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EMBEDDING $\mathcal{I}_N$ IN A 2-GENERATOR INVERSE SUBSEMIGROUP OF $\mathcal{I}_{N+2}$

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Abstract Given an integer n, we show that $\mathcal{I}_n$ embeds in a 2-generated subsemigroup of $\mathcal{I}_{n+2}$, which is an inverse semigroup. An immediate consequence of this result is the following, which is analogous to the case for groups and semigroups: every finite inverse semigroup may be embedded in a finite 2-generated semigroup which is an inverse semigroup.

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1. Introduction

It is well known that any countable semigroup can be embedded in a 2-generator semigroup. Apparently, the first proof of this result is due to Evans [**2**]. The proof of Evans was combinatorial in nature. Later, Neumann [**6**] gave an alternative proof based on the wreath product construction. A corollary to Neumann's proof is that any finite semigroup can be embedded in a finite 2-generator semigroup. This result is also a consequence of yet another proof of the result of Evans, which is due to Subbiah [**7**].

Since every countable semigroup can be embedded in the full transformation semigroup T on the positive integers, Subbiah begins her proof by assuming that the semigroup S in question is a subsemigroup of T. On the basis of an enumeration of the elements of S, she explicitly produces a pair of elements α, β of T such that S is contained in the subsemigroup $\langle \alpha, \beta \rangle$ generated by α and β. Ash (see Hall [**3**]) showed that Subbiah's proof could be modified to prove that every countable inverse semigroup can be embedded in a 2-generator inverse semigroup and that every finite inverse semigroup can be embedded in a finite 2-generator inverse semigroup.

Hall remarks that Ash's construction can be modified to obtain Neumann's result: that every finite semigroup S can be embedded in a finite 2-generator semigroup. To do this, and to get the result for finite inverse semigroups, one regards the finite semigroup S as

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a semigroup of partial transformations on a finite set X and embeds S as a semigroup of partial transformations on the finite set $X \times Z_{4m}$, where m is the number of elements of S. Thus, for example, if S is the full transformation semigroup on $X = \{1, 2, \ldots, n\}$, the set $X \times Z_{4m}$ has order $4n^{n+1}$.

In fact, it is easy to obtain a direct embedding of a finite semigroup of transformations of a set X, of order n, in a 2-generator subsemigroup of the full transformation semigroup on $n+1$ letters. To see this, it suffices to produce a 2-generator subsemigroup of the full transformation semigroup $\mathcal{T}_{n+1}$ on $\{1, 2, \ldots, n+1\}$ that contains $\mathcal{T}_n$ as a submonoid.

To this end, define transformations α, β of $\{1, \ldots, n+1\}$ as follows. The transformation β is the cyclic permutation of $\{1, \ldots, n+1\}$ which maps i to $i + 1 \bmod (n+1)$ while

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n & n+1 \\ 2 & 1 & 3 & \cdots & n-1 & n+1 & n+1 \end{pmatrix},$$

so that α interchanges 1 and 2, maps n to $n+1$, and leaves the other members of $\{1, 2, \ldots, n+1\}$ unchanged. The subsemigroup S of the full transformation semigroup $T = \mathcal{T}_{n+1}$ on $\{1, \ldots, n+1\}$ generated by α, β also contains the idempotent $\varepsilon = \alpha^2$. Each element of $\varepsilon T \varepsilon$ has the same value at both n and at $n+1$ and its image is contained in $\{1, 2, \ldots, n-1, n+1\}$. Indeed, it is easy to see that the restriction of each element of $\varepsilon T \varepsilon$ to $\{1, 2, \ldots, n-1, n+1\}$ gives an isomorphism of $\varepsilon T \varepsilon$ onto the full transformation semigroup on $\{1, 2, \ldots, n-1, n+1\}$. The inverse isomorphism is obtained by extending maps of $\{1, 2, \ldots, n-1, n+1\}$ to $\{1, 2, \ldots, n-1, n, n+1\}$ by defining the value at n to be the same as that at $n+1$. It follows that $\varepsilon T \varepsilon$ is generated by the images, under this inverse isomorphism, of the n cycle $(1, 2, \ldots, n-1, n+1)$, the transposition $(1, 2)$, and any idempotent of rank $n-1$ (see [**8**] or [**1, 4, 5**] for this and other standard results in semigroup theory). Thus $\varepsilon T \varepsilon$ is generated by

$$\kappa = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & n+1 \\ 2 & 3 & \cdots & n & 1 & 1 \end{pmatrix},$$

by α and the idempotent

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & n+1 \\ 1 & 2 & \cdots & n+1 & n+1 & n+1 \end{pmatrix}.$$

But $\kappa = \varepsilon \beta \varepsilon$ so that $\kappa \in S$. Furthermore, it is easy to verify that $\phi = \beta \varepsilon \beta^{-1} \varepsilon$ so that S contains the three generators β, κ, ϕ of $\varepsilon T \varepsilon$. Thus S contains a subsemigroup isomorphic to the full transformation semigroup of n letters. Indeed, $\varepsilon S \varepsilon = \varepsilon T \varepsilon$ is an isomorphic copy of the full transformation semigroup on $\{1, \ldots, n\}$. Thus S contains the full transformation semigroup on $\{1, 2, \ldots, n\}$ as a local submonoid.

