REMARKS ON THE UNIVALENCE CRITERION OF PASCU AND PASCU

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Abstract

We consider a recent work of Pascu and Pascu [‘Neighbourhoods of univalent functions’, Bull. Aust. Math. Soc. 83(2) (2011), 210–219] and rectify an error that appears in their work. In addition, we study certain analogous results for sense-preserving harmonic mappings in the unit disc $|z| < 1$. As a corollary to this result, we derive a coefficient condition for a sense-preserving harmonic mapping to be univalent in $|z| < 1$.


1. Introduction and preliminaries

The well-known Noshiro–Warschawski–Wolff criterion (see [3, page 47]) for univalency asserts the following.

Theorem A. If $f : D \to \mathbb{C}$ is analytic in a convex domain $D$ and $\Re f'(z) > 0$ for all $z \in D$, then $f$ is univalent in $D$.

As a counterpart of this result Pascu and Pascu [6] proved the following lemma.

Lemma B [6, Proposition 2.1]. Let $f : D \to \mathbb{C}$ be an analytic function in the domain $D$ and define

$$K(f, D) = \inf_{a, b \in D} \left| \frac{f(a) - f(b)}{a - b} \right|.$$

(1) If $K(f, D) > 0$, then $f$ is univalent in $D$.
(2) Conversely, if $f$ is univalent in $D$ and $\Omega \subset \overline{\Omega} \subset D$ is a domain strictly contained in $D$, then $K(f, \Omega) > 0$. 

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It is worth pointing out that the converse result, namely item (2) in Lemma B, is not necessarily true. For example, consider \( f(z) = e^z \) in the strip \( D = \{ z : -\pi < \text{Im} z < \pi \} \). It is a simple exercise to see that \( f \) is univalent in \( D \). Also let \( \Omega = \{ z : -\pi/2 < \text{Im} z < \pi/2 \} \) so that \( \Omega \subset \bar{\Omega} \subset D \) and \( \{-n : n \in \mathbb{N}\} \subset \Omega \). Moreover, since the sequence \( \{e^{-n}\} \) converges to 0, given \( \epsilon > 0 \) we can find a stage \( N \in \mathbb{N} \) such that

\[
\left| e^{-n} - e^{-m} \right| \leq |e^{-n} - e^{-m}| < \epsilon \quad \text{for all } n, m \geq N.
\]

This observation shows that

\[
K(f, \Omega) = \inf_{a, b \in \Omega} \left| \frac{e^{-a} - e^{-b}}{a - b} \right| = 0,
\]

from which we obtain that the converse part of Lemma B fails. The main mistake in the proof of part (2) of Lemma B comes from the fact that Pascu and Pascu implicitly assumed in their argument that the domain \( D \) is bounded. If this were made an explicit condition then their result would be correct.

In addition, the authors in [6] proved the following result.

**Theorem C** [6, Theorem 2.4]. Let \( f : D \to \mathbb{C} \) be a nonconstant analytic function in the convex domain \( D \). If there exists an analytic function \( g : D \to \mathbb{C} \) univalent in \( D \) such that

\[
|f'(z) - g'(z)| \leq K(g, D), \quad z \in D,
\]

then the function \( f \) is also univalent in \( D \).

As a consequence of Theorem C, they obtained the following corollary.

**Corollary D** [6, Corollary 2.6]. If \( f : D \to \mathbb{C} \) is nonconstant and analytic in the convex domain \( D \) and there exists \( c > 0 \) such that

\[
|f''(z) - c| \leq c, \quad z \in D, \quad (1.1)
\]

then \( f \) is univalent in \( D \).

Moreover, Pascu and Pascu remarked [6, Remark 2.7] that Corollary D is equivalent to Theorem A. It can easily be seen that Theorem A implies Corollary D, but again the converse is not necessarily true as the next example demonstrates.

**Example 1.1.** Let \( D \) be the right half-plane \( \{ z \in \mathbb{C} : \text{Re} z > 0 \} \) and consider the function \( f(z) = z^2 \). Then \( f'(z) = 2z \) and \( \text{Re} f'(z) > 0 \) in \( D \). Clearly, by the Noshiro–Warschawski–Wolff univalence criterion \( f \) is univalent in \( D \). On the other hand, univalency of \( f \) in \( D \) does not follow from Corollary D, because we cannot find a universal constant \( c > 0 \) satisfying (1.1). Thus the observation made by the authors in [6] about the converse of Corollary D is not true in general.

