1. Introduction. If X is a topological space then S(X) will denote the semigroup, under composition, of all continuous functions from X into X. An element f in a semigroup is regular if there is an element g such that fgf = f. The regular elements of S(X) will be denoted by R(X). Elements f and g are inverses of each other if fgf = f and gfg = g. Every regular element has an inverse [1]. If every element in a semigroup has a unique inverse then the semigroup is an inverse semigroup. In this paper we examine maximal inverse subsemigroups of S(X).

For certain idempotents e we will define a set I_e and show that I_e is a maximal inverse subsemigroup of S(X) with e as its smallest idempotent. N. R. Reilly [5], J. W. Nichols [4] and B. M. Schein [7] have looked at maximal inverse subsemigroups of T(X), the full transformation semigroup on the set X. By letting X have the discrete topology we can apply our theorems about 0-dimensional spaces to yield the results of Nichols and Reilly. Further results give conditions on X which ensure that G(X), the group of units of S(X), is a maximal inverse subsemigroup. Other theorems will give results for X a Euclidean n-cell or Euclidean n-space.

2. Preliminary results. Throughout the paper we will use the notation and basic results about semigroups from Clifford and Preston [1]. A retract is the range of an idempotent in S(X), f|_A will denote the restriction of the map f to the set A. The juxtaposition fg will mean the composition f ° g. We begin with a result of R. D. Hofer [2] which gives conditions for f and g to be inverses of each other.

**Proposition 1.** Let f ∈ R(X). Then g is an inverse for f if and only if there exist retracts A, B of X such that B = range of f, A = range of g, f|_A is a homeomorphism onto B, g|_B is a homeomorphism onto A, f|_B = id|_B (identity map on B) and g|_A = id|_A.

Note that if f ∈ R(X) then the set B above is uniquely determined; we will denote it by B_f. If the set A is also uniquely determined (for example, if f belongs to an inverse semigroup) then it will be denoted by A_f. If f is an idempotent then we will say A_f = B_f. Finally, if f belongs to an inverse semigroup J then the unique algebraic inverse of f (in J) will be denoted by f^{-1}. We will also occasionally use the symbol f^{-1} for the inverse image of the map f; no confusion should result from this.

The next lemma is concerned with composing two elements in R(X).

**Lemma 2.** Suppose f, g ∈ R(X) with inverses f', g' respectively. Let A = g'(B_f ∩ B_g) and B = fg(A).

1. If range of fg = B then (fg)(g'(fg)) = fg, fg ∈ R(X) and fg maps A homeomorphically onto B.
2. If range of fg = B and range of g'f = A then g'f' is an inverse for fg.
Proof. (1) Suppose $\text{range of } fg = B$. If $y \in A$ then $y \in B_g$, $g(y) \in B_f$, and so $f'fg(y) = g(y)$ ($f'|_{B_f} = \text{id}|_{B_f}$) and hence $g'f'fg(y) = g'g(y) = y$. But now, if $x \in X$ then $fg(x) = fg(y)$ for some $y$ in $A$. Thus

$$(fg)(g'f')fg(x) = (fg)(g'f')(fg)(y) = fg(y) = fg(x).$$

Thus $fg \in \text{R}(X)$ and $fg$ maps $A$ homeomorphically onto $B$ ($g|_{A_e}$ and $f|_{B_f}$ are both homeomorphisms).

(2) Assume range of $fg = B$ and range of $g'f' = A$. If we show that $A = g'f'(f'(B_g \cap B_f))$ then we can apply (1) to the element $g'f'$ to conclude that $(g'f')(fg)(g'f') = g'f'$. But this is true since $A = g'(B_f \cap B_g)$ and $f'|_{B_f} = \text{id}|_{B_f}$.

We now introduce a new notion.

**Definition.** Let $e$ be an idempotent in $S(X)$. We say that an element $f \in S(X)$ respects $A_e$ if there exists an inverse $f'$ of $f$ with $A_e \subseteq B_f \cap B_f$ and $f|_{A_e}$ is a homeomorphism onto $A_e$. If we wish to emphasize the role of $f'$ we will say $f$ respects $A_e$ via $f'$.

Next we consider Green’s relation $\mathcal{H}$. Let $H_e$ denote the $\mathcal{H}$-class of an idempotent $e \in S(X)$. Then by using results of K. D. Magill, Jr. and S. Subbiah [3] we see that

$$H_e = \{ f \in \text{R}(X): \text{there exists an inverse } f' \text{ of } f \text{ such that } B_f = B_f, e(x) = e(y) \text{ if and only if } f(x) = f(y) \}.$$

Note that every element of $H_e$ respects $A_e$ and that if $f \in H_e$ then $e(x) = e(y)$ if and only if $ef(x) = ef(y)$ ($e$ is the identity on $B_f$). We now state a result pertaining to these notions (the proof will be omitted).

**Lemma 3.** Let $e$ be an idempotent in $S(X)$ and suppose that $h$ respects $A_e$. Then $he \in H_e$ and $he|_{A_e} = h|_{A_e}$.

**Lemma 4.** Suppose $e$ and $f$ are idempotents in $S(X)$ which commute.

(1) If $A_e = A_f$ then $e = f$.

(2) If $e(x), f(x) \in A_e \cap A_f$ then $e(x) = f(x)$. In particular, if $A_e \subseteq A_f$ and $f(x) \in A_e$ then $e(x) = f(x)$.

**Proof.** The proof is straightforward and will be omitted.

Recall that in an inverse semigroup $J$ all idempotents commute. $J$ has a smallest idempotent $e$ if $fe = ef = e$ for all idempotents $f$ in $J$. If this is the case then $A_e \subseteq A_f$, with equality occurring only if $e = f$ (by the last lemma).

