GROUPS WITH RESTRICTION ON THEIR INFINITE SUBNORMAL SUBGROUPS

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Dedicated to B. Huppert on the occasion of his sixtieth birthday

Introduction

A group G is called a B_n-group, if all its subnormal subgroups are of defect n. If all *infinite* subnormal subgroups of G have defect n, we say that G is an IB_n-group. DeGiovanni and Franciosi [2] have considered soluble IB_n groups and characterized them.

The object of this note is the class of IB_n-groups in general, in particular in comparison to B_n-groups. We are able to show, that infinite IB_n-groups are extensions of an abelian group by a B_n-group (Proposition 1), where the structure of the abelian normal subgroup can be restricted (Proposition 2), furthermore infinite IB_n-groups are B_{n+1}-groups (Corollary 3).

Changing the notation we will call B_1-groups and IB_1-groups T-groups and IT-groups respectively. Gaschütz [3] considered finite T-groups while Robinson [5] was able to classify the soluble T-groups. De Giovanni and Franciosi [1] characterized soluble IT-groups. For IT-groups we have a result somewhat dual to Proposition 1: If G is an infinite IT-group, it is an extension of a T-group by a residually finite T-group which in turn is metabelian or almost abelian of finite exponent (Proposition 3). Results for the soluble case are given separately. Finally, examples are given to show that some of the statements on IT-groups cannot be strengthened.

1. Preparations

We will make use of the following notation.

$$X^{\mathfrak{F}} = \bigcap_{\substack{N \triangleleft X \\ |X:N| \text{finite}}} N$$

$$W(x) = \langle U \, | \, U \text{ finite subnormal subgroup of } X \rangle$$

If T is a subnormal subgroup of G, we define the so-called canonical sequence

$$T_0 = G, \; T_{i+1} = T^{T_i} \quad \text{for all } i \geqq 0,$$

moreover T is of defect n if and only if $T_n = T$.

It is well known that T_k is the intersection of all subnormal subgroups of defect k in G which contain T. So

$$(A \cap B)_k \subseteq A_k \cap B_k \quad \text{for all } k.$$

The following consequence will be needed.

$$\text{If } x^{-1}Ax = A \quad \text{then} \quad x^{-1}A_k x = A_k \quad \text{for all } k \geq 0. \tag{+}$$

Looking for an abelian normal subgroup of restricted structure in IB_n-groups we require

Lemma 1. *Suppose*

  (i) *G is an infinite IB_n-group*

 (ii) *A is a subnormal subgroup of G which is not of defect n*

(iii) *$K \neq 1$ is a divisible abelian normal subgroup of G.*

Then K is a locally cyclic p-group for some prime p.

Proof. Hypotheses (i) and (ii) imply that A is finite, on the other hand, (ii) and (iii) yield that AK is an infinite subnormal subgroup of G. By (i),

$$A_n \subseteq (AK)_n = AK.$$

Since A is finite and subnormal in AK and K is divisible, we have in addition

$$K \subseteq Z(AK),$$

and, for every infinite subgroup L of K,

$$A_n \subseteq AL.$$

If L is torsionfree, A is the torsion subgroup of AL and hence characteristic in $AL = (AL)_n$. This implies $A_n = A$, a contradiction. We have obtained

$$K \text{ is a torsion group.} \tag{1}$$

Assume now that K is not a p-group. Then there are subgroups $U \simeq C_{p^\infty}$ and $V \simeq C_{q^\infty}$ of K, where p and q are distinct primes.

Now

$$A_n \subseteq AU \cap AV = A,$$

again a contradiction; this proves

$$K \text{ is a } p\text{-group for some prime } p. \tag{2}$$

If K is not locally cyclic, there is a divisible subgroup Q of rank two in K. There are elements b and c in Q such that

$$A \cap Q = \langle b \rangle x \langle c \rangle,$$

and we find subgroups B and C of Q such that

$$BC = Q \qquad\qquad B \cap C = 1$$
$$A \cap B = \langle b \rangle \quad \text{and} \quad A \cap C = \langle c \rangle.$$

Now

$$A_n \subseteq AB \cap AC = A,$$

a final contradiction showing

$$K \text{ is locally cyclic.} \tag{3}$$

Lemma 3 now follows from (2) and (3).

