A CENSUS OF SLICINGS

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1. Introduction. A *band* is a closed connected set in the 2-sphere, bounded by one or more disjoint simple closed curves.

Consider a band B with bounding curves $J_1, J_2, \ldots, J_k$. On each curve J_i let there be chosen $m_i \geqslant 0$ points to be called *vertices*, with the restriction that the sum of the k integers m_i is to be even. Write

$$\sum_{i=1}^{k} m_i = 2n. \tag{1}$$

Next consider a set of n disjoint open arcs in the interior of B which join the $2n$ vertices in pairs and partition the remainder of the interior of B into simply connected domains. We call the resulting dissection of B a *slicing* with respect to the given set of vertices. The arcs are the *internal edges* of the slicing and the simply connected domains are its *internal faces*, or *slices*.

The *external edges* of a slicing are the open segments into which the vertices separate the curves J_i. If $n_i = 0$ we count the complete curve J_i as a "singular" external edge. It is clear, however, that this happens only in the *singular case* ($k = 1, n = 0$). The *external faces* of a slicing are the components of the complement of B.

In the non-singular case the number of external edges is $2n$ and the number of external faces is k. The number f of internal faces can be calculated from the Euler polyhedron formula.

$$f = n - k + 2. \tag{2}$$

In the singular case $n = 0$ and $k = 1$. There is just one internal face, the interior of B, and just one external one. So formula (2) is still valid.

Figure 1 shows a slicing of a region bounded by four simple closed curves. The external faces are shaded.

Two slicings of B are *equivalent* if one can be transformed into the other by a topological mapping of R onto itself which leaves each vertex invariant.

We propose the problem of determining the number of inequivalent slicings of a band corresponding to a given sequence of numbers m_i. In what follows we dispose of the case in which the numbers m_i are all even. We call this the case of *even* slicings.

In this special case we write $m_i = 2n_i$ for each i, and we denote the number

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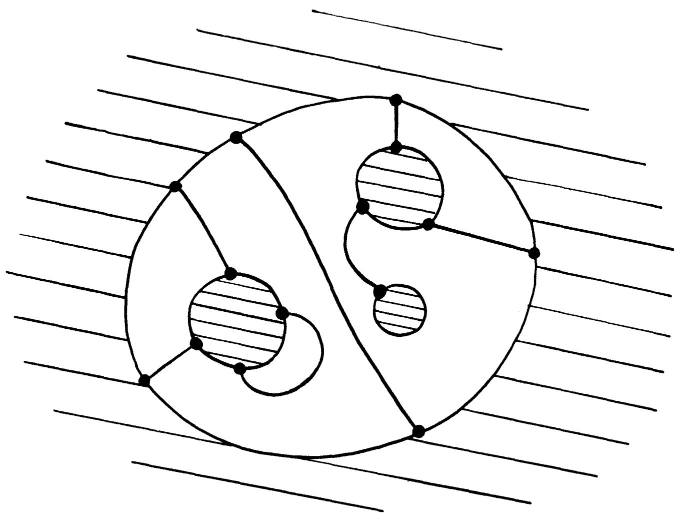

FIGURE 1.

of inequivalent slicings with respect to the given set of vertices by $\gamma(n_1, n_2, \ldots, n_k)$. This number is determined by the integers n_i and is not otherwise dependent on B. The main theorem of this paper states that

$$\gamma(n_1, n_2, \ldots, n_k) = \frac{(n-1)!}{(n-k+2)!} \prod_{i=1}^{k} \frac{(2n_i)!}{n_i!(n_i-1)!}, \tag{1.1}$$

where the n_i are non-zero. If one of the integers n_i is zero we have of course $\gamma(n_1, n_2, \ldots, n_k) = 0$, except that in the singular case $\gamma(0) = 1$.

Formula (1.1) provides a new tool for the enumerator of planar maps. For example, if each external face of an even slicing is contracted to a point we get an *Eulerian map*, that is, a map in which each vertex is incident with an even number of edges (loops being counted twice). The reader is invited to use (1.1) to verify that the number of combinatorially distinct oriented* Eulerian maps having n edges and $k \geqslant 3$ labelled vertices $v_1, v_2, \ldots, v_k$ of specified valencies $2n_1, 2n_2, \ldots, 2n_k$ respectively is

$$\frac{(n-1)!}{(n-k+2)!} \prod_{i=1}^{k} \frac{(2n_i-1)!}{n_i!(n_i-1)!}.$$

In the case $k \leqslant 2$ complications arise due to the possible existence of non-trivial isomorphisms of an Eulerian map which leave all the vertices invariant and preserve the orientations of the regions.

According to the above formula there are 9 oriented Eulerian maps corresponding to the case $(n_1 = n_2 = 2, n_3 = 1)$. They are shown in Figure 2.

*A map is "oriented" when its regions are consistently oriented.