**Lemma 5.** Let $J$ be an inverse subsemigroup of $S(X)$ with smallest idempotent $e$ and suppose $g \in J$. Then $g$ respects $A_e$, $g^{-1}eg = e$, $ge = eg$ and for all $x, y \in X, e(x) = e(y)$ if and only if $eg(x) = eg(y)$.

**Proof.** The elements $geg^{-1}$ and $g^{-1}eg$ are idempotents in $J$ and so $A_e \subseteq g(A_e) \subseteq B_g$ and $A_e \subseteq g^{-1}(A_e) \subseteq A_g$. But then $g|_{A_e}$ maps onto $A_e$ and so $g$ respects $A_e$. Now $A_{geg^{-1}} =
MAXIMAL INVERSE SUBSEMIGROUPS OF $S(X)$

$A_{e^{-1}e} = A_e$ and so, by Lemma 4, $ge^{-1} = e$ and thus $eg = geg^{-1}g = gg^{-1}ge = ge$. Now 
\[ e(x) = e(y) \iff ge(x) = ge(y) \quad (g \text{ is one-to-one on } A_e) \]
\[ \iff eg(x) = eg(y). \]

The next corollary shows us that every maximal inverse subsemigroup with a smallest idempotent $e$ must contain $H_e$ (also proved by Reilly [5]).

**Corollary 6.** Let $J$ be an inverse subsemigroup of $S(X)$ with smallest idempotent $e$ and let $g \in J$. If $f \in H_e$ then $fg, gf \in H_e$; if $J$ is maximal then $He \subseteq J$.

**Proof.** Suppose $f$ respects $A_e$ via $f'$. Then we apply Lemmas 2 and 5 to show that $fg$ and $gf$ are in $R(X)$ and that $B_{g^{-1}f} = B_{fg} = B_{f^{-1}g^{-1}} = B_{gf} = A_e$. Now if $f \in H_e$ then $ef = fe = f$. Thus 
\[ fg(x) = fg(y) \iff feg(x) = feg(y) \]
\[ \iff eg(x) = eg(y) \quad (f \text{ is one-to-one on } A_e) \]
\[ \iff e(x) = e(y) \quad (\text{by the last Lemma}). \]

Also, 
\[ gf(x) = gf(y) \iff f(x) = f(y) \quad (g \text{ is one-to-one on } B_f) \]
\[ \iff e(x) = e(y) \quad (f \in H_e). \]

Thus $fg$ and $gf$ both belong to $H_e$. Now suppose $J$ is maximal. Then $H_e \cup J$ is a subsemigroup by the above. Clearly idempotents in $H_e \cup J$ commute and so $H_e \cup J$ is an inverse subsemigroup [1]. Hence $H_e \subseteq J$ by maximality of $J$.

Later in the paper we will define several maximal inverse subsemigroups with smallest idempotent $e$. This last corollary then tells us that each of these maximal inverse subsemigroups contains $H_e$. The next results indicate when such a smallest idempotent is present.

**Definition.** Let $J$ be an inverse subsemigroup of $S(X)$. Then we define $A_J = \cap \{A_f : f \in J\}$. (Note that the collection $\{A_f : f \in J\}$ satisfies the finite intersection property; if $X$ is compact then $A_J \neq \emptyset$.)

**Lemma 7.** Let $J$ be an inverse subsemigroup of $S(X)$ and suppose $f \in J$. Then $A_J \subseteq A_f \cap B_f$ and $f|_{A_J}$ is a homeomorphism onto $A_f$. If there exists an idempotent $e \in J$ such that $A_e = A_J$ then $e$ is the smallest idempotent of $J$.

**Proof.** $A_J \subseteq A_f$ by definition and since there exists $f^{-1} \in J$ with $A_{f^{-1}} = B_f$ we have $A_J \subseteq B_f$ also. Thus $f|_{A_J}$ is a homeomorphism. If $x \in A_J$ and $f(x) \notin A_f$ then there exists $g \in J$ such that $f(x) \notin B_g$. Without loss of generality we may assume $g$ is an idempotent and $B_g \subseteq B_f$ (ff$^{-1}$gg$^{-1} \in J$). Now $f^{-1}gf$ is an idempotent in $J$ and so $f^{-1}gf(x) = x$ ($x \in A_J \subseteq A_{f^{-1}gf}$. But then $f^{-1}gf(x) = f(x)$. Since $B_g \subseteq B_f$ we have $ff^{-1}gf(x) = gf(x)$. Thus $gf(x) = f(x)$ but $f(x) \notin B_g$. This is a contradiction. Hence $f(x) \in A_f$. This means that $f$ maps
Apply this result to \( f^{-1} \) to conclude that \( f \) maps \( A_J \) onto \( A_J \). Now suppose \( e \) is an idempotent in \( J \) with \( A_e = A_J \). Then if \( f \) is any other idempotent, \( f(x) = x \) for all \( x \in A_e = A_J \) (\( A_J \subseteq A_J \)). Thus \( ef = fe = e \) and so \( e \) is the smallest idempotent.

**Corollary 8.** Let \( J \) be an inverse subsemigroup of \( S(X) \), \( e \) an idempotent in \( J \). Suppose the following condition is satisfied: if \( B \) is any retract of \( X \) with \( B \subseteq A_e \) then there exists \( f \in J \) such that \( f(B) \cap B = \emptyset \). Then \( A_J = A_e \), \( e \) is the smallest idempotent in \( J \) and if \( g \in J \) then \( g \) respects \( A_e \).