The next statement is probably well known, a proof is added for the convenience of the reader.

Lemma 2. *Suppose*

  (i) *R is a finite subgroup of the subgroup S of G.*

 (ii) *$\mathcal{M}$ is a set of subgroups of G such that*

    (a) *$X \cap Y$ belongs to $\mathcal{M}$ if X and Y do,*

    (b) *$S = R(S \cap X)$ for all X in $\mathcal{M}$.*

Then $S = R(S \cap \bigcap_{X \in \mathcal{M}} X)$.

Proof. The index $|S:(S \cap X)| = |R(S \cap X):(S \cap X)| = |R:R \cap X|$ is bounded by the order $|R|$ of R. So among the intersections $S \cap X$ with subgroups X from $\mathcal{M}$ there is a minimal one, $S \cap Y$, say.

For all X in $\mathcal{M}$ we have

$$S \cap Y = S \cap (Y \cap X) = (S \cap Y) \cap X$$

and

$$S \cap Y = (S \cap Y) \bigcap_{X \in \mathcal{M}} X = S \cap \bigcap_{X \in \mathcal{M}} X,$$

$$S = R(S \cap Y) = R(S \cap \bigcap_{X \in \mathcal{M}} X).$$

Corollary 1. *Suppose*

(i) *G is an infinite IB_n-group,*

(ii) *A is a subnormal subgroup of G, not of defect n,*

(iii) *L is an infinite subnormal subgroup of G which is normalized by A.*

Then $A_n \subseteq AL^{\mathfrak{G}}$.

Proof. From (i) and (ii) we deduce that A is finite. The statement $(+)$ implies that every L_i is normalized by A, so AL is an infinite subnormal subgroup of G and $A_n \subseteq AL$.

Every normal subgroup K of finite index in L contains an A-invariant normal subgroup K^* of L which is still of finite index in L. We conclude $A_n \subseteq AK^*$ and $A_n = A(A_n \cap K^*) = A(A_n \cap K)$.

We apply Lemma 2 for $R = A$ and $S = A_n$, while $\mathcal{M}$ is the set of all normal subgroups of finite index in L; now Corollary 1 follows.

2. The abelian normal subgroup

In a first step, we find an abelian normal subgroup of our IB_n-group G with a B_n-group as quotient group. Later on refinements lead to the abelian normal subgroup of restricted structure.

Proposition 1. *If G is an infinite IB_n-group and A is a subnormal subgroup of G, then $AZ(W(G))$ is subnormal of defect n.*

Proof. $AZ(W(G))$ is subnormal of the same defect as A since $Z(W(G))$ is a characteristic subgroup of G. Therefore the statement of the Proposition is nontrivial for finite subnormal subgroups only. Denote by V any finite subnormal subgroup of G. By Wielandt [6; statement (10), p. 218] the subgroup $R = \langle A, V \rangle$ is also finite and subnormal in G. Considering the canonical sequence for R, we see that there is an index i such that R_i is infinite while R_{i+1} is finite. Since R_{i+1} is normal in R_i, we conclude that $C(R_{i+1}) \cap R_i$ is an infinite subnormal subgroup and

$$A_n \subseteq A(C(R_{i+1}) \cap R_i).$$

Thus

$$A_n = A(A_n \cap C(R_{i+1})) = (A_n \cap C(R)) = A(A_n \cap C(V)).$$

If U and V are two finite subnormal subgroups of G, the subgroup $\langle U, V \rangle$ is also finite and subnormal and

$$C(U) \cap C(V) = C(\langle U, V \rangle).$$

We apply Lemma 2 to A, A_n, and the set $\mathcal{M}$ of all centralizers of finite subnormal

subgroups and find

$$A_n = A(A_n \cap C(W(G))) = A(A_n \cap Z(W(G)))$$

since $A_n \subseteq W(G)$.

Now Proposition 1 follows.

Corollary 2. *If G is an infinite IB_n-group and not a B_n-group, then $Z(W(G))$ is finite or the direct product of a finite group with a group of Prüfer type.*

Proof. Consider the set S of all elements of square-free order in $Z(W(G))$. This is a normal subgroup of G, not of defect n. Then, by Corollary 1,

$$A_n \subseteq AS^{\mathfrak{F}} = A,$$

a contradiction. We have

(1) The socle S of $Z(W(G))$ is finite.