In Section 2, we extend Theorem C for sense-preserving harmonic univalent mappings and present a number of corollaries, remarks and examples.
2. Main results

A complex-valued function \( f = u + iv \) in a simply connected domain \( D \) is said to be harmonic if the real and imaginary parts of \( f \) satisfy Laplace’s equation. In \( D \), \( f \) has the canonical decomposition \( f = h + g \), where \( h \) and \( g \) are analytic in \( D \). The Jacobian \( J_f \) of \( f \) is given by

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2.
\]

We say that \( f \) is sense-preserving in \( D \) if \( J_f(z) > 0 \), for all \( z \in D \). If the Jacobian of \( f \) is nonvanishing in \( D \), then by the inverse mapping theorem it follows that \( f \) is locally univalent in \( \mathbb{D} \). For harmonic functions the converse is also true as asserted by Lewy’s theorem [5] (see also [4, page 20]). We refer to Clunie and Sheil-Small [2] and Duren [4] for many important results on harmonic univalent mappings.

In [7], the authors considered the class

\[
C_H^1 := \{ f = h + g, f(0) = f_2(0) = 1 \text{ and } f_2(0) = 0 : \text{Re } h'(z) > |g'(z)|, \, z \in \mathbb{D} \},
\]

where \( \mathbb{D} = \{ z : |z| < 1 \} \) is the open unit disc in \( \mathbb{C} \). They proved that the functions in \( C_H^1 \) are not only univalent in \( \mathbb{D} \) but also close-to-convex in \( \mathbb{D} \) (see [7, Lemma 1.1]). This result is regarded as a harmonic analogue of the Noshiro–Warschawski–Wolff criterion.

**Theorem 2.1.** Let \( f : D \to \mathbb{C} \) be a sense-preserving harmonic function in a convex domain \( D \) with the canonical decomposition \( f = h + g \). If there exists an analytic univalent function \( \phi : D \to \mathbb{C} \) such that

\[
|h'(z) - \phi'(z)| + |g'(z)| \leq K(\phi, D), \quad z \in D,
\]

then \( f \) is univalent in \( D \).

**Proof.** Assume that \( f \) is not univalent in \( D \). Then there are points \( z_1, z_2 \in D \) such that \( z_1 \neq z_2 \) and \( f(z_1) = f(z_2) \). Since \( D \) is convex, the line segment joining \( z_1 \) and \( z_2 \) lies completely in \( D \), that is, \( \{ z(t) = (1-t)z_1 + tz_2 : 0 \leq t \leq 1 \} \subset D \). An integration along this line segment, together with (2.1), yields

\[
|\phi(z_2) - \phi(z_1)| = |(f(z_2) - \phi(z_2)) - (f(z_1) - \phi(z_1))| = \left| \int_0^1 \frac{d}{dt}(f(z(t)) - \phi(z(t))) \, dt \right|
\]

\[
= \left| \int_0^1 ((h'(z(t)) - \phi'(z(t)))(z_2 - z_1) + g'(z(t))(z_2 - z_1)) \, dt \right|
\]

\[
\leq \int_0^1 (|h'(z(t)) - \phi'(z(t))| + |g'(z(t))|)|z_2 - z_1| \, dt 
\]

\[
\leq K(\phi, D)|z_2 - z_1|.
\]
Since $z_1 \neq z_2$, from the above inequality and the definition of $K(\phi, D)$, as in [6],

$$K(\phi, D) = \left| \frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1} \right|. \tag{2.2}$$

Again following the method of proof of [6], we consider the auxiliary function $P$ defined on $D \setminus \{z_2\}$ by

$$P(z) = \frac{\phi(z) - \phi(z_2)}{z - z_2}, \quad z \in D \setminus \{z_2\}. \tag{2.3}$$

As $\phi$ is analytic in $D$, it follows that $P$ is analytic in $D \setminus \{z_2\}$ and we see that the limit

$$\lim_{z \to z_2} P(z) = \lim_{z \to z_2} \frac{\phi(z) - \phi(z_2)}{z - z_2} = \phi'(z_2)$$

exists and is finite. Therefore, we can extend the function $P$ to an analytic function in $D$, which we also denote by $P$. Since

$$\inf_{z \in D} |P(z)| = \inf_{z \in D} |P(z)| = \inf_{z \not\in \{z_2\}} \left| \frac{\phi(z) - \phi(z_2)}{z - z_2} \right| \geq \inf_{a \neq b, a, b \in D} \left| \frac{\phi(a) - \phi(b)}{a - b} \right| = K(\phi, D),$$

it follows from (2.2) that

$$\inf_{z \in D} |P(z)| \geq K(\phi, D) = \left| \frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1} \right| = |P(z_1)| \geq \inf_{z \in D} |P(z)|.$$}

Thus, the minimum modulus value of $P$ in $D$ is attained at $z_1$.