**Proof.** We know \( A_J \subseteq A_e \). If \( A_e \not\subseteq A_J \) then there exists an idempotent \( g \in J \) such that \( A_g \not\subseteq A_e \). Then by the condition there exists an \( f \in J \) such that \( f(A_g) \cap A_e = \emptyset \). Then \( fgf^{-1} \) is an idempotent in \( J \) and so \( g(fgf^{-1}) = (fgf^{-1})g \). But \( B_{fgf^{-1}} \subseteq A_g \), \( B_{fgf^{-1}} \subseteq f(A_g) \) and \( f(A_g) \cap A_e = \emptyset \). This is a contradiction. Thus \( A_e = A_J \). The rest of the corollary follows from Lemmas 7 and 5.

**3. Main results.** We first prove several results about maximal inverse subsemigroups of \( S(X) \) where \( X \) is 0-dimensional. The symbol \( c_y \) will signify the constant map in \( S(X) \) which sends everything to the point \( y \).

**Theorem 9.** Let \( X \) be \( T_1 \) and 0-dimensional and suppose \( e = c_y \) for some fixed \( y \in X \). Let

\[
I_e = \{ f \in R(X) : f(y) = y, \text{ there exists an inverse } f' \text{ of } f \text{ such that } \{y\} \subseteq B_f \cap B_{f'}, \text{ and if } f(x) \neq y \text{ then } |\{z : f(z) = f(x)\}| = 1 \}. 
\]

Then \( I_e \) is a maximal inverse subsemigroup of \( S(X) \) with smallest idempotent \( e \).

**Proof.** We initially note that if \( f \in I_e \) with inverse \( f' \) then \( f \) respects \( A_e \) via \( f' \), if \( x \in B_f \) and \( x \neq y \) then \( f(x) \neq y \), and if \( x \notin B_f \) then \( f(x) = y \). This, coupled with the fact that \( X \) is \( T_1 \), means that the boundaries of \( B_f \) and \( B_{f'} \) are contained in \( \{y\} \). We can now show that if \( f \in I_e \) then \( f \) has an inverse \( k \in I_e \); define \( k \) by

\[
k(x) = \begin{cases} f'(x) & \text{if } x \in B_f \\ y & \text{otherwise}. \end{cases}
\]

Note that \( k \) is continuous by the above remarks and it is straightforward to show that \( k \in I_e \). Now suppose \( f, g \in I_e \) with inverses \( f', g' \in I_e \). Let \( h = fg \). If \( A = g'(B_f \cap B_g) \) and \( B = h(A) \) we show that \( B = \text{range of } h \). Let \( x \in X \). Then there exists \( z \) such that \( g(z) = g(x) \) and \( g'(z) = z \). If \( g(z) \in B_f \) then \( z \in A \) and \( h(z) = h(x) \). If \( g(z) \notin B_f \) then \( fg(x) = y \) and \( h(y) = h(x) \) with \( y \in A \). Thus range of \( h = B \). Now by Lemma 2, \( h \in R(X) \). Clearly \( h \) respects \( A_e \) since \( h(y) = fg(y) = y \). It is also clear that if \( h(x) \neq y \) then \( |\{z : h(z) = h(x)\}| = 1 \). Hence \( h \in I_e \) and so \( I_e \) is a subsemigroup. We have already shown that \( I_e \) contains inverses. Note that if \( f \) is an idempotent in \( I_e \) then

\[
f(x) = \begin{cases} x & \text{if } x \in A_f \\ y & \text{otherwise}. \end{cases}
\]

Two such idempotents commute and so \( I_e \) is an inverse subsemigroup of \( S(X) \).
To show that $I_e$ is maximal suppose $I_e \subseteq J$ where $J$ is an inverse subsemigroup. By Corollary 8 we have that $e$ is the smallest idempotent in $J$ and if $f \in J$ then $f$ respects $A_e$. Now suppose $f(w) \neq y$ and
\[ |\{z : f(z) = f(w)\}| > 1.\]
We may assume $w \in A_e$. Then there exists $z \notin A_e$ such that $f(w) = f(z)$. Choose a clopen (closed and open) set $G$ so that $z, y \in G$ and $w \notin G$. Define $g \in S(X)$ by
\[ g(x) = \begin{cases} 
  x & \text{if } x \in G, \\
  y & \text{otherwise}. 
\end{cases} \]
It is easy to see that $g$ is an idempotent in $I_e$, hence in $J$. Thus $gf^{-1}f = f^{-1}fg$. But $gf^{-1}f(z) = g(w) = y$ and $f^{-1}fg(z) = f^{-1}f(z) = w$ and $w \neq y$. This is a contradiction. Hence if $f(w) \neq y$ then $|\{z : f(z) = f(w)\}| = 1$ and so $f \in I_e$. Thus $J \subseteq I_e$ and $I_e$ is maximal with smallest idempotent $e$.

If we let $X$ be discrete then $S(X) = T_X$, the full transformation semigroup on the set $X$. We may then apply the last theorem to obtain the result of Nichols [4]. The next theorem is also concerned with 0-dimensional spaces. Recall that a space $X$ is homogeneous if for every two points $x$ and $y$ there exists a homeomorphism $h$ of $X$ onto $X$ such that $h(x) = y$.

**Theorem 10.** Let $X$ be a homogeneous, 0-dimensional space and suppose $e$ is an idempotent in $S(X)$ such that $A_e$ is open. Let $I_e = \{f \in R(X) : f$ respects $A_e, B_f$ is open, if $f(x) \notin A_e$ then $|\{y : f(y) = f(x)\}| = 1$ and for all $x, y \in X, e(x) = e(y)$ if and only if $ef(x) = ef(y)\}$. Then $I_e$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent $e$.