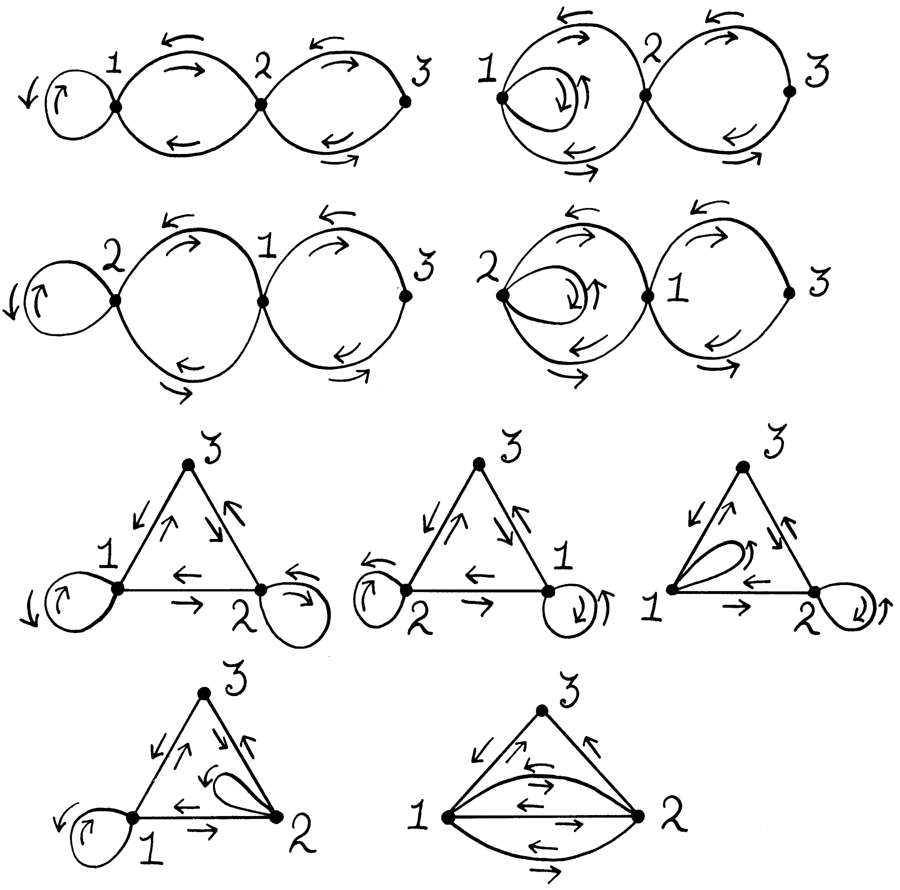

FIGURE 2.

2. A recursion formula. Consider a band B with boundary curves $J_1, J_2, \ldots, J_k$. We can dissect it into 2-cells as follows. If $k = 1$ we select distinct points v_1 and w_1, not necessarily vertices, on J_1 and join them by an open arc L_1 in the interior of B. If $k > 1$ we select points $v_1 \in J_1$ and $w_2 \in J_2$, and join them by an open arc L_1 in the interior of B. In the latter case J_1 and J_2 are different components of the boundary of B. We can therefore choose any point $v_2 \neq w_2$ on J_2 and join it to any point w_3 on J_3, or $w_1 \neq v_1$ on J_1 if $k = 2$, by an open arc L_2 in the interior of B which does not meet L_1. We continue in this way until finally we join $v_k \in J_k$ to $w_1 \neq v_1$ on J_1 by an open arc L_k in the interior of B, the arcs $L_1, L_2, \ldots, L_k$ being disjoint.

Let the two closed arcs in J_i joining v_i and w_i be denoted by K_i and K_i'. $(1 \leqslant i \leqslant k.)$ We can adjust the notation so that the simple closed curve made up of the arcs L_i, K_j, has a residual domain D contained in B. Then the curve made up of the arcs L_i, K_j', has a residual domain D' contained in B. In fact we have constructed a new slicing of B with the points v_i, w_j as vertices, the arcs L_i as internal edges, the arcs K_j and K_j' as external edges, and the domains D and D' as the only internal faces.

It follows that we can replace B by the topologically equivalent and geometrically simple model of Figure 3. Here J_1 is represented by a large square, and the other curves J_i are represented by small congruent squares lying inside J_1 with their centres evenly spaced on a segment joining the midpoints V and W of two opposite sides of J_1. The small squares have their sides parallel and perpendicular to the segment VW. V is taken as v_1 and W as w_1.

The construction can of course be modified so as to make the outer square represent any desired boundary curve J_i, and so as to make the representative squares of the other boundary curves lie in any desired order along VW.

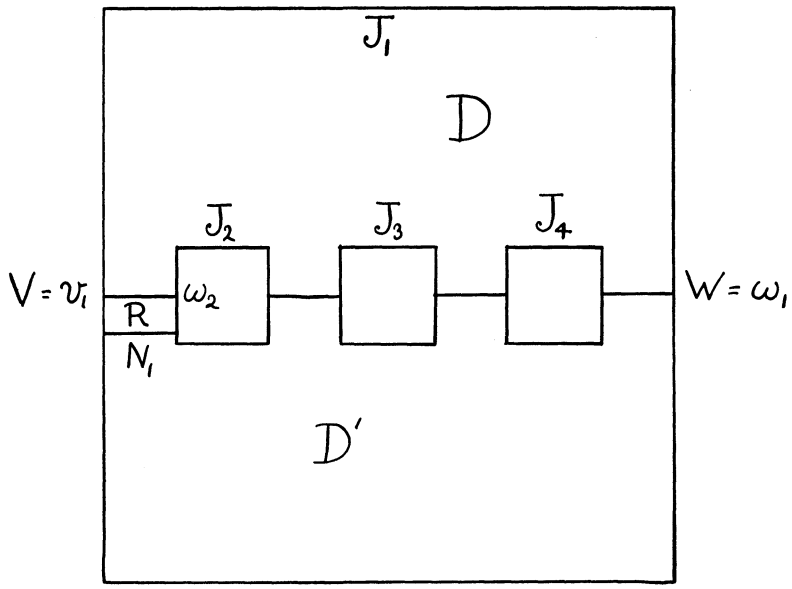

FIGURE 3.

It follows from elementary topological considerations that the number $\gamma(n_1, n_2, \ldots, n_k)$ of §1 is a function of the numbers $n_1, n_2, \ldots, n_k$ only, and is independent of the order in which they are written.

We proceed to establish a recursion formula for $\gamma(n_1, n_2, \ldots, n_k)$. We assume that $2n_i$ points are distinguished as vertices on J_i. $(1 \leqslant i \leqslant k.)$

Choose a vertex X on J_1. In any slicing of B, with respect to the given set of vertices, X will be incident with just one internal edge E. Let the other end of E be Y. Let the number of inequivalent slicings for which Y lies on J_i $(1 \leqslant i \leqslant k)$ be β_i.

Consider the case in which Y is on J_1. Distinguish an external edge A incident with X. Let N and N' be the two closed arcs in J_1 joining X and Y,

N being the one containing A. The arc E separates the interior of J_1, that is, the residual domain of J_1 meeting B, into two simply connected domains R and R' with boundaries $E \cup N$ and $E \cup N'$ respectively. Thus E induces an ordered partition $\{P, \bar{P}\}$ of the class $S = \{J_2, J_3, \ldots, J_k\}$ into two complementary subclasses P and $\bar{P}$, the members of P being those curves J_i which are contained in R.