Since $\phi$ is univalent in $D$, it follows that $P$ is a nonvanishing analytic function in $D$ which attains its minimum modulus value in the interior of $D$. Hence, by the minimum modulus principle for nonvanishing analytic functions, it follows that $P$ must be constant in $D$.

Thus,

$$\phi(z) = c(z - z_2) + \phi(z_2), \quad z \in D, \tag{2.4}$$

for a certain constant $c \in \mathbb{C}$. From the definition of $P$, one can easily see that $c = \phi'(z_2)e^{i\theta}$ for some $\theta \in \mathbb{R}$. From (2.3) we see that $\phi$ is a linear function and so a simple computation shows that $K(\phi, D) = |c|$ in this case.

As a consequence of the above discussion, (2.1) becomes

$$|h'(z) - c + \overline{g'(z)}| \leq |h'(z) - c| + |g'(z)| \leq |c|, \quad z \in D. \tag{2.5}$$

We need to deal with two cases.

**Case (i).** Suppose that equality holds in both the inequalities in (2.5) for a particular point, say at $z_0 \in D$. Now, by the maximum modulus principle for complex-valued harmonic functions (see [1, Corollary 1.11, page 8]),

$$h'(z) = l - \overline{g'(z)}, \quad z \in D,$$

where $l \in \mathbb{C}$. Since $h'$ is an analytic function, it follows that $g'$ is constant and so is $h'$. Further, from the sense-preserving property of $f$, we get $f(z) = az + \beta \overline{z} + \gamma$ for some $\alpha, \beta$ and $\gamma \in \mathbb{C}$ with $|\alpha| > |\beta|$. 

Case (ii). Suppose Case (i) does not happen. Now, repeating the above proof with \( \phi(z) = cz \),

\[
|cz_2 - cz_1| = |(f(z_2) - cz_2) - (f(z_1) - cz_1)|
\]

\[
= \left| \int_0^1 \frac{d}{dt}(f(z(t)) - cz(t)) \, dt \right|
\]

\[
= \left| \int_0^1 (h'(z(t)) - c)(z_2 - z_1) + e^{i\theta}g'(z(t))(z_2 - z_1) \, dt \right| \quad \text{for some } \theta \in \mathbb{R},
\]

\[
\leq \int_0^1 |h'(z(t)) - c + e^{i\theta}g'(z(t))| |z_2 - z_1| \, dt
\]

\[
< |c| |z_2 - z_1|,
\]

which is a contradiction, where in the above \( \theta = 2 \arg(z_2 - z_1) \). Indeed, if we have equality in the last inequality, then as in Case (i) it is easy to see that \( f \) is an affine mapping. This contradiction shows that the function \( f \) is univalent in \( D \).

**Remark 2.2.** The sense-preserving assumption about \( f \) cannot be removed in Theorem 2.1. For example, consider the harmonic function \( f(z) = \text{Re } z, z \in \mathbb{D} \). The Jacobian of \( f \) is zero on \( \mathbb{D} \), which shows that \( f \) is not even sense-preserving. Now take \( \phi(z) = z/2 \); then (2.1) is satisfied with \( K(\phi, \mathbb{D}) = 1/2 \) but \( f \) is not univalent in \( \mathbb{D} \).

**Remark 2.3.** The right-hand side in (2.1) cannot be replaced by a larger quantity, as can be seen by the function \( f(z) = z + az^2 \) in the unit disc \( \mathbb{D} \), where \( a \in \mathbb{D} \). For if we take \( \phi(z) = z \), then \( K(\phi, \mathbb{D}) = 1 \) and hence, using Theorem 2.1, we get that \( f \) is univalent in \( \mathbb{D} \) if \( |2az| \leq 1 \) for all \( z \in \mathbb{D} \), that is, if \( |2a| \leq 1 \). But using a direct computation, one can see that \( f \) is univalent in \( \mathbb{D} \) if and only if \( |2a| \leq 1 \). Hence inequality (2.1) in Theorem 2.1 is sharp. Here we note that if \( |2a| \leq 1 \) then \( f \in C^1_H \) and hence \( f \) is close-to-convex on \( \mathbb{D} \).

**Corollary 2.4.** Let \( f : D \to \mathbb{C} \) be a sense-preserving harmonic function in a convex domain \( D \) with the canonical decomposition \( f = h + \overline{g} \). If there exists a constant \( c > 0 \) such that

\[
|h'(z) - c| + |g'(z)| \leq c, \quad z \in D,
\]

then \( f \) is univalent in \( D \).