**Proof.** Note that if $f \in I_e$ respects $A_e$ via $f'$ and $x \notin B_f$, then $f(x) \in A_e$. Thus $B_f = (f^{-1}(X - A_e) \cup A_e)$ and so $B_f$ is clopen. We first show that if $f \in I_e$ then there exists an inverse $g$ of $f$ which also belongs to $I_e$. Define $g$ by
\[ g(x) = \begin{cases} 
  f'(x) & \text{if } x \in B_f, \\
  f'e(x) & \text{otherwise}. 
\end{cases} \]
Since $B_f$ is clopen we have that $g \in S(X)$. Clearly $g$ is an inverse for $f$, $B_g$ is open, $g$ respects $A_e$ and if $g(x) \notin A_e$ then $|\{y : g(y) = g(x)\}| = 1$. To show the last condition for membership in $I_e$ we consider several cases:

1. $x, y \in B_f : eg(x) = eg(y) \Leftrightarrow ef'(x) = ef'(y) \Leftrightarrow ef'(x) = ef'(y) \Leftrightarrow e(x) = e(y)$.
2. $x \notin B_f, y \notin B_f : eg(x) = eg(y) \Leftrightarrow e'f(x) = e'f(y) \Leftrightarrow e'f(x) = e'f(y) \Leftrightarrow e(x) = e(y)$.
3. $x \in B_f, y \notin B_f : eg(x) = eg(y) \Leftrightarrow e'f'(x) = e'f'(y) \Leftrightarrow e'f'(x) = e'f'(y) \Leftrightarrow e(x) = e(y)$.

Thus $g \in I_e$.

We now show $I_e$ is a subsemigroup. Let $h = fg$ with $f, g \in I_e$ and inverses $f', g' \in I_e$. Let $h = fg$, let $A = g'(B_g \cap B_f)$ and $B = h(A)$. We show $B = \text{range of } h$. Let $x \in X$. Then there exists $y$ such that $g(x) = g(y)$ and $g'g(y) = y$. If $g(y) \in B_f$, then $y \in A$, $g(x) = g(y)$ and hence $h(x) = h(y)$. If $g(y) \notin B_f$, then $fg(x) \in A_e$ and so there exists $z \in A_e \subseteq A$ such that $h(z) = h(z)$. Now we use Lemma 2 to conclude that $h \in R(X)$. Clearly $B_h$ is open and $h$ respects
Now suppose \( h(x) = h(y) \) where \( h(x) \notin A_e \). Then \( fg(x) = fg(y) \) with \( fg(x) \notin A_e \). This means that \( g(x) = g(y) \). Now \( g(x) \notin A_e \) (otherwise \( fg(x) \in A_e \)) and so \( x = y \). Thus if \( h(x) \notin A_e \) then \( |\{y : h(x) = h(y)\}| = 1 \). Finally, note that for any \( x, y \in X \),

\[
e(x) = e(y) \iff eg(x) = eg(y) \iff ef(x) = ef(y) \iff eh(x) = eh(y).
\]

Thus \( h \in I_e \).

To show that \( I_e \) is an inverse subsemigroup we need only show that idempotents in \( I_e \) commute. But note that if \( f \) is an idempotent in \( I_e \) then

\[
f(x) = \begin{cases} 
  x & \text{if } x \in A_f, \\
  e(x) & \text{otherwise}.
\end{cases}
\]

Thus any two idempotents in \( I_e \) will commute and so \( I_e \) is an inverse subsemigroup.

For maximality suppose that \( I_e \subseteq J \) where \( J \) is an inverse subsemigroup. We first show that \( A_e \subseteq A_f \). If not, then there exists an idempotent \( f \in J \) and \( y \in X \) such that \( y \in A_e - A_f \). But then \( ef(y) = fe(y) \in A_f \cap A_e \) and so \( fe(y) \neq y \). By the homogeneity of \( X \) choose a homeomorphism \( h \) from \( X \) onto \( X \) such that \( h(y) = fe(y) \). Now choose clopen disjoint sets \( U, V \) of \( X \) so that \( y \in U, fe(y) \in V, U \cup V \subseteq A_e, U \cap A_f = \emptyset \) and \( h(U) = V \). Now define a homeomorphism \( k \) from \( X \) onto \( X \) by

\[
k(x) = \begin{cases} 
  h(x) & \text{if } x \in U, \\
  h^{-1}(x) & \text{if } x \in V, \\
  e(x) & \text{otherwise}.
\end{cases}
\]

Then \( B_k = A_e \) and \( ke = ek \). Thus \( k \in I_e \), hence \( k \in J \). Now \( k^{-1}fk \) is an idempotent of \( J \). So

\[
(fk^{-1}fk)(fe) = (k^{-1}fk)(fe).
\]

But

\[
(fk^{-1}fk)(fe)(y) = (fk^{-1}fk)(fe)(h(y)) = (fk^{-1}fkfe)(y) = (fk^{-1}fkfe)(y) = fe(y)
\]

and \( (k^{-1}fk)(fe)(y) \in k^{-1}f(U) \). Now \( f(U) \cap U = \emptyset \) since \( U \cap A_f = \emptyset \). Thus \( k^{-1}f(U) \cap V = \emptyset \). But \( fe(y) \in V \) and this is a contradiction. Thus \( A_e \subseteq A_f \) and so, by Lemma 7, \( e \) is the smallest idempotent of \( J \). Now by Lemma 5, if \( g \in J \) then \( g \) respects \( A_e, ge = eg \) and

\[
e(x) = e(y) \iff eg(x) = eg(y).
\]