From the model of Figure 3 we see that when X and Y are given an arc E can be constructed to correspond to any assigned ordered partition $\{P, \bar{P}\}$ of S. Let t denote the number of vertices, other than X and Y, on N.

Then the corresponding number for N' is $2n_1 - t - 2$. If $P = \{J_{s(1)}, J_{s(2)}, \ldots, J_{s(r)}\}$ we make the abbreviation

$$\gamma(q, P) = \gamma(q, n_{s(1)}, n_{s(2)}, \ldots, n_{s(r)}). \tag{3}$$

We observe that E decomposes B into two bands W and W'. The bounding curves of W are $E \cup N$ and the members of P, while those of W' are $E \cup N'$ and the members of $\bar{P}$. By simple topological considerations any other edge E corresponding to the same ordered partition of S would determine a topologically equivalent decomposition of B.

Any slicing of B with E as an internal edge must be completed by adjoining a slicing of W and a slicing of W', each with respect to the appropriate vertices of B other than X and Y. It is thus necessary that t shall be an even number $2j$. We deduce that

$$\beta_1 = \sum_P \sum_{j=0}^{n_1-1} \gamma(j, P)\gamma(n_1 - j - 1, \bar{P}). \tag{4}$$

Now suppose $Y \in J_2$. We can use the model of Figure 3, with $L_1 = E$, $v_1 = X$, and $w_2 = Y$. We construct a segment N_1 in B, parallel to E and joining a point a on K_1' to a point b on K_2'. We thus mark off a strip R of B bounded by E, N_1, and the closed arcs Xa and Yb of K_1' and K_2' respectively. By taking N_1 sufficiently near E we arrange that R has no vertices other than X and Y in its boundary.

We form a new band B' from B by uniting the interior of R with the open segments Xa and Yb and the external faces corresponding to J_1 and J_2 so as to form a new external face, with bounding curve J_0 say. We assign to B' the same vertices as for B, with the omission of X and Y. Given a slicing of B' we can obtain one of B by uniting the interior of R with the open arc N_1 and the adjacent internal face, taking E as an internal edge, and adjusting the external edges accordingly. Conversely any slicing of B with E as an internal edge gives rise to one of B'. Since there are $2n_2$ possible choices for Y we deduce that

$$\beta_2 = 2n_2\gamma(n_1 + n_2 - 1, n_3, n_4, \ldots, n_k), \tag{5}$$

provided that $n_1 > 0$. If $n_1 = 0$ we have of course $\beta_2 = 0$. Similar formulae hold for $\beta_3, \beta_4, \ldots, \beta_k$.

Combining these results with (4) we obtain

$$\begin{aligned}(2.1)\qquad \gamma(n_1, n_2, \ldots, n_k) &= \sum_P \sum_{j=0}^{n_1-1} \gamma(j, P)\gamma(n_1 - j - 1, \bar{P}) \\ &\quad + \sum_{r=2}^{k} 2n_r\gamma(n_1 + n_r - 1, S - \{J_r\}),\end{aligned}$$

provided $n_1 > 0$.

This formula makes possible the computation of $\gamma(n_1, n_2, \ldots, n_k)$ for small values of k and the n_i. At an early stage of this research numerical values so obtained suggested the general formula (1.1).

3. Generating functions. We now fix the numbers $n_2, n_3 \ldots, n_k$, but allow n_1 to vary. We introduce the generating function

$$G(P, x) = \sum_{n_1=0}^{\infty} \gamma(n_1, P)x^{m-1}, \tag{6}$$

where P runs through the subclasses of $S = \{J_2, J_3, \ldots, J_k\}$, and

$$m = n_1 + \sum_{J_j \epsilon P} n_j. \tag{7}$$

Using this generating function we can rewrite (2.1) as

$$\begin{aligned}(3.1)\qquad G(S, x) &= x^2 \sum_P G(P, x)G(\bar{P}, x) \\ &\quad + \sum_{r=2}^{k} 2n_r \sum_{n_1=1}^{\infty} \gamma(n_1 + n_r - 1, S - \{J_r\})x^{n-1}\end{aligned}$$

when $k > 1$, and as

$$G(\phi, x) = x^2\{G(\phi, x)\}^2 + x^{-1} \tag{3.2}$$

when $k = 1$. From (3.2) we deduce

$$G(\phi, x) = \frac{1}{2x^2}\{1 - (1 - 4x)^{\frac{1}{2}}\}. \tag{3.3}$$

4. A combinatorial identity. We turn from the contemplation of equation (2.1) to forge the tool that will be needed for its solution. Let λ, μ, $f_1, f_2, \ldots, f_s$ be arbitrary functions (sufficiently often differentiable) of x. Let S denote the set $\{1, 2, \ldots, s\}$ of integers. We use the symbol P to denote a subset of S and we write $\bar{P}$ for its complementary subset. We write $\alpha(P)$ for the number of members of P, and we write $\langle P \rangle$ for the product of the functions f_i with $i \in P$. If $\alpha(P) = 0$ we take $\langle P \rangle$ to be 1.

We propose to establish the identity

$$(4.1)\quad \sum_P D^{\alpha(P)-k}\{\lambda\langle P\rangle\}.D^{\alpha(\bar{P})-l}\{\mu\langle\bar{P}\rangle\}$$
$$= \sum_{\alpha(P)=0}^{k-1}\sum_{i=0}^{k-1-\alpha(P)}(-1)^i\binom{\alpha(\bar{P})-l}{i}D^{\alpha(\bar{P})-l-i}\{D^{-k+\alpha(P)+i}\{\lambda\langle P\rangle\}.\mu\langle\bar{P}\rangle\}$$
$$+\sum_{\alpha(\bar{P})=0}^{l-1}\sum_{i=0}^{l-1-\alpha(\bar{P})}(-1)^i\binom{\alpha(P)-k}{i}D^{\alpha(P)-k-i}\{\lambda\langle P\rangle.D^{-l+\alpha(\bar{P})+i}\{\mu\langle\bar{P}\rangle\}\}.$$

Here k and l are arbitrary positive integers such that

$$(8)\qquad s \geqslant k+l-1.$$

The summations are taken over all relevant sets P. Thus the first summation on the right is over all P satisfying $\alpha(P) < k$.

