, \Gamma_2] = 1$. We note that $\langle S_2, L_4\rangle$ as a subgroup of $\mathrm{Spin}(V_0)$ is $\mathrm{Spin}(V_1 + \langle xv_5 + \bar{x}v_6\rangle)$. On the other hand, $\langle S_2, L_4\rangle$ regarded as a subgroup of M acts on the hermitian space $\{u_3, u_4, u_5, u_6\}$. The isomorphism between $\mathrm{Spin}(V_1 + \langle xv_5 + \bar{x}v_6\rangle)$ and $SU(4,q)$ maps Q to a totally degenerate subspace U_0 of dimension 2 [Dieudonné (1955)] in $\{u_3, u_4, u_5, u_6\}$ with $U_0 \cap \langle u_4, u_5\rangle = \langle w_4\rangle \neq 0$ and $U_0 \cap \langle u_3, u_6\rangle = \langle w_3\rangle \neq 0$ and $U_0 = \langle w_3, w_4\rangle$. Let $U_1 = \theta\langle w_3, w_4\rangle = \langle w_3, \theta(w_4)\rangle$ since θ fixes

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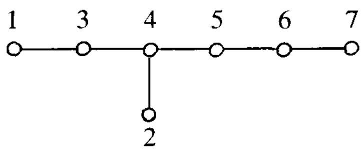
elementwise $\langle u_3, u_6\rangle$. It follows $\langle U_0, U_1\rangle = \langle w_3, w_4, \theta(w_4)\rangle$ is a 3-dimensional totally degenerate space as $\theta(w_4) \in \langle u_1, u_7\rangle$. Set $S_1 = \theta S_2 \theta^{-1}$. Clearly $\langle S_1, S_2\rangle \cong SL(3, q^2)$ and $\langle S_1, L_{36}\rangle = \langle L_1, L_{36}, L_6\rangle$ and $\langle S_1, S_2, L_4\rangle = M$.

We have shown $[\theta; L_2] = 1 = [\theta, L_0]$ and $\theta n_4 \theta^{-1} = n_{17}$. So $\theta z \theta^{-1} = z$ and because $\theta N_0 \theta^{-1} = N_0$, therefore $\theta \Gamma_2 \theta^{-1} = \Gamma_2$ by (2.3). It follows $[S_1, \Gamma_2] = 1$.

We now look at the following chain of subgroups Γ_2, L_4, S_2, S_1. First they generate G since $\langle S_1, S_2, L_4\rangle = M$ and $L_2 \subseteq \langle L_4, \Gamma_2, L_{36}, L_{17}\rangle$. We also have the following relations $\langle \Gamma_2, L_4\rangle \cong SL(3, q)$; $[\Gamma_2, S_2] = 1 = [\Gamma_2, S_1]$; $\langle L_4, S_2\rangle \cong SU(4, q)$; $[L_4, S_1] = 1$ (as $\langle L_1, L_{36}, L_6\rangle$ centralizes L_4); $\langle S_2, S_1\rangle \cong SL(3, q^2)$. It is now easy to see that the conditions of Curtis' Theorem 1.4 [Curtis (1965)] are satisfied. So $G \cong {}^2E_6(q^2)$, the group of fixed points in the universal Chevalley group of type E_6 over F of a 'twisting' automorphism.

THEOREM 2.7. *Let G be a universal group of type E_7 generated by $SU(3, q)$'s. Then G is isomorphic to the universal Chevalley group of type E_7 over F_0.*

PROOF. Let the graph of G be as follows



The subgroup $P = \langle L_1, L_i \mid 3 \leqq i \leqq 7\rangle$ is isomorphic to $SU(7, q)$ and $R = \langle L_i \mid 1 \leqq i \leqq 6\rangle \cong {}^2E_6(q^2)$. In the proof of (2.6), we have found the subgroup L_0 is joined to L_2 unitarily and commutes elementwise with L_1, L_i $3 \leqq i \leqq 6$. The subgroup $N_0 = \langle L_{36}, L_{45}, L_2, L_0, L_{17}\rangle$ is universal of type D_4. We also found the element $n_2 n_{45} n_{36} n_2 n_{36}^{-1} n_{45}^{-1}$ which interchanges L_0 and L_{17} by conjugation (see 2.1)). Because L_{17} is joined to $L_7 = L_{78}$ unitarily, L_0 is joined to L_{78} unitarily. The subgroup $S = \langle L_1, L_0, L_i \mid 3 \leqq i \leqq 7\rangle$ is a group generated by $SU(3, q)$'s of type A_7 and so by (2.3) of I S is a homomorphic image of $SU(8, q)$. We now use the notation introduced just prior to (2.6) and so $L_0 = L_{89}$. As P is a subgroup of S, the previous notation for subgroups and elements of P in (2.6) is consistent with the present one.