Assume \( f \) is an idempotent in \( J \). Suppose there exists \( z \in A_f - A_e \) such that \( f(z) = z = f(y) \) with \( y \neq z \). Choose clopen \( U \) so that \( y \in U, z \notin U \) and \( U \cap A_e = \emptyset \) (note \( y \notin A_e \)). define \( g \in S(X) \) by

\[
g(x) = \begin{cases} 
  e(x) & \text{if } x \in U, \\
  x & \text{if } x \notin U.
\end{cases}
\]
Then $g$ is an idempotent in $I_e$ and so $fg = gf$. But $fg(y) = fe(y) = e(y) \in A_e$ and $gf(y) = g(z) = z$ with $z \not\in A_e$. This is a contradiction. Thus if $f(z) \not\in A_e$ then $|\{x : f(x) = f(z)\}| = 1$. This means that if $x \not\in A_f$ then $f(x) \in A_e$. But then $X - A_f = f^{-1}(A_e) \cap (X - A_e)$ which is closed. Thus $A_f$ (and hence $B_f$) is open. But then $f \in I_e$.

Now suppose $g \in I$. Then $B_g = A_{gg^{-1}}$ is open, $g$ respects $A_e$ and

$$e(x) = e(y) \iff eg(x) = eg(y).$$

If $g(x) \not\in A_e$ and $g(x) = g(y)$ then $g^{-1}g(x) \not\in A_e$ ($g^{-1}$ respects $A_e$) and $g^{-1}g(x) = g^{-1}g(y)$. Thus $x = y$ ($g^{-1}g \in I_e$). But then $|\{y : g(x) = g(y)\}| = 1$. This shows $g \in I_e$. Thus $J \subseteq I_e$ and so $I_e$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent $e$.

**Corollary 11.** Let $X$ be a homogeneous 0-dimensional space. Then $G(X)$, the group of units of $S(X)$, is a maximal inverse subsemigroup of $S(X)$.

**Proof.** Let $e$ be the identity map on $X$ in the previous theorem.

To see that homogeneity is necessary in this corollary let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Then $G(X) \cup \{e_0\}$ is an inverse subsemigroup of $S(X)$. If $X$ is discrete then we apply the last theorem to yield the result of Reilly [5]. We now consider other types of maximal inverse subsemigroups of $S(X)$. This will result in applications to $\mathbb{R}^n$ (Euclidean $n$-space) and $T^n$ (Euclidean $n$-cell). We first make several definitions.

**Definition.** Suppose $e$ is an idempotent in $S(X)$ and $\mathcal{R}$ is a decomposition of $X - A_e$ ($\mathcal{R}$ is an equivalence relation on $X - A_e$). We will call $\mathcal{R}$ a ray decomposition of $X - A_e$ if the following conditions are satisfied:

1. for any $x \in X - A_e$, if $[x]$ denotes the $\mathcal{R}$-equivalence class of $x$ in $X - A_e$ then $\overline{[x]} = [x] \cup \{x_e\}$ where $x_e$ is an element of $A_e$ ($[x]$ denotes the closure of the set $[x]$ in $X$),
2. for any $x \in X - A_e$, $\overline{[x]}$ is homeomorphic to $[0, 1]$ or $[0, 1)$ via a homeomorphism $h$ such that $h(x_e) = 0$.

When we write $[x]$ we shall understand that $x \in X - A_e$. If $a \in [x]$ we will use the notation $[x_e, a]$ to mean $h^{-1}[0, h(a)]$ and we will say $y > a$ ($y \geq a$) if $a, y \in [x]$ and $h(y) > h(a)$ ($h(y) \geq h(a)$).

**Definition.** Suppose $e$ is an idempotent in $S(X)$, $\mathcal{R}$ is a ray decomposition of $X - A_e$ and for every $x \in X - A_e$, $e$ is constant on $[x]$. A function $f \in R(X)$ is said to be $e$-admissible if the following are satisfied:

1. there exists an inverse $f'$ of $f$ such that $f$ respects $A_e$ via $f'$,
2. for every $x \in X - A_e$, either $f$ is constant on $[x]$ or $f[x] \subseteq [z]$ for some $z \in X - A_e$,
3. for every $x \in X - A_e$, either $[x] \subseteq B_f$ or there exists $x_f \in [x]$ such that $[x_e, x_f] \subseteq B_f$ (may have $x_f = x_e$) and $f$ is constant on all $y \geq x_f$. As before, we will also say $f$ is $e$-admissible via $f'$.

Note that if $f$ is $e$-admissible via $f'$, $[x] \subseteq B_f$ and $f[x] \subseteq [z]$ then $f|_{[x]}$ is a homeomorphism into $[z]$ with $f(x_e) = z_e$; and if $[x_e, x_f] \subseteq B_f$ then $f$ is constant on all $y \geq x_f$.

**Theorem 12.** Suppose $X$ is a topological space, $e$ is an idempotent in $S(X)$, $\mathcal{R}$ is a ray
decomposition of $X - A_e$ and the following conditions are satisfied:

1. For every $x \in X - A_e$, $e$ is constant on $[x]$.
2. If $a \in [x]$ then there exists an idempotent $h \in R(X)$ such that $h$ is $e$-admissible, $h|_{[x,a)} = \text{id}|_{[x,a)}$ and $h(z) = a$ for all $z \geq a$. If, in addition, there exists $y$ such that $[y] \neq [x]$ then $h$ can be chosen so that $h|_{[y]} = \text{id}|_{[y]}$.
3. If $A$ is a retract of $X$ and $A \subseteq A_e$ then there exists $h \in R(X)$ such that $h$ respects $A_e$ and $h(A) \cap A = \emptyset$.