Applying Lemma 1 to the divisible part of $Z(W(G))$ yields

(2) the divisible part of $Z(W(G))$ is trivial or a locally cyclic p-group.

Now the Corollary follows.

Corollary 3. *If G is an infinite IB_n-group and A is a subnormal subgroup of G, then*

(i) *A' is subnormal of defect n,*

(ii) *A is of defect $n+1$,*

(iii) *A is of finite index in A_n.*

Proof. All three statements are nontrivial only for those subnormal subgroups A which are not of defect n.

For this case we deduce from Proposition 1

$$A_n = A(A_n \cap Z(W(G))) = AZ(A_n).$$

Now A is normal in A_n and $A'_n = (AZ(A_n))' = A'$.

This shows (i) and (ii).

By (ii) we know that A_n is generated by some conjugates of A which are all normal in A_n. We deduce that the exponents of $A_n/(A_n)'$ and of A/A' are the same and the exponent of A_n is finite.

Now $Z(A_n)^{\mathfrak{F}} = 1$, and we deduce from Corollary 1 that $Z(A_n)$ cannot be infinite. Now (iii) follows easily.

Corollary 3(ii) generalizes results of De Giovanni and Franciosi [2; Corollary]. We can now do our final reduction.

Proposition 2. *If G is an infinite IB_n-group, there exists an abelian normal subgroup $\ddot{K}$ of G such that*

(i) *G/K is a B_n-group,*

(ii) *K is finite or a group of Prüfer type.*

Proof. We have to choose $K \neq 1$ only if G is not a B_n-group. In this case, by Proposition 1, we know that $AZ(W(G))$ is subnormal of defect n for all subnormal subgroups A of G. If $Z(W(G))$ is finite, we use $K = Z(W(G))$ for Proposition 2.

If $Z(W(G))$ is infinite, we use Corollary 1 and find that $A(Z(W(G))^{\mathfrak{G}}$ is subnormal of defect n for all subnormal subgroups A of G. We take $K = (Z(W(G)))^{\mathfrak{G}}$ since this is a group of Prüfer type by Corollary 2.

Proposition 2 should be compared with results of De Giovanni and Franciosi [1, Theorem 1.11 and 2, Theorems A and B]: In the soluble case K can be chosen of Prüfer type.

3. Normal T-subgroups of IT-groups

Before we looked for a comparatively small normal subgroup A of the IB_n-group G such that G/A is a B_n-group. In this section we want to construct another normal subgroup M which is itself a T-group and whose quotient group G/M is restricted. The form of restriction depends on the abelian normal subgroup mentioned in Proposition 2.

Proposition 3. *Assume that G is an infinite IT-group and A is an abelian normal subgroup of G such that G/A is a T-group. Then:*

(a) *There is a normal T-subgroup M of G such that G/M is a T-group and residually finite.*

(b) *If A is finite of exponent r, G/M is an extension of an abelian group of exponent r by a finite group,*

(c) *If A is cyclic or locally cyclic, G/M is metabelian.*

Proof. If S is any subnormal subgroup of G, it is either normal or contained in a finite normal subgroup of G. So in any case $N_G(S)$ is of finite index in G. We deduce that

$$M = \bigcap_{S \triangleleft \triangleleft G} N_G(S)$$

is a normal subgroup with quotient group G/M residually finite. M is a T-group by construction. Since

$$Z(W(G)) \subseteq N_G(S) \quad \text{for all } S \triangleleft \triangleleft G,$$

we have $Z(W(G)) \subseteq M$, and G/M is a T-group by Proposition 1. So we have shown (a).

For statements (b) and (c) we have to consider the quotient group

$$G \Big/ \bigcap_{g \in G} g^{-1} N_G(S) g$$

for some non-normal subnormal subgroup S of G.

Since S is subnormal and G/A is a T-group, AS is a normal subgroup of G and $AS = AS^G$. We deduce from the modular law $S^G = SK$, where $K = A \cap S^G$. Assume $r = \exp K$. We have seen earlier that $(S^G)' = S'$, using this we obtain

$$(S^G)'(S^G)^r = (S^G)'S^rK^r = S'S^r$$

and $S'S^r$ is a normal subgroup of G.