The symbol D stands for differentiation with respect to x. In (4.1) it sometimes occurs, operating on $\lambda\langle P\rangle$ or $\mu\langle\bar{P}\rangle$, with a negative index. But the indices $\alpha(\bar{P}) - l - i$ and $\alpha(P) - k - i$ are never negative, by (8) and the limitations $i \leqslant k-1-\alpha(P)$ and $i \leqslant l-1-\alpha(\bar{P})$ of the summations concerned. When D does occur with a negative index it is to be treated as an operation of repeated integration. To avoid difficulties with arbitrary constants we suppose that for each function $X = \lambda\langle P\rangle$ or $\mu\langle\bar{P}\rangle$ concerned $D^{-1}(X)$ is fixed as a particular integral of X. Then $D^{-2}(X)$ is fixed as a particular integral of $D^{-1}(X)$, and so on as far as may be necessary. We shall in particular fix $D^{-1}(\lambda')$ as λ.

We refer to the identity of (4.1) as $H(s; k, l)$.

Lemma I. *$H(s; k, l)$ is true whenever $s = k+l-1$.*

Proof. In this case we have for any P either $\alpha(P) < k$ or $\alpha(\bar{P}) < l$, but not both. Suppose first that $\alpha(P) < k$. Then

$$\sum_{i=0}^{k-1-\alpha(P)}(-1)^i\binom{\alpha(\bar{P})-l}{i}D^{\alpha(\bar{P})-l-i}\{D^{-k+\alpha(P)+i}\{\lambda\langle P\rangle\}.\mu\langle\bar{P}\rangle\}$$
$$=\sum_{i=0}^{k-1-\alpha(P)}(-1)^i\binom{k-1-\alpha(P)}{i}D^{k-1-\alpha(P)-i}\{D^{-k+\alpha(P)+i}\{\lambda\langle P\rangle\}.\mu\langle\bar{P}\rangle\}$$
$$=\sum_{i=0}^{k-1-\alpha(P)}(-1)^i\binom{k-1-\alpha(P)}{i}\sum_{j=0}^{k-1-\alpha(P)-i}\binom{k-1-\alpha(P)-i}{j}\times D^{-1-j}\{\lambda\langle P\rangle\}D^j\{\mu\langle\bar{P}\rangle\}$$
$$=\sum_{j=0}^{k-1-\alpha(P)}D^{-1-j}\{\lambda\langle P\rangle\}D^j\{\mu\langle\bar{P}\rangle\}\sum_{i=0}^{k-1-\alpha(P)-j}(-1)^i\frac{(k-1-\alpha(P))!}{i!\,j!\,(k-1-\alpha(P)-i-j)!}$$
$$=\sum_{j=0}^{k-1-\alpha(P)}\left(\frac{(k-1-\alpha(P))!}{j!\,(k-1-\alpha(P)-j)!}D^{-1-j}\{\lambda\langle P\rangle\}D^j\{\mu\langle\bar{P}\rangle\}\times\sum_{i=0}^{k-1-\alpha(P)-j}(-1)^i\binom{k-1-\alpha(P)-j}{i}\right)$$
$$=D^{\alpha(P)-k}\{\lambda\langle P\rangle\}D^{\alpha(\bar{P})-l}\{\mu\langle\bar{P}\rangle\},$$

since the sum

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$$\sum_{i=0}^{k-1-\alpha(P)-j} (-1)^i \binom{k-1-\alpha(P)-j}{i}$$

is 1 if $j = k - 1 - \alpha(P)$, and is 0 if $j < k - 1 - \alpha(P)$.

On the other hand the empty sum

$$\sum_{i=0}^{l-1-\alpha(\bar{P})} (-1)^i \binom{\alpha(P)-k}{i} D^{\alpha(P)-k-i}\{\lambda\langle P\rangle\} D^{-1+\alpha(\bar{P})+i}\{\mu\langle\bar{P}\rangle\}$$

must be interpreted as 0.

A similar argument applies if $\alpha(\bar{P}) < l$. It follows that each subset P makes equal contributions to the two sides of (4.1). The lemma follows.

LEMMA II. *Let s, k, and l be positive integers satisfying* (8) *and such that $H(s; k, l)$ and $H(s+1; k+1, l)$ are both true. Then $H(s+1; k, l)$ is true.*

Proof. We operate on $H(s; k, l)$ as follows. We introduce a new arbitrary function f_{s+1} and replace each term $\lambda\langle P\rangle$ occurring in $H(s; k, l)$ by $\lambda f_{s+1}'\langle P\rangle$. The identity remains valid since the change corresponds merely to a new choice of the arbitrary function λ.

Again we can rewrite $H(s; k, l)$ using as suffices all the numbers $1, 2, \ldots, s+1$ except an arbitrarily chosen j, and operate as before with j replacing $s+1$. If this is done for each j, including $s+1$, and the resulting identities are added the effect is to replace each sum

$$\sum_{\alpha(P)=t} D^a\{D^b\{\lambda\langle P\rangle\}D^c\{\mu\langle\bar{P}\rangle\}\}$$

occurring in $H(s; k, l)$ by the corresponding sum

$$\sum_{\alpha(Q)=t+1} D^a\{D^b\{\lambda(f_{q1}'f_{q2}\ldots f_{qt+1} + f_{q1}f_{q2}'f_{q3}\ldots f_{qt+1} + \ldots + f_{q1}f_{q2}\ldots f_{qt}f_{qt+1}')\}D^c\{\mu\langle\bar{Q}\rangle\}\},$$

where $Q = \{q_1, q_2, \ldots, q_{t+1}\}$ denotes a subset of $Z = \{1, 2, \ldots, s+1\}$ and $\bar{Q}$ is its complementary subset in Z. But this sum is

$$\sum_{\alpha(Q)=t+1} D^a\{D^{b+1}\{\lambda\langle Q\rangle\}D^c\{\mu\langle\bar{Q}\rangle\}\} - \sum_{\alpha(Q)=t+1} D^a\{D^b\{\lambda'\langle Q\rangle\}D^c\{\mu\langle\bar{Q}^c\rangle\}\}.$$