Now let $I_e = \{ f \in R(X) : \text{there exists an inverse } f \text{ such that } f \text{ is } e\text{-admissible via } f \}$ and $I_e$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent $e$.

**Proof.** We first show $I_e$ is a subsemigroup. Let $f, g \in I_e$ with inverses $f', g' \in I_e$, $h = fg$, $A = g'(B_f \cap B_g)$ and $B = h(A)$. We show simultaneously that range of $h = B$ and that $h$ satisfies conditions (2) and (3) of the definition of $e$-admissibility. We can then apply these results to inverses $f'$ and $g'$ of $f$ and $g$ and use Lemma 2 to conclude that $h \in R(X)$, $h' = g'f'$ is an inverse for $h$ and both $h$ and $h'$ are $e$-admissible (clearly $h$ and $h'$ respect $A_e$). This will then show that $I_e$ is a subsemigroup. So consider $x \in X$. If $x \in A_e$ then $x \in A$ and $h(x) \in B$. If $x \notin A_e$ and $g$ is constant on $[x]$ then $h$ is constant on $[x]$ and $h(x) \subseteq A_e \subseteq B$. Now suppose $g(x) \subseteq [y]$. If $f$ is constant on $[y]$ then $h$ is constant on $[x]$ and $h(x) \subseteq A_e \subseteq B$. Now suppose $f(y) \subseteq [z]$. Then $h(x) \subseteq [z]$. If $[x] \subseteq B_g$ and $[y] \subseteq B_f$ then $[x] \subseteq A$, $h(x) \subseteq B$ and $h$ is a homeomorphism on $[x]$. If $[x] \subseteq B_g$ and there exists $y_t$ such that $[y_t, y_t] \subseteq B_f$ with $f$ constant on all $w \geq y_t$, then let $x_h = g'(y_t)$. Then $[x_h, x_h] \subseteq A$ and $h$ is constant on all $w \geq x_h$. Thus $h(x) \subseteq B$. Now suppose there exists $x_t$ such that $[x_t, x_t] \subseteq B_g$. If $[x] \subseteq B_f$ or if there exists $y_t \geq g(x_t)$ such that $[y_t, y_t] \subseteq B_f$ then $[x_t, x_t] \subseteq A$, $h$ is constant on all $w \geq x_t$, and $h(x) \subseteq B$. If there exists $y_t < g(x_t)$ such that $[y_t, y_t] \subseteq B_f$ and $f$ is constant on all $w \geq y_t$ then let $x_t = g'(y_t)$. Then $[x_t, x_t] \subseteq A$ and $h$ is constant on all $w \geq x_t$ and again $h(x) \subseteq B$. This completes the proof that $I_e$ is a subsemigroup.

To show that $I_e$ is an inverse subsemigroup we need only prove that idempotents commute. Let $f, g$ be idempotents in $I_e$ and suppose $x \in X$. If $x \in A_e$ then $f(x) = x = g(x)$ and so $fg(x) = gf(x)$. If $x \in X - A_e$, then either $f|_{[x,e]} = \text{id}|_{[x,e]}$ or $f(x) = x_t$ with $x_t < x$. If $f(x) = x_t$ then $g(x) = g(x_t) = x_t = f(x) = gf(x)$. If $g(x) = x_t$ then $g(x) = x_t = f(x) = gf(x)$. If $g(x) = x_t$ then $g(x) = x_t = f(x) = gf(x)$. If $y > a$. By condition (3) of the hypothesis choose $g$ an
idempotent such that $g$ is $e$-admissible, $g|_{[x, a]} = \text{id}|_{[x, a]}$ and $g(z) = a$ for $z > a$. If $f(y) \notin [x]$ then $f(y) \notin A_e$ (otherwise $f(y) = e(y) = x_e$ by Lemma 4) and so we can also choose $g$ so that $g|_{(f(y))} = \text{id}|_{(f(y))}$. Then $g$ is in $I_e$; hence in $J$ and so $fg = gf$. If $f(y) \notin [x]$ then $gf(y) = f(y) \notin [x]$ but $fg(y) = f(a) = a \in [x]$, which is a contradiction. Hence $f(y) \in [x]$. Note that this means that $f(y) \leq a(f|x| \subseteq [x]$ and so $A_f \cap [x]$ must be an interval). Now $a = f(a) = fg(y) = gf(y)$. Thus $f(y) \geq a$. Hence $f(y) = a$ and this shows that $f \in I_e$.

Now let $g \in J$. We know that $g$ respects $A_e$. Let $x \in X - A_e$. Note that if $g(x) \in A_e$ then since $eg = ge$ by Lemma 5 we have $g(x) = eg(x) = ge(x) = g(x_e)$. Consider $[x]$. If $g^{-1}g$ is constant on $[x]$ and $y \in [x]$ then $g^{-1}g(y) = g^{-1}g(x_e) = x_e$. But then $g(y) = g(x_e)$ and so $g$ is constant on $[x]$. Now suppose there exists $a > x_e$ such that $[x_e, a] \subseteq A_e^{-1}\frac{g}{g}$ and let $x_e < y \leq a$. Then $g(y) \notin A_e$ ($y \notin A_e - A_e$). If $g(y) \notin [g(a)]$ then choose an idempotent $e$ so that $e$ is the identity on $[g(a)]$ and constant on $[g(y)]$. Then $g^{-1}fg$ is an idempotent in $J$, hence in $I_e$. Now $g^{-1}fg(a) = a$ and so $g^{-1}fg(y) = y$ also ($y \leq a$). But $g^{-1}fg(y) \in A_e$. This is a contradiction. Thus $g(y) \in [g(a)]$ for all $y$ with $x_e < y \leq a$. Thus if $[x] \subseteq A_{e^{-1}}g$ then $[x] \subseteq A_g$ and $g[x] \subseteq [g(x)]$. Now suppose $g^{-1}g$ is such that there exists $a \in [x]$ such that $g^{-1}g$ is the identity on $[x_e, a]$ and constant thereafter. Then $[x_e, a] \subseteq A_g$ and $g[x_e, a] \subseteq [g(a)]$ by the above. Now let $y > a$. Then $g^{-1}g(y) = g^{-1}g(a) = a$ and hence $g(y) = gg^{-1}g(y) = g(a)$. Thus $g \in I_e, J \subseteq I_e$ and so $I_e$ is a maximal inverse subsemigroup.