Let

$$D_S = C(A) \cap \{x \,|\, [x, y] \in S'S^r \text{ for all } y \in S^G\}$$

and

$$E_S = C(A) \cap \{x \,|\, [x, y] \in S'S^rK \text{ for all } y \in S^G = SK\}.$$

The elements of $C(A)$ induce in $SK/S'S^rK \simeq SA/S'S^rA$ only power automorphisms since G/A is a T-group.

We deduce

$$C(A)/E_S \text{ is abelian and isomorphic to a subgroup of } \mathrm{Aut}(C_r). \tag{1}$$

On the other hand, the elements of E_S stabilize the chain

$$SK \supseteq S'S^rK \supseteq S'S^r,$$

and

$$E_S/D_S \text{ is abelian of exponent } r. \tag{2}$$

Now if A is finite, $C(A)$ is of finite index in G, and we have

$$\begin{array}{c} \text{If } A \text{ is finite of exponent } r, G/D_S \text{ is an extension of} \\ \text{an abelian group of exponent } r \text{ by a finite group,} \end{array} \tag{3a}$$

If, on the other hand, A is cyclic or locally, cyclic, $G/C(A)$ is abelian, and

$$\text{If } A \text{ is cyclic or locally cyclic, } G/D_S \text{ is soluble.} \tag{3b}$$

In both cases (a) and (b) we have $C(A) \supseteq Z(FC(G))$ and therefore

$$D_S \supseteq Z(FC(G)), \tag{4}$$

so

$$D = \bigcap_{S \lhd \lhd G} D_S$$

is a normal subgroup with T-quotient group G/D. From (3a) and (3b) we obtain

> If A is finite of exponent r, G/D is an extension of
> an abelian group of exponent r by a finite group   (5a)

and

> If A is cyclic or locally cyclic, G/D is metabelian.   (5b)

Now $D_S \supseteq \bigcap_{g \in G} g^{-1} N(S) g$, and so $D \subseteq M$. This shows (a) and (b).

4. Soluble IT-groups

Closer knowledge of soluble IT-groups allows the strenthening of the results for. this special case.

Proposition 4. *If G is an infinite soluble IT-group, then G possesses a normal T-subgroup of finite index.*

Proof. If G is periodic, G is abelian by finite (see De Giovanni and Franciosi [1, Theorem 1.10]), so the statement is true.

If G is nonperiodic, $W(G)$ is an extension of a Prüfer group $A \simeq C_{p^\infty}$ by a finite abelian group (see [1, Theorem 1.11]).

All non-normal subnormal subgroups of G are subgroups of $W(G)$, and finite subnormal subgroups S of order prime to p are normal since they are characteristic subgroups of the normal subgroup AS.

The maximal p-subgroup of $W(G)$ is then generated by A and some finite p-group L of exponent, say, p^k. If A^+ is the subgroup of order p^k of A, A^+L is a characteristic subgroup of AL and hence normal in G, and $C(A^+L)$ is a normal subgroup of finite index in G. In order to show that $C(A^+L)$ is a T-group it suffices to show that subnormal subgroups K of G which are contained in AL are normal in $C(A^+L)$. Now K^{p^k} is contained in $(AL)^{p^k} = A$ and is normal in G.

Choose B in A such that $B^{p^k} = K^{p^k}$. Then $K \subseteq BL$ and $[BL, C(A^+L)] \subseteq B^{p^k} \leq K^{p^k}$. This shows that K is normalized by $C(A^+L)$ and $C(A^+L)$ is a T-group.

Proposition 5. *If G is an infinite soluble IT-group, then G possesses a finite normal subgroup U such that G/U is a T-group.*

Proof. It follows from [1, Theorem 1.10 and 1.11] that $W(G)$ is the extension of a Prüfer group A by finite group. Since $A \subseteq Z(W(G))$, and A is a subgroup of finite index, by a well-known theorem of Schur we have that $(W(G))'$ is finite, and there is a finite characteristic subgroup R of $W(G)$ such that $W(G)/R$ is a Prüfer subgroup. If X/R is a

non-normal subnormal subgroup of G/R, then X is also non-normal and subnormal in G and contained in $W(G)$. Now X/R is a subgroup of $W(G)/R$ which is a normal subgroup possessing characteristic subgroups only. So G/R is a T-group.