We thus derive from $H(s; k, l)$ the following identity.

$$\begin{aligned}
&\sum_{\alpha(Q)=1}^{k} D^{\alpha(Q)-k}\{\lambda\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\} - \sum_{\alpha(Q)=1}^{k} D^{\alpha(Q)-k-1}\{\lambda'\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\} \\
&= \sum_{\alpha(Q)=1}^{k} \sum_{i=0}^{k-\alpha(Q)} (-1)^i \binom{\alpha(\bar{Q})-l}{i} D^{\alpha(\bar{Q})-l-i}\{D^{-k+\alpha(Q)+i}\{\lambda\langle Q\rangle\}\cdot\mu\langle\bar{Q}\rangle\} \\
&+ \sum_{\alpha(\bar{Q})=0}^{l-1} \sum_{i=0}^{l-1-\alpha(\bar{Q})} (-1)^i \binom{\alpha(Q)-k-1}{i} D^{\alpha(Q)-k-1-i}\{D\{\lambda\langle Q\rangle\}D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\} \\
&- \sum_{\alpha(Q)=1}^{k} \sum_{i=0}^{k-\alpha(Q)} (-1)^i \binom{\alpha(\bar{Q})-l}{i} D^{\alpha(\bar{Q})-l-i}\{D^{-k-1+\alpha(Q)+i}\{\lambda'\langle Q\rangle\}\cdot\mu\langle\bar{Q}\rangle\} \\
&- \sum_{\alpha(\bar{Q})=0}^{l-1} \sum_{i=0}^{l-1-\alpha(\bar{Q})} (-1)^i \binom{\alpha(Q)-k-1}{i} D^{\alpha(Q)-k-1-i}\{\lambda'\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}.
\end{aligned}$$

On the left-hand side of this equation we include the terms for $\alpha(Q) = 0$ in the summations, subtracting them again afterwards. On the right-hand side we write separately the terms for $i = k - \alpha(Q)$ in the first summation, and in the second summation we replace

$$D^{\alpha(Q)-k-1-i}\{D\{\lambda\langle Q\rangle\}D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}$$

by

$$D^{\alpha(Q)-k-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\} - D^{\alpha(Q)-k-1-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i+1}\{\mu\langle\bar{Q}\rangle\}\}.$$

We thus obtain

$$\begin{aligned}
(9)\quad & \sum_Q D^{\alpha(Q)-k}\{\lambda\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\} \\
& - \sum_Q D^{\alpha(Q)-k-1}\{\lambda'\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\} \\
& - D^{-k}\{\lambda\}D^{s+1-l}\{\mu\langle Z\rangle\} \\
& + D^{-k-1}\{\lambda'\}D^{s+1-l}\{\mu\langle Z\rangle\} \\
= & \sum_{\alpha(Q)=1}^{k}\sum_{i=0}^{k-1-\alpha(Q)}(-1)^i\binom{\alpha(\bar{Q})-l}{i}D^{\alpha(\bar{Q})-l-i}\{D^{-k+\alpha(Q)+i}\{\lambda\langle Q\rangle\}.\mu\langle\bar{Q}\rangle\} \\
& + \sum_{\alpha(Q)=1}^{k}(-1)^{k+\alpha(Q)}\binom{\alpha(\bar{Q})-l}{k-\alpha(Q)}D^{s+1-k-l}\{\lambda\langle Q\rangle\mu\langle\bar{Q}\rangle\} \\
& + \sum_{\alpha(\bar{Q})=0}^{l-1}\sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^i\binom{\alpha(Q)-k-1}{i}D^{\alpha(Q)-k-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\} \\
& - \sum_{\alpha(\bar{Q})=0}^{l-1}\sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^i\binom{\alpha(Q)-k-1}{i}D^{\alpha(Q)-k-1-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i+1}\{\mu\langle\bar{Q}\rangle\}\} \\
& - \sum_{\alpha(Q)=1}^{(k+1)-1}\sum_{i=0}^{(k+1)-\alpha(Q)-1}(-1)^i\binom{\alpha(\bar{Q})-l}{i}D^{\alpha(\bar{Q})-l-i}\{D^{-k-1+\alpha(Q)+i}\{\lambda'\langle Q\rangle\}.\mu\langle\bar{Q}\rangle\} \\
& - \sum_{\alpha(\bar{Q})=0}^{l-1}\sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^i\binom{\alpha(Q)-k-1}{i}D^{\alpha(Q)-k-1-i}\{\lambda'\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}.
\end{aligned}$$

The last two terms on the left of the preceding identity cancel.

Consider the identity $H(s+1; k+1, l)$, which is valid by hypothesis. We apply it to the same functions as $H(s; k, l)$, except that f_{s+1} is adjoined and λ' replaces λ. Adding it to the identity (9) we obtain

$$\begin{aligned}
& \sum_Q D^{\alpha(Q)-k}\{\lambda\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\} \\
= & \sum_{\alpha(Q)=0}^{k-1}\sum_{i=0}^{k-1-\alpha(Q)}(-1)^i\binom{\alpha(\bar{Q})-l}{i}D^{\alpha(\bar{Q})-l-i}\{D^{-k+\alpha(Q)+i}\{\lambda\langle Q\rangle\}.\mu\langle\bar{Q}\rangle\} \\
& - \sum_{i=0}^{k-1}(-1)^i\binom{s+1-l}{i}D^{s-l-i+1}\{D^{-k+i}\{\lambda\}.\mu\langle Z\rangle\}
\end{aligned}$$

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$$+\sum_{j=1}^{k}(-1)^{k+j}\binom{s+1-l-j}{k-j}\sum_{\alpha(Q)=j}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}$$
$$+\sum_{\alpha(\bar{Q})=0}^{l-1}\ \sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^{i}\binom{\alpha(Q)-k-1}{i}D^{\alpha(Q)-k-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}$$
$$-\sum_{\alpha(\bar{Q})=0}^{l-1}\ \sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^{i}\binom{\alpha(Q)-k-1}{i}D^{\alpha(Q)-k-1-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i+1}\{\mu\langle\bar{Q}\rangle\}\}$$
$$+\sum_{i=0}^{k}(-1)^{i}\binom{s-l+1}{i}D^{s+1-l-i}\{D^{-k+i}\{\lambda\}.\mu\langle Z\rangle\}.$$