**Proof.** The proof follows from Theorem 2.1 by taking \( \phi(z) = cz \) with \( c > 0 \).

**Corollary 2.5.** Let \( \phi : \mathbb{D} \to \mathbb{C} \) be an analytic univalent function with Taylor series expansion

\[
\phi(z) = \sum_{n=0}^{\infty} k_nz^n, \quad z \in \mathbb{D}.
\]

Let \( f \) be a sense-preserving harmonic mapping with the canonical decomposition

\[
f(z) = \sum_{n=1}^{\infty} a_nz^n + \sum_{n=1}^{\infty} \overline{b_n}z^n, \quad z \in \mathbb{D}.
\]
If the coefficients in (2.5) satisfy
\[
\sum_{n=1}^{\infty} n|a_n - k_n| + \sum_{n=1}^{\infty} n|b_n| \leq K(\phi, D),
\] (2.6)
then \( f \) is univalent in \( D \).

**Proof.** Let \( h(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \). Then \( f = h + g \). Now
\[
|h'(z) - \phi'(z)| + |g'(z)| = \left[ \sum_{n=1}^{\infty} na_n z^{n-1} - \sum_{n=1}^{\infty} nk_n z^{n-1} \right] + \left[ \sum_{n=1}^{\infty} nb_n z^{n-1} \right]
\leq \sum_{n=1}^{\infty} n|a_n - k_n| |z|^{n-1} + \sum_{n=1}^{\infty} n|b_n| |z|^{n-1}
< \sum_{n=1}^{\infty} n|a_n - k_n| + \sum_{n=1}^{\infty} n|b_n|
\leq K(\phi, D),
\]
for all \( z \in D \). Thus, by Theorem 2.1, we conclude that \( f \) is univalent in \( D \). \(\square\)

**Example 2.6.** If we take \( \phi(z) = z \) in Corollary 2.5, then it follows easily that the harmonic function \( f(z) = z + a \bar{z}^n \) \((n \geq 2)\) is univalent in \( D \) whenever \( |a| \leq 1/n \) (as pointed out in Remark 2.3).

**Example 2.7.** Let \( \alpha \) be such that \( \alpha \in (0, 1) \) and consider the function
\[
\varphi(z) = \frac{z - \alpha}{1 - \alpha z}, \quad z \in D.
\]
It is well known that \( \varphi \) is an analytic automorphism of the unit disc and
\[
K(\varphi, D) = \inf_{a \neq b, a, b \in D} \frac{\varphi(a) - \varphi(b)}{a - b} = \inf_{a \neq b, a, b \in D} \left| \frac{1 - \alpha^2}{(1 - \alpha a)(1 - \alpha b)} \right| = \frac{1 - \alpha}{1 + \alpha}.
\]
Now we consider the harmonic function \( f(z) = \varphi(z) + g(z) \), where \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) and the coefficients of \( g \) satisfy the condition
\[
\sum_{n=1}^{\infty} n|b_n| \leq \frac{1 - \alpha}{1 + \alpha} \leq \frac{1 - \alpha}{1 + \alpha}.
\] (2.7)
We can easily see that (2.7) implies \( f \) is sense-preserving in \( D \). For
\[
|g'(z)| = \left| \sum_{n=1}^{\infty} nb_n z^{n-1} \right| \leq \sum_{n=1}^{\infty} n|b_n| \leq \frac{1 - \alpha}{1 + \alpha} \leq \frac{1 - \alpha^2}{|1 - \alpha z|^2} = |\varphi'(z)|.
\]
By Corollary 2.5, it follows that $f$ is univalent in $\mathbb{D}$. We observe that $\varphi$ is a convex function and, by (2.7), $f$ is sense-preserving. Thus, by a result of Clunie and Sheil-Small [2, Theorem 5.17], we conclude that the function $f$ in this case is close-to-convex in $\mathbb{D}$.

**Example 2.8.** For $0 < \alpha < 1$, consider the harmonic function

$$f_{a,\alpha}(z) = \frac{z - \alpha}{1 - \alpha z} + a e^{i\beta} z + \left(\frac{1 - \alpha}{1 + \alpha} - a\right) e^{i\gamma} \frac{z^2}{2}, \quad z \in \mathbb{D}$$

where $\beta, \gamma$ are real, and $0 < a < (1 - \alpha)/(1 + \alpha)$. As in Example 2.7, it can be easily seen that $f_{a,\alpha}(z)$ is sense-preserving in the unit disc $\mathbb{D}$ and a simple computation shows that (2.6) is satisfied. Thus, by Corollary 2.5, $f_{a,\alpha}(z)$ is univalent and close-to-convex in $\mathbb{D}$.

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