5. Examples

Our first example is an IT-group which is a W-group and no quotient group modulo a finite normal subgroup is a T-group; also no normal subgroup of finite index is a T-group.

Example 1. Let p be a prime. For every natural number i we choose a pair X_i, Y_i of groups such that

$$X_i' \supset Z(X_i), \ Y_i' \supset Z(Y_i),$$

$$X_i'/Z(X_i) \text{ and } Y_i'/Z(Y_i) \text{ are nonabelian simple,}$$

$$X_i/X_i' \simeq Y_i/Y_i' \simeq C_{p^{i+1}}, \ Z(X_i) \simeq Z(Y_i) \simeq C_{p^i}$$

(If $q^t - 1$ is divisible by p^{i+1}, a quotient group of a normal subgroup of $GL(p^{i+1}, p^t)$ will do for X_i and Y_i).
Let

$$X_i = \langle X_i', x_i \rangle, \quad Y_i = \langle Y_i', y_i \rangle, \quad Z(X_i) = \langle u_i \rangle, \quad Z(Y_i) = \langle v_i \rangle,$$

and consider

$$W_i = \langle s_i, t_i \mid s_i^{p^{i+1}} = t_i^{p^{i+1}} = [[s_i, t_i], t_i] = [[s_i, t_i]s_i] = 1 \rangle.$$

We have $Z(W_i) = \langle [s_i, t_i] \rangle \simeq C_{p^{i+1}}$. We form the direct product $X_i \times Y_i \times W_i$ and choose a quotient group of some subgroup, namely

$$D_i = \langle X_i', Y_i', x_i s_i, y_i t_i \rangle / \langle \mu_i v_i, u_i [s_i, t_i]^p \rangle.$$

Again D_i has a cyclic centre of order p^{i+1}, it is generated by the image d_i of $[s_i, t_i]$.
The IT-group we want to construct is

$$G = (D_1 \times D_2 \times \cdots)/\langle d_1^{-1} d_2^p, \ldots, d_i^{-1} d_{i+1}^p, \ldots \rangle$$

$G/Z(G)$ is a T-group and $Z(G)$ is a Prüfer group. If L is a finite normal subgroup of G, there is an integer m such that D_m is not mapped onto a T-group under the canonical mapping of G onto G/L. A corresponding statement holds for normal subgroups of finite index. The commutator subgroup G' is a T-group, for instance. (The groups D_i are counterexamples to Satz 5 of [3]. Its incorrectness was noticed earlier by G. Stroth, see U. Kassens [4, p. 3 and Beispiel 4.19, p. 54].)

The following is an example of an IT-group which is a FC-group such that the quotient group modulo a finite normal subgroup is a T-group.

Example 2.   The central product of a finite extra-special 2-group with infinitely many groups isomorphic to $SL(2,5)$ is an IT-group. Modulo its centre of order 2 it is a T-group. Its maximal perfect subgroup is a normal T-subgroup of finite index.

Our last example shows that the finite abelian normal subgroup B in Proposition 2 can be of any form, also the quotient group G/M mentioned in Proposition 3 can be any finite T-group.

Example 3.   Let H be a finite T-group and a subgroup of the Galois group of some normal extension F of the field Q of rational numbers. We denote by L the group of all invertible linear mappings of an F-vector space of countably infinite dimension onto itself, and by L^* the set of those mappings fixing a subspace of finite codimension elementwise.

Then $L^+ = L'L^*/L^*$ is simple modulo its centre, and $Z(L^*)$ is isomorphic to the multiplicative group of F.

Once a basis of the vector space is fixed, there is a canonical extension K of L^+ by H, using the automorphism description of every element in H. We consider H and L^+ as subgroups of K, and obtain

$$FC(K) = Z(L^+).$$

If $T \triangleleft K$ is of finite index in $FC(K)$, the quotient group K/T is an IT-group, further L^+/T is a normal T-subgroup of K/T such that

$$(K/T)(L^+/T) \simeq K/L^+ = H.$$

On the other hand, K/T possesses a finite abelian normal subgroup $W(K)/T$ such that

$$(K/T)/(W(K)/T) \simeq K/W(K)$$

is a T-group.

Close analysis of the construction method will show that even the way H operates on $W(K)/T$ can be prescribed, once the field F possesses a correspondingly big Galois group.

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