In the fifth summation on the right-hand side replace i by $\nu - 1$ and separate out the term for which $\nu = l - \alpha(\bar{Q})$. The remainder of this summation can then be combined with the preceding summation, and the right-hand side becomes

$$\sum_{\alpha(Q)=0}^{k-1}\ \sum_{i=0}^{k-1-\alpha(Q)}(-1)^{i}\binom{\alpha(\bar{Q})-l}{i}D^{\alpha(\bar{Q})-l-i}\{D^{-k+\alpha(Q)+i}\{\lambda\langle Q\rangle\}.\mu\langle\bar{Q}\rangle\}$$
$$+\sum_{j=0}^{k}(-1)^{k+j}\binom{s+1-l-j}{k-j}\sum_{\alpha(Q)=j}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}$$
$$-(-1)^{k}\binom{s+1-l}{k}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}$$
$$+\sum_{\alpha(\bar{Q})=0}^{l-1}\ \sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^{i}\binom{\alpha(Q)-k}{i}D^{\alpha(Q)-k-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}$$
$$+\sum_{j=0}^{l-1}(-1)^{l+j}\binom{s-k-j}{l-1-j}\sum_{\alpha(\bar{Q})=j}D^{s-k-l+1}\{\lambda\mu\langle Z\rangle\}$$
$$+(-1)^{k}\binom{s+1-l}{k}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}.$$

Rearranging we obtain

$$\text{(10)}\quad \sum_{Q}D^{\alpha(Q)-k}\{\lambda\langle Q\rangle\}D^{\alpha(\bar{Q})-l}\{\mu\langle\bar{Q}\rangle\}$$
$$-\sum_{\alpha(Q)=0}^{k-1}\ \sum_{i=0}^{k-1-\alpha(Q)}(-1)^{i}\binom{\alpha(\bar{Q})-l}{i}D^{\alpha(\bar{Q})-l-i}\{D^{-k+\alpha(Q)+i}\{\lambda\langle Q\rangle\}.\mu\langle\bar{Q}\rangle\}$$
$$-\sum_{\alpha(\bar{Q})=0}^{l-1}\ \sum_{i=0}^{l-1-\alpha(\bar{Q})}(-1)^{i}\binom{\alpha(Q)-k}{i}D^{\alpha(Q)-k-i}\{\lambda\langle Q\rangle D^{-l+\alpha(\bar{Q})+i}\{\mu\langle\bar{Q}\rangle\}\}$$
$$=\sum_{j=0}^{k}(-1)^{k+j}\binom{s+1-l-j}{k-j}\sum_{\alpha(Q)=j}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}$$
$$+\sum_{j=0}^{l-1}(-1)^{l+j}\binom{s-k-j}{l-1-j}\sum_{\alpha(\bar{Q})=j}D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}$$
$$=D^{s+1-k-l}\{\lambda\mu\langle Z\rangle\}\left\{\sum_{j=0}^{k}(-1)^{k+j}\binom{s+1-l-j}{s+1-k-l}\binom{s+1}{j}\right.$$
$$\left.+\sum_{j=0}^{l-1}(-1)^{l+j}\binom{s-k-j}{s+1-k-l}\binom{s+1}{j}\right\}.$$

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Now

$$\binom{s+1-l-j}{s-k-l+1}(-1)^{k+j}$$

is the coefficient of x^{k-j} in $(1+x)^{-s+k+l-2}$ and

$$\binom{s+1}{j}$$

is the coefficient of x^j in $(1+x)^{s+1}$. Hence

$$\sum_{j=0}^{k}(-1)^{k+j}\binom{s+1-l-j}{s-k-l+1}\binom{s+1}{j}$$

is the coefficient of x^k in $(1+x)^{k+l-1}$, which is

$$\binom{k+l-1}{k}.$$

A similar argument, in which k and $l-1$ are interchanged shows that

$$\sum_{j=0}^{l-1}(-1)^{l-1+j}\binom{s-k-j}{s+1-k-l}\binom{s+1}{j}=\binom{k+l-1}{l-1}=\binom{k+l-1}{k}.$$

Hence the right-hand side of (10) is identically zero. Since we can now recognize (10) as $H(s+1;k,l)$ the Lemma is established.

We complete the proof of (4.1) as follows. If possible choose s, k, l and the functions λ, μ, f_i so that the identity is false, and so that $s-k-l+1$ has the least value consistent with this condition.

By Lemma I we have $s-k-l+1>0$. Hence the identities $H(s-1;k,l)$ and $H(s;k+1,l)$ are defined. They are valid by the choice of s, k, and l. Applying Lemma II with s replaced by $s-1$ we find that $H(s,k,l)$ is also valid. But this contradicts the choice of s, k, and l. This contradiction establishes the theorem.

5. Enumeration of the even slicings. We write

$$\lambda = 2(1-4x)^{-3/2},$$

(11)
$$D^{-1}\{\lambda\} = (1-4x)^{-\frac{1}{2}},$$

$$D^{-2}\{\lambda\} = -\tfrac{1}{2}(1-4x)^{\frac{1}{2}}.$$

For $n \geqslant 1$ and C any constant we put

(12)
$$D^{-1}\{Cx^n(1-4x)^{-3/2}\} = \int_0^x Ct^n(1-4t)^{-3/2}dt.$$

We also make the abbreviation

(13)
$$c(n) = \frac{(2n)!}{n!(n-1)!}.$$

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LEMMA III. *If n is any positive integer, then*

$$n\sum_{i=1}^{\infty} c(n+i-1)x^{n+i-2}$$

$$= -\lambda D^{-2}\{\lambda\}D\{c(n)x^n\} + \lambda D^{-1}\{\lambda\}c(n)x^n - \lambda D^{-1}\{\lambda c(n)x^n\}.$$