Let $\psi = n_{17} n_{36} n_{45} n_{89}$. We compute that ψ interchanges the elements of the sets $\{L_1, L_6\}$; $\{L_3, L_5\}$, $\{L_{19}, L_7\}$ and fixes L_4, L_{89} by conjugation. As $\psi \in N_0$, we compute that ψ fixes L_2 by conjugation (see also (2.1) (viii)). Therefore $[L_{19}, L_2] = 1$ and so L_{19} commutes elementwise with L_i, $2 \leqq i \leqq 5$ and is joined to L_1 and $L_{89} = L_0$ unitarily. Thus $M_1 = \langle L_{19}, L_1, L_i \mid 2 \leqq i \leqq 5\rangle \cong \mathrm{Spin}^+(12, q)$. Similarly $M_2 = \langle L_i \mid 2 \leqq i \leqq 7\rangle$ and $M_3 = \langle L_{78}, L_{89}, L_i \mid 2 \leqq i \leqq 5\rangle$ are isomorphic to $\mathrm{Spin}^+(12, q)$.

Let Γ be the dual of L_2 in M_3. By (2.1), $[L_i,\Gamma]=1$ $i=2,3,5,7$ and Γ is joined to L_4 and L_{89} unitarily. But Γ is the dual of L_3 and L_5 in M_1, M_2 respectively. Hence $[L_{19},\Gamma]=1$ and Γ is joined to L_1 and L_6 unitarily. Let $\phi = n_{13}n_{49}n_{67}n_{58}$. We compute that ϕ interchanges the elements of the sets $\{L_3, L_{19}\}$; $\{L_4, L_8\}$; $\{L_5, L_7\}$ by conjugation. Let t be the involution in L_2. Set $N_1=\langle L_i \mid 2\leqq i\leqq 5\rangle$ and $N_2=\langle L_2, L_{89}, L_{19}, L_7\rangle$. These groups are isomorphic to $\mathrm{Spin}^+(8,q)$ and $\phi N_1\phi^{-1}=N_2$. As $C_{N_1}(t)'=\Gamma L_2L_3L_5$ and $C_{N_2}(t)=\Gamma L_2L_{19}L_7$ it follows $\phi\Gamma\phi^{-1}=\Gamma$. Let $L=n_{13}n_{49}\Gamma n_{49}^{-1}n_{13}^{-1}$. Then $L=n_{58}^{-1}n_{67}^{-1}\Gamma n_{67}n_{58}$. Since $n_{13}n_{49}\in N(L_1)$ and $n_{58}n_{67}\in N(L_6)$, L is joined to L_1 and L_6 unitarily since Γ is.

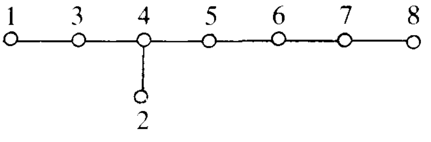
In G we have defined the elements h_i, n_i such that $n_ih_in_i^{-1}=h_i^{-1}$ and $n_ih_jn_i^{-1}=h_ih_j=n_jh_in_j^{-1}$ if $\{i,j\}$ is an edge. Let $h_\Gamma = x_1h_5x_1^{-1}$, $n_\Gamma = x_1n_5x_1^{-1}$; $h_L = x_2h_7x_2^{-1}$ and $n_L = x_2n_7x_2^{-1}$ where $x_1 = n_4n_3n_2n_4n_2^{-1}n_3^{-1}$ and $x_2 = n_6n_5n_\Gamma n_6n_\Gamma^{-1}n_5^{-1}$. So by (2.1), we have $n_\Gamma h_4n_\Gamma^{-1}=h_4h_\Gamma^{-1}$; $n_\Gamma h_6n_\Gamma^{-1}=h_6h_\Gamma$ and $n_Lh_6n_L^{-1}=h_6h_L^{-1}$. Similarly working with the subgroup $\langle L_5, L_6, L_7, \Gamma, L_1, L_3\rangle$ we obtain that $n_Lh_1n_L^{-1}=h_1h_L$, $n_{19}h_1n_{19}^{-1}=h_1h_{19}^{-1}$. Also by (2.1) we have the identities $h_2h_3h_4^2h_8=h_\Gamma$; $h_5h_6^2h_\Gamma h_7=h_L$ and $h_3h_1^2h_\Gamma h_L=h_{19}$.