We have several corollaries.

**Corollary 13.** Let $X = I$ (the unit interval) or $\mathbb{R}$ (the reals) and let $e$ be defined by

$$
e(x) =
\begin{cases}
x & \text{if } a \leq x \leq b, \\
a & \text{if } x \leq a, \\
b & \text{if } x \geq b,
\end{cases}
$$

where $0 \leq a \leq b \leq 1$ if $X = I$ and $a \leq b$ if $X = \mathbb{R}$. Then $e$ is an idempotent and if $I_e = \{f \in R(X) : \text{there exists an inverse } f' \text{ of } f \text{ such that } f \text{ respects } A_e \text{ via } f', \text{ if } B_f = [c, d] \text{ then } f(x) = f(c) \text{ for all } x \leq c \text{ and } f(x) = f(d) \text{ for all } x \geq d\}$ we have that $I_e$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent $e$.

**Corollary 14.** Let $X = \mathbb{R}^n$ or $I^n$ and let $D$ be an $n$-dimensional disk in $\mathbb{R}^n$ (or $I^n$) with centre $y$. Define an idempotent $e$ as follows: if $x \in D$, $e(x) = x$; if $x \in \mathbb{R}^n - D$, $e(x) = x_b$, where $x_b$ is the unique element on the boundary of $D$ which intersects the line segment from $y$ to $x$.

If $x, z \in X - A_e$ then we say $x$ is $\mathbb{R}$-equivalent to $z$ if $x$ and $z$ lie on the same line segment beginning at $y$. Then this gives a ray decomposition of $X - A_e$ and if $I_e = \{f \in R(X) : \text{there exists an inverse } f' \text{ of } f \text{ such that } f \text{ is } e\text{-admissible via } f' \text{ and } f' \text{ is } e\text{-admissible via } f\}$ then $I_e$ is a maximal inverse subsemigroup of $S(X)$.

**Proof.** It is straightforward to see that conditions (1) and (2) of the theorem are satisfied. To see condition (3) note that if $A$ is a retract of $X$ and $A \subseteq D$ then there exists a point $x$ in the boundary of $D$ but not in $A$. Since $A$ is closed there exists an open neighborhood $U$ of $x$ such that $U$ is homeomorphic to $\mathbb{R}^n$ and $\bar{U} \cap A = \emptyset$. Now there exists $h \in R(X)$ with inverse $h'$ such that $B_h = B_{h'} = D$ and $h(A) \subseteq \bar{U} \cap D$. 

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Corollary 15. Let $X = \mathbb{R}^n$ or $I^n$, $e = c_y$ for fixed $y \in X$ and let the ray decomposition of $X - \{y\}$ be defined by $z \in [x]$ if and only if $z$, $x$ and $y$ all lie on a line segment beginning at $y$. Then $I_e = \{f \in R(X) : \text{there exists an inverse } f' \text{ of } f \text{ such that } f \text{ is } e\text{-admissible via } f' \text{ and } f' \text{ is } e\text{-admissible via } f\}$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent $c_y$.

Note that for the above corollary we could have chosen a different ray decomposition of $X - \{y\}$ and this would have resulted in a different maximal inverse subsemigroup, still with the same smallest idempotent $c_y$.

Corollary 16. Let $X = I^n$. Then $G(X)$, the group of units of $S(I^n)$, is a maximal inverse subsemigroup of $S(X)$.

Proof. Let $e$ be the identity on $X$ in Theorem 12.

Corollaries 11 and 16 give situations where $G(X)$, the group of units of $S(X)$, forms a maximal inverse subsemigroup. This is not always the case. For instance, if $X$ is a triod then every homeomorphism of $X$ will fix the same point $y$ and so $G(X) \cup \{c_y\}$ is an inverse subsemigroup which properly contains $G(X)$. However, we do have the following result (also proved by Reilly [6]):

Proposition 17. Suppose $X$ is a homogeneous, compact space. Then $G(X)$, the group of units of $S(X)$, is a maximal inverse subsemigroup of $S(X)$.

Proof. Clearly $G(X)$ is an inverse subsemigroup. Suppose $G(X) \subseteq J$ where $J$ is an inverse subsemigroup. Then $A_j \neq \emptyset$ since $X$ is compact. Suppose $A_j \neq X$. Then by the homogeneity of $X$ choose $f \in G(X)$ and $x \in X$ so that $x \in A_j$ and $f(x) \neq A_j$. Then $f \in J$ but $f(A_j) \subseteq A_j$. This contradicts Lemma 7. Thus $A_j = X$ and so $J = G(X)$ and $G(X)$ is maximal.