*Proof.** We need to prove that

$$n\sum_{i=1}^{\infty} c(n+i-1)x^{n+i-2} = (1-4x)^{-1}nc(n)x^{n-1} + 2(1-4x)^{-2}c(n)x^n - 4(1-4x)^{-3/2}\int_0^x c(n)t^n(1-4t)^{-3/2}dt. \tag{14}$$

Putting

$$P(n) = \sum_{i=1}^{\infty} c(n+i-1)x^{n+i-2}$$

we can rewrite (14) as

$$P(n) = \frac{c(n)}{n}(1-4x)^{-3/2}\left\{nx^{n-1}(1-4x)^{\frac{1}{2}} + 2x^n(1-4x)^{-\frac{1}{2}} - 4\int_0^x t^n(1-4t)^{-3/2}dt\right\}. \tag{15}$$

We assume $n \geqslant 2$, and integrate by parts twice in (15). The right-hand side then simplifies to

$$M(n) = (n-1)c(n)(1-4x)^{-3/2}\int_0^x t^{n-2}(1-4t)^{\frac{1}{2}}dt. \tag{16}$$

Now it is easily verified that

$$(2n+3)\int_0^x t^n(1-4t)^{\frac{1}{2}}dt = -\tfrac{1}{2}x^n(1-4x)^{3/2} + \tfrac{1}{2}n\int_0^x t^{n-1}(1-4t)^{\frac{1}{2}}dt. \tag{17}$$

It follows that

$$M(n+2) = -c(n+1)x^n + M(n+1). \tag{18}$$

But by the definition of $P(n)$ we have

$$P(n+2) = -c(n+1)x^n + P(n+1). \tag{19}$$

By (16) we have

$$M(2) = c(2)(1-4x)^{-3/2}\int_0^x (1-4t)^{\frac{1}{2}}dt,$$

which reduces to $2(1-4x)^{-3/2} - 2$, of which $P(2)$ is the power series expansion. Since $P(2) = M(2)$ it follows from (18) and (19) that $P(n) = M(n)$ for all $n \geqslant 2$.

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*The author wishes to thank the referee for this proof, which is considerably shorter than the original.

If $n = 1$ the right-hand side of (15) reduces to $2(1-4x)^{-3/2}$, of which the left-hand side, $P(1)$, is the power series expansion. Hence Lemma III is valid for $n \geqslant 1$.

We now return to the function $G(S, x)$ defined in §3. We make a minor change of notation by taking S to represent the set $\{1, 2, \ldots, k-1\}$ of integers instead of the set $\{J_2, J_3, \ldots, J_k\}$ of bounding curves. Correspondingly a symbol P denoting a set $\{J_{s(1)}, \ldots, J_{s(r)}\}$ of curves (with each $s(i) \geqslant 2$) will now be interpreted as the subset $\{s(1)-1, \ldots, s(r)-1\}$ of S. We write

$$f_i = c(n_{i+1})x^{n_i+1}, \qquad (1 \leqslant i \leqslant k-1), \tag{20}$$

and we define $\langle P \rangle$ as in §4.

THEOREM. *If $k \geqslant 1$ and the integers $n_1, n_2, \ldots, n_k$ are all positive, then*

$$\gamma(n_1, n_2, \ldots, n_k) = \frac{(n-1)!}{(n-k+2)!} \prod_{i=1}^{k} (c_i),$$

where $n = n_1 + n_2 + \ldots + n_k$.

Proof. We have the expansion

$$\lambda = 2(1-4x)^{-3/2} = \sum_{r=1}^{\infty} \frac{(2r)!}{r!(r-1)!} x^{r-1}.$$

Hence, in the case $k \geqslant 2$ the theorem can be rewritten as

$$G(S, x) = x^{k-3} D^{k-3}\{\lambda \langle S \rangle\}. \tag{21}$$

If possible choose k so that the theorem fails for some set of values of the n_i, and so that it has the least value consistent with this condition. We then have $k \geqslant 2$, for when $k = 1$ the theorem can be verified by applying the binomial theorem to (3.3). From (3.1) we have

$$G(S, x)(1 - 2x^2 G(\phi, x)) = x^2 \sum_{\alpha(P)=1}^{k-2} G(P, x) G(\bar{P}, x)$$
$$+ 2 \sum_{j=2}^{k} n_j \sum_{n_1=1}^{\infty} \gamma(n_1 + n_j - 1, S - \{j-1\}) x^{n-1},$$

where $n = n_1 + n_2 + \ldots + n_k$. Hence by (3.3),

$$x^{-k+3}(1-4x)^{\frac{1}{2}} G(S, x) - \sum_{\alpha(P)=1}^{k-2} D^{\alpha(P)-2}\{\lambda \langle P \rangle\} D^{\alpha(\bar{P})-2}\{\lambda \langle \bar{P} \rangle\}$$
$$= 2x^{-k+3} \sum_{j=2}^{k} n_j \sum_{n_1=1}^{\infty} \frac{(n-2)!}{(n-k+2)!} c(n_1 + n_j - 1) x^{n_1+n_j-1} \langle S - \{j-1\} \rangle$$
$$= 2 \sum_{j=2}^{k} D^{k-4}\left\{ n_j \sum_{n_1=1}^{\infty} c(n_1 + n_j - 1) x^{n_i+n_j-2} \langle S - \{j-1\} \rangle \right\}$$
$$= 2 \sum_{i=1}^{k-1} D^{k-4}\{(-\lambda D^{-2}\{\lambda\} D\{f_i\} + \lambda D^{-1}\{\lambda\} f_i - \lambda D^{-1}\{\lambda f_i\}) \langle S - \{i\} \rangle\},$$

by Lemma III,

$$= 2D^{k-4}\Big\{ - \lambda D^{-2}\{\lambda\}D\{\langle S\rangle\} + (k-1)\lambda D^{-1}\{\lambda\}\langle S\rangle - \sum_{i=1}^{k-1} \lambda D^{-1}\{\lambda f_i\}\langle S - \{i\}\rangle\Big\}$$

$$= 2D^{k-4}\Big\{ - D\{\lambda D^{-2}\{\lambda\}\langle S\rangle\} + (k-3)\lambda D^{-1}\{\lambda\}\langle S\rangle - \sum_{i=1}^{k-1} \lambda D^{-1}\{\lambda f_i\}\langle S - \{i\}\rangle\Big\},$$