Consequently, we have the following proposition.

Proposition 1.1. *Every finite semigroup can be embedded in a 2-generator finite semigroup which is generated by a pair of group elements.*

Proof. It is well known that every finite semigroup S can be embedded in the full transformation semigroup on a finite set. By the remarks above, this full transformation semigroup can be embedded in a finite 2-generator semigroup. Hence so can S. Finally, it is clear that each of the elements a, b in the discussion above is a group element; that is, it belongs to a subgroup of S. This is the end of the proof. □

The main result of this paper shows that a similar construction can be used to prove that the analogous result holds for finite inverse semigroups. More precisely, let n be a positive integer and denote by $\mathcal{I}_n$ the symmetric inverse semigroup on $\{1, 2, \dots, n\}$. We shall show that there is a 2-generator subsemigroup S of $\mathcal{I}_{n+2}$ that contains an isomorphic copy of $\mathcal{I}_n$ as a local submonoid. The semigroup S is inverse and is generated by a pair of group elements. Since, by the Wagner–Preston Theorem, every finite inverse semigroup can be embedded in some $\mathcal{I}_n$, it follows that every finite inverse semigroup can be embedded in a finite 2-generator inverse semigroup.

2. The result

Theorem 2.1. *The inverse semigroup $\mathcal{I}_n$ may be embedded, as a local submonoid, in a 2-generated inverse subsemigroup of $\mathcal{I}_{n+2}$.*

Proof. Let $X = \{1, 2, \dots, n\}$ and $Y = X \cup \{a, b\}$, where $a, b \notin X$. Let $\mathcal{I}_X$ (respectively, $\mathcal{I}_Y$) denote the semigroup of all one-to-one partial mappings on X (respectively, on Y). Notice that $\mathcal{I}_X$ (respectively, $\mathcal{I}_Y$) is isomorphic to $\mathcal{I}_n$ (respectively, $\mathcal{I}_{n+2}$). There is a natural isomorphism between the semigroup $\mathcal{I}_X$ and the semigroup of those one-to-one partial mappings on Y that are not defined on the elements a and b and contain neither a nor b in their image. To simplify our notation, we shall assume that the two semigroups literally coincide. Let us consider two mappings $\alpha, \beta \in \mathcal{I}_Y$ defined as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & a & b \\ 2 & 3 & \cdots & n & 1 & a & - \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n & a & b \\ a & b & 3 & 4 & \cdots & n-1 & n & 2 & 1 \end{pmatrix}.$$

In other words, α contains the cycle $(1, 2, \dots, n)$, maps a to itself and is not defined on b; whereas β contains the cycle $(1, a, 2, b)$, and maps each of the elements $3, 4, \dots, n$ to itself. We shall prove that the semigroup $\mathcal{I}_X$ is a subsemigroup of the 2-generator subsemigroup of $\mathcal{I}_Y$ generated by α and β. For $i = 1, \dots, n-1$ let the mapping $\gamma_i \in \mathcal{I}_X$ be defined as follows:

$$\gamma_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & i+1 & i & \cdots & n-1 & n \end{pmatrix}.$$

For $j = 1, \dots, n$ let the mapping $\delta_j \in \mathcal{I}_X$ be defined as follows:

$$\delta_j = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ 1 & 2 & \cdots & - & \cdots & n-1 & n \end{pmatrix}.$$

The semigroup $\mathcal{I}_X$ is obviously generated by the mappings γ_i, $i = 1, \ldots, n-1$, and δ_j, $j = 1, \ldots, n$. Therefore, in order to prove that $\mathcal{I}_X$ is a subsemigroup of the semigroup generated by α and β, it suffices to note that each of γ_i and δ_j can be factorized into a product of α and β. Indeed, it is routine to check that

$$\gamma_i = \alpha^n \beta^2 \alpha^n \beta^2 \alpha^{n-i} \beta^2 \alpha^i$$

and

$$\delta_j = \alpha^n \beta^2 \alpha^n \beta^2 \alpha^{n-j+1} \beta^3 \alpha^n \beta \alpha^{j-1}.$$

The described embedding of $\mathcal{I}_n$ into $\mathcal{I}_{n+2}$ is efficient in the sense that $\mathcal{I}_n$ is not spread all over the embedding semigroup $\mathcal{I}_{n+2}$. To see this we note that $\mathcal{I}_n = \mathcal{I}_X$ is contained in the local submonoid of $\mathcal{I}_{n+2} = \mathcal{I}_Y$ with identity e on $1, \ldots, n$. Since this submonoid is just $\mathcal{I}_n$, again we have $\mathcal{I}_n = e\mathcal{I}_{n+2}e$.

This concludes our proof. □

Corollary 2.2. *A finite inverse semigroup may be embedded in a 2-generated finite inverse semigroup.*

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