Corollary 18. Let $X = S^n$ (the $n$-dimensional sphere). Then $G(X)$ is a maximal inverse subsemigroup of $S(X)$.

We now consider one last type of maximal inverse subsemigroup of $S(I)$.

Theorem 19. Let $e$ be an idempotent in $S(I)$ such that if $A_e = [a, b]$ (where possibly $a = 0$ or $b = 1$) then $e$ is a homeomorphism on $[0, a]$ and $e$ is a homeomorphism on $[b, 1]$. Define $I_e = \{f \in R(I) : \text{there exists an inverse } f' \text{ of } f \text{ such that } B_f = [0, b], [0, 1], [a, b] \text{ or } [a, 1], B_{f'} \text{ is also one of these sets, } f \text{ respects } A_e \text{ via } f', \text{ and } e(x) = e(y) \text{ if and only if } ef(x) = ef(y)\}$. Then $I_e$ is a maximal inverse subsemigroup of $S(I)$ with smallest idempotent $e$.

Proof. Suppose $f \in I_e$ with inverse $f'$. We define an inverse $g$ for $f$ by

$$g(x) = \begin{cases} f'(x) & \text{if } x \in B_f, \\ f' e(x) & \text{if } x \notin B_f. \end{cases}$$

It is straightforward to check that $g$ is continuous. Clearly $g$ is an inverse for $f$, $g$ respects $A_e$ and satisfies the conditions on $B_f$ and $B_{g'}$. The proof for the last condition follows the
corresponding proof in Theorem 10. Now suppose \( f, g \in I_e \) with inverses \( f', g' \in I_e \) and let \( h = fg \). Then \( h \in R(X) \), \( h \) respects \( A_e \) and \( B_n, B_{e'} \) are of the desired form. Now

\[
e(x) = e(y) \iff eg(x) = eg(y) \iff efg(x) = efg(y) \iff eh(x) = eh(y).
\]

So \( h \in I_e \). We now show idempotents commute. Suppose \( f \) is an idempotent in \( I_e, f \neq e \) and \( f \) is not the identity on \( I \). Without loss of generality assume \([0, a] \cap A_f = \emptyset\). Then \( f \) is one-to-one on \([0, a]\) (if \( f(x) = f(y) \) then \( ef(x) = ef(y) \) and hence \( e(x) = e(y) \), but \( e \) is one-to-one on \([0, a]\) ). Furthermore, if \( x \in [0, a] \) then \( f(x) = e(x) \) (if \( f(x) \in A_e \) then \( f(x) = f(y) \) for some \( y \in A_e \), hence \( e(x) = e(y) = f(y) = f(x) \); if \( f(x) = e(x) = b \) then \( x = 0 \)). This means that if \( f \) is an idempotent in \( I_e \) then

\[
f(x) = \begin{cases} x & \text{if } x \in A_f, \\ e(x) & \text{if } x \notin A_f. \end{cases}
\]

Clearly two such idempotents commute. Thus \( I_e \) is an inverse subsemigroup of \( S(I) \).

To show that \( I_e \) is maximal suppose \( I_e \subseteq J \) where \( J \) is an inverse subsemigroup and \( g \in J \). It is straightforward to show that \( A_e \) and \( J \) satisfy the conditions of Corollary 8 and hence \( e \) is the smallest idempotent for \( J \). Now apply Lemma 5 to conclude that \( g \) respects \( A_e \) and \( e(x) = e(y) \) if and only if \( eg(x) = eg(y) \). To show the remaining conditions we may assume, without loss of generality, that \( g \) is an idempotent and \( A_g = [c, d] \) with \( 0 < c < a \). But then \( g(x) = g(y) \) for some \( x, y \in [0, a] \) where \( x \neq y \). Thus \( eg(x) = eg(y) \) and hence \( e(x) = e(y) \), which is a contradiction. Thus \( g \in I_e \) and so \( I_e \) is a maximal inverse subsemigroup of \( S(I) \) with smallest idempotent \( e \).

Note that it is possible to make slight modifications and prove a similar theorem if \( X \) is the reals.

As an example of this last theorem let \( X = [-1, 1] \) and suppose \( e(x) = |x| \). Then \( I_e = \{f \in S(X) : f \text{ maps } [0, 1] \text{ homeomorphically onto } [0, 1] \text{ and either } f \text{ is an odd function } \} \) is a maximal inverse subsemigroup of \( S(X) \). Or, let \( X \) be the reals and again let \( e(x) = |x| \). Then \( I_e = \{f \in S(X) : f \text{ is a homeomorphism from } [0, \infty) \text{ onto } [0, \infty) \text{ and } f \text{ is either an odd or even function} \} \) is a maximal inverse subsemigroup of \( S(X) \).

All of the maximal inverse subsemigroups we have considered thus far have contained a smallest idempotent \( e \). As Reilly [5] remarks, this is not always the case for \( S(X) \), where \( X \) is discrete. Since every inverse subsemigroup is contained in a maximal inverse subsemigroup, to produce examples of inverse subsemigroups with no smallest idempotent one needs to find subsemigroups \( J \) of \( S(X) \) of commuting idempotents such that \( A_J = \emptyset \). For instance, if \( X \) is the reals, define \( f_n \) for \( n = 1, 2, \ldots \) as follows:

\[
f_n(x) = \begin{cases} n & \text{if } x \leq n, \\ x & \text{if } x > n. \end{cases}
\]

Then \( J = \{f_n : n = 1, 2, \ldots \} \) is a subsemigroup of commuting idempotents but \( \bigcap_{n=1}^{\infty} A_{f_n} = \emptyset \) and so \( A_J = \emptyset \).
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