since $D\{\lambda D^{-2}\{\lambda\}\} = -D\{(1-4x)^{-1}\} = -4(1-4x)^{-2} = -2\lambda D^{-1}\{\lambda\}$. (It may happen that the index in D^{k-4} is negative.) We now have

$$(22)\quad x^{-k+3}(1-4x)^{\frac{1}{2}}G(S,x) = \sum_{\alpha(P)=1}^{k-2} D^{\alpha(P)-2}\{\lambda\langle P\rangle\}D^{\alpha(\bar{P})-2}\{\lambda\langle\bar{P}\rangle\} - 2D^{k-3}\{D^{-2}\{\lambda\}\lambda\langle S\rangle\} + 2(k-3)D^{k-4}\{D^{-1}\{\lambda\}\lambda\langle S\rangle\} - 2\sum_{i=1}^{k-1} D^{k-4}\{D^{-1}\{\lambda f_i\}\lambda\langle S - \{i\}\rangle\}.$$

We now dispose of the special cases $k = 2$ and $k = 3$. If $k = 2$ we have, by (22),

$$\begin{aligned} x(1-4x)^{\frac{1}{2}}G(S,x) &= -2D^{-1}\{D^{-2}\{\lambda\}\lambda\langle S\rangle\} - 2D^{-2}\{D^{-1}\{\lambda\}\lambda\langle S\rangle\} - 2D^{-2}\{D^{-1}\{\lambda\langle S\rangle\}\lambda\} \\ &= -2D^{-1}\{D^{-2}\{\lambda\}\lambda\langle S\rangle\} - 2D^{-1}\{D^{-1}\{\lambda\}D^{-1}\{\lambda\langle S\rangle\}\} \\ &= -2D^{-2}\{\lambda\}D^{-1}\{\lambda\langle S\rangle\} \\ &= (1-4x)^{\frac{1}{2}}D^{-1}\{\lambda\langle S\rangle\}, \end{aligned}$$

whence (21) follows for this case. (Any arbitrary constants introduced in these transformations affect only terms of degrees 0 and 1 in the expansions. Terms of these degrees balance in the final equation, by (12).)

If $k = 3$ we have instead

$$\begin{aligned} (1-4x)^{\frac{1}{2}}G(S,x) &= 2D^{-1}\{\lambda f_1\}D^{-1}\{\lambda f_2\} - 2D^{-2}\{\lambda\}\lambda\langle S\rangle - 2D^{-1}\{D^{-1}\{\lambda f_1\}\lambda f_2\} \\ &\qquad - 2D^{-1}\{D^{-1}\{\lambda f_2\}\lambda f_1\} \\ &= -2D^{-2}\{\lambda\}\lambda\langle S\rangle \\ &= (1-4x)^{\frac{1}{2}}\lambda\langle S\rangle, \end{aligned}$$

and again (21) is verified.

In the remaining case $k \geqslant 4$. Then we may set the integers k and l of (4.1) equal to 2 and substitute the present $k-1$ for s. Taking $\mu = \lambda$ we then obtain

$$\sum_P D^{\alpha(P)-2}\{\lambda\langle P\rangle\}D^{\alpha(\bar{P})-2}\{\lambda\langle\bar{P}\rangle\}$$
$$= 2\sum_{\alpha(P)=0}^{1}\sum_{i=0}^{1-\alpha(P)}(-1)^i\binom{k-3-\alpha(P)}{i}D^{k-3-\alpha(P)-i}\big\{D^{-2+\alpha(P)+i}\{\lambda\langle P\rangle\}\lambda\langle\bar{P}\rangle\big\}$$
$$= 2D^{k-3}\big\{D^{-2}\{\lambda\}\lambda\langle S\rangle\big\} - 2(k-3)D^{k-4}\big\{D^{-1}\{\lambda\}\lambda\langle S\rangle\big\}$$
$$+ 2\sum_{i=1}^{k-1} D^{k-4}\big\{D^{-1}\{\lambda f_i\}\lambda\langle S - \{i\}\rangle\big\}.$$

Combining this with (22) we have

$$x^{-k+3}(1-4x)^{\frac{1}{2}}G(S,x) = -2D^{-2}\{\lambda\}D^{k-3}\{\lambda\langle S\rangle\},$$
$$G(S,x) = x^{k-3}D^{k-3}\{\lambda\langle S\rangle\}.$$

This completes the proof of the theorem, since the definition of k is now contradicted. Accordingly formula (1.1) is established.

6. Odd slicings. The case in which some of the integers m_i of §1 are odd seems to be more difficult. It is still possible to obtain recursion formulae like (2.1), and there are apparently relevant generalizations of (4.1). For example, only minor modifications of the proof of (4.1) are required to show that the formula remains valid when the summations are restricted to subsets P with an odd (or even) number of elements. Using the generalized formulae the author has found that if only two of the m_i are odd then the number of slicings is still given by the right-hand side of (1.1) provided that we replace the factor

$$\frac{(2n_i)!}{n_i!(n_i-1)!} \quad \text{by} \quad \frac{(2n_i+1)!}{(n_i!)^2}$$

whenever m_i is an odd number $2n_i+1$. But this rule does not apply when four odd numbers occur. In the case $(m_1 = 3, m_2 = m_3 = m_4 = 1)$, for example, the number of slicings is

$$6 \neq (3-1)\frac{3!}{(1!)^2}\left\{\frac{1!}{(0!)^2}\right\}^3.$$

Added in proof. The identity $H(s; k, l)$ can be deduced from Lagrange's Theorem. For example, to prove $H(s; 1, 1)$ we write $F(x) = D^{-1}(\lambda)$, $G(x) = D^{-1}(\mu)$ and define ξ by $\xi = a + x\phi(\xi)$, where $\phi(x) = u_1f_1 + u_2f_2 + \ldots + u_sf_s$. We then equate the coefficient of $u_1u_2\ldots u_s$ in the product of the expansions of $F(\xi)$ and $G(\xi)$ to the corresponding coefficient in the expansion of the function $F(\xi)G(\xi)$ of ξ.

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