Now set $z_1=n_Ln_{19}n_\Gamma n_3$ (resp. $h_1h_Ln_Ln_{19}n_\Gamma n_3$); $z_4=n_\Gamma^{-1}n_3n_5n_2$ (resp. $h_4h_5h_\Gamma^{-1}n_\Gamma^{-1}n_3n_5n_2$) and $z_6=n_\Gamma n_5n_L^{-1}n_7$ (resp. $h_6^{-1}h_\Gamma^{-1}h_Ln_\Gamma n_5n_L^{-1}n_7$) when $q\equiv 1(4)$ (resp. $q\equiv -1(4)$). We compute that z_1, z_4, z_6 are commuting involutions such that z_1z_4, $z_4z_6z_1z_6$ are not conjugate to z_1, z_4, z_6 in M_1, M_2 and $\langle L_5, L_6, L_7, \Gamma, L_1, L_3\rangle$ respectively. Therefore there exist subgroup Γ_1, Γ_4, Γ_6 isomorphic to $SL(2,q)$ with $Z(\Gamma_i)=\langle z_i\rangle$ $i=1,4,6$ such that $[\Gamma_1,\Gamma_4]=[\Gamma_4,\Gamma_6]=[\Gamma_1,\Gamma_6]=1$; Γ_1 is joined to L_2 linearly; Γ_4 is joined to L_2, L_3, L_5 linearly and Γ_6 joined to L_5, L_7 linearly by (2.5). Clearly we also have $[\Gamma_1, L_i]=1$ $i=2,5,7$; $[\Gamma_4,L_7]=1$ and $[\Gamma_6,L_j]=1$ $j=2,3,4$.

We can now apply Curtis' Theorem 1.4 to the chain of subgroups Γ_1, L_2, L_3, Γ_4, L_5, Γ_6, L_7 which generate G and get that $G\cong E_7(q)$, the universal Chevalley group of type E_7 over F_0.

THEOREM 2.8. *Let G be a universal group of type E_8 generated by $SU(3,q)$'s. Then G is isomorphic to the universal group of type E_8 over F_0.*

PROOF. Let the graph of G be as follows



We shall use the notation in the proof of (2.7) as $\langle L_i \mid 1\leqq i\leqq 7\rangle\cong E_7(q)$. There we have defined subgroups $L_0=L_{89}$, L_{19}, Γ, L and $\psi=n_{17}n_{36}n_{45}n_{89}$. The

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element ψ interchanges the elements of the sets $\{L_3, L_5\}$; $\{L_1, L_6\}$ and $\{L_{19}, L_7\}$ by conjugation. Since $\psi L_8 \psi^{-1} = L_8$, L_8 is joined to L_{19} unitarily. Also $[L_0, L_8] = 1$ because $L_0 \subseteq \langle L_i \mid 1 \leqq i \leqq 6\rangle$.

Next we note that $Q_1 = \langle L_{19}, L_i \mid 2 \leqq i \leqq 8\rangle$ is a universal group of type D_8 with $\langle L_i \mid 2 \leqq i \leqq 7\rangle$ as a subgroup of type D_6. In the proof of (2.7), we found that L is the dual of L_7 in Q_1 and therefore by (2.1), L is joined to L_8 unitarily. Let Γ_0 be the dual of L_{19} in Q_1. We note that $\Gamma_0 \subseteq \langle L_{19}, L, L_7, L_8\rangle$, a group of type D_4. Let z_1, z_4, z_6, Γ_1, Γ_4, Γ_6 be as defined in (2.7). Let $z_8 = h_8 h_{19} h_L n_L^{-1} n_{19} n_7 n_0$ where $n_0 = x n_7 x^{-1}$, $x = n_8 n_L n_{19} n_8 n_{19}^{-1} n_L^{-1}$ and $h_0 = x h_7 x^{-1}$. We compute that $n_0 h_8 n_0^{-1} = h_8 h_0^{-1}$; $n_L h_8 n_L^{-1} = h_8 h_L = n_8 h_L n_8^{-1}$; $n_{19} h_8 n_{19}^{-1} = h_8 h_{19} = h_8 h_{19} n_8^{-1}$. Together with the relations found in (2.7), we compute that z_1, z_8, z_6 are commuting involutions such that $z_1 z_8$, $z_8 z_6$ are not conjugate to z_1, z_8 in $\langle L_3, L_1, \Gamma, L, L_8, L_7\rangle$ and $\langle L_5, L_6, \Gamma, L_7, L_8, L_{19}\rangle$ respectively. It follows by (2.5). $[\Gamma_1, \Gamma_8] = [\Gamma_8, \Gamma_6] = 1$ where Γ_8 is the unique subgroup isomorphic to $SL(2, q)$ in $\langle L_{19}, \Gamma_0, L_7, L_8, L\rangle$ and also Γ_8 is joined to L_7 linearly. From the proof of (2.7), $\langle L, L_{19}\rangle$ centralizes $\langle L_i \mid 2 \leqq i \leqq 5\rangle$; hence $[\langle L, L_{19}, L_8, L_7\rangle, \langle L_2, L_3, L_4, L_5\rangle] = 1$ and therefore $[\Gamma_8, L_i] = 1 = [\Gamma_8, \Gamma_4]$ $i = 2, 3, 4$ because $\Gamma_4 \subseteq \langle L_2, L_3, L_4, L_5\rangle$ and $\Gamma_8 \subseteq \langle L, L_{19}, L_8, L_7\rangle$. Finally we compute that the chain of subgroups Γ_1, L_2, L_3, Γ_4, L_5, Γ_6, L_7, Γ_8 generates G and by Curtis' Theorem 1.4 [Curtis, 1965], $G \cong E_8(q)$, the universal Chevalley group of type E_8 over F_0. This completes the proof.

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