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ON COMPLETENESS WITH RESPECT TO THE CARATHÉODORY METRIC*

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1. **Introduction.** Let X be a bounded region in the plane. Define

$$d(x, y) = \sup_f \tfrac{1}{2} \log \left(\frac{1 + \left| \dfrac{f(x)-f(y)}{1-\overline{f(y)}f(x)} \right|}{1 - \left| \dfrac{f(x)-f(y)}{1-\overline{f(y)}\,f(x)} \right|} \right),$$

where f is an analytic function on X bounded in modulus by 1. We call d the Carathéodory metric.

In this note we give necessary and sufficient conditions to ensure that this metric be complete.

2. **Preliminaries.** Before a statement of the main result is given, we present some background material.

Endowed with the pointwise operations and the supremum norm, $H^\infty(X)$, the set of bounded analytic functions on X, becomes a Banach algebra with unit 1. Denoting by M the compact Hausdorff space of nontrivial complex homomorphisms of $H^\infty(X)$, we see that $H^\infty(X)$ is isometrically isomorphic to a (uniformly closed) subalgebra of $C(M)$. This follows in the usual manner from the equality $\|f^2\|_\infty = \|f\|_\infty^2$, valid for all $f \in H^\infty(X)$.

Gleason pointed out that an equivalence relation could be defined on the elements of M via $\phi_1 \sim \phi_2$ if and only if $\|\phi_1 - \phi_2\| < 2$, where

$$\|\phi\| = \sup_f |\phi(f)|, \qquad \begin{matrix} f \in H^\infty(X) \\ \|f\|_\infty \leq 1 \end{matrix}.$$

We will use the following notation. For $\lambda \in \bar{X}$, we set $X_\lambda = \{\phi \mid \phi \in M, \phi(z) = \lambda\}$. X_λ is called the fibre over λ. It is called a peak fibre if there is a bounded analytic function f such that $\phi(f) = 1$ for all $\phi \in X_\lambda$ while $|\phi(f)| < 1$ with $\phi \notin X_\lambda$. We denote by X^*, the union of all those fibres corresponding to (interior) points of X.

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3. The main result.

THEOREM. *Let X be a bounded region in $\mathbb{C}$ and d be the Carathéodory metric. The following are equivalent*:

(a) *X is complete with respect to d.*

(b) *X^* is an entire Gleason part.*

(c) *X_λ is a peak fibre for all boundary points λ.*

(d) *The d-closed and bounded sets are compact.*

(e) *X^* is closed with respect to $\|\quad\|$ as a subset of the maximal ideal space.*

Before we proceed to give a proof, we give some facts which will be useful. We define

$$\rho(\phi_1, \phi_2) = \sup\{|\phi_1(f)| : \|f\|_\infty \leq 1, \qquad f \in H^\infty(X)\phi_2(f) = 0\}.$$

It can be shown that ρ is a metric on M with $\rho(\phi_1, \phi_2) \leq 1$ and $\phi_1 \sim \phi_2$ if and only if $\rho(\phi_1, \phi_2) < 1$.

It is also true that d yields the relative topology that X receives as a subset of $\mathbb{C}$ with ordinary euclidean metric $|\quad|$. We should also note that $\|\phi_{x_n} - \phi_x\| \to 0$ if and only if $|x_n - x| \to 0$ where $\phi_x(f) = f(x)$, $x \in X$.

Outline of proof: Note that X^* is contained in a Gleason part. We now show $(b) \Rightarrow (a)$. Let $\{x_n\}$ be a Cauchy sequence with respect to d. Hence $\{\phi_{x_n}\}$ is a Cauchy with respect to $\|\quad\|$, the norm on the homomorphisms. However, the homomorphisms are complete with respect to this norm. Therefore, there exists a ϕ such that $\|\phi - \phi_{x_n}\| \leq 1$, $n \geq N_0$. Because X^* is the entire Gleason part, $\phi = \phi_x$ for some $x \in X$. Since $\|\phi_{x_n} - \phi_x\| \to 0$, we have $|x_n - x| \to 0$; therefore $d(x_n, x) \to 0$.

We now show $(c) \Rightarrow (b)$. Suppose $\phi \notin X^*$. Then $\phi(z) = \lambda$, λ a boundary point. By (c), there exists $f \in H^\infty(X)$ such that $|\phi(f)| = 1$, while $|\phi_x(f)| < 1$, $x \in X$. We form $g = (f - f(x))/(1 - \overline{f(x)}f)$. Since $|\phi(g)| = 1$ while $\phi_x(g) = 0$, $\rho(\phi, \phi_x) = 1$. Hence, ϕ_x is not equivalent to ϕ. Therefore, X^* is an entire Gleason part.

To show (a)$\Rightarrow$(c) we need theorem 3.5 of [1]. We suppose that there is a boundary point λ, with X_λ not a peak fibre. By theorem 3.5 [1], we can choose $\{x_n\}$ with $\rho(\phi_\lambda, \phi_{x_x}) < 1/n$ where ϕ_λ is the distinguished homomorphism [1] ($\phi_\lambda \in X_\lambda$ and its representing measure is supported on the complement of X_λ). Hence $\{\phi_{x_n}\}$ is Cauchy with respect to ρ. Therefore, $\{x_n\}$ is Cauchy with respect to d. If X were complete, then there would be $x_0 \in X$ with $d(x_n, x_0) \to 0$. This would imply $\rho(\phi_{x_n}, \phi_{x_0}) \to 0$. Hence, we would have $\phi_{x_0} = \phi_\lambda$. Therefore, $\phi_{x_0}(z) = \phi_\lambda(z)$, i.e. $x_0 = \lambda$. This is a contradiction.

We have now shown (c)$\Rightarrow$(b)$\Rightarrow$(a)$\Rightarrow$(c). Clearly (d)$\Rightarrow$(a). To show (a)$\Rightarrow$(d), we suppose X is complete. Let B be a d-closed and bounded set. Then B is relatively closed. It now suffices to show B is compactly contained in X.

If this were false, then there would be a sequence $\{x_n\}$ with $x_n \in B$ and $x_n \to \lambda$, a boundary point. By assumption, $d(x_n, x_m) \leq M$ for all n and m.

Since X is complete, X_λ is a peak fibre. Hence, there exists $f \in H^\infty(X)$ with $|f(x)|<1$, $x \in X$ and $\phi(f)=1$ $\phi \in X_\lambda$. However, if α is a cluster value of f at λ, there exists $\phi \in X_\lambda$ such that $\phi(f)=\alpha$. Therefore f has only one cluster value at λ, namely 1. Hence f has a continuous extension to $X \cup \{\lambda\}$ with $f(\lambda)=1$. By composing with a proper Möbius transformation, we may assume $f(\lambda)=1$, $|f(x)|<1$ and $f(x_1)=0$. Since $d(x_1, x_n) \geq \frac{1}{2} \log[(1+|f(x_n)|)/(1-|f(x_n)|)]$, $d(x_1, x_n) \to \infty$. This contradicts $d(x_1, x_n) \leq M$ for all n.

Finally, let $\|\phi_{x_n}-\phi\| \to 0$. If we assume X is complete, then X^* is an entire Gleason part. Therefore, $\phi \in X^*$ because $\|\phi_{x_n}-\phi\|<2$ for $n \geq N_0$. So X^* is closed. Hence, (a)$\Rightarrow$(e). We assume (e). If there is a boundary point λ with X_λ not a peak fibre, then there is a sequence $\{x_n\}$ such that $\rho(\phi_{x_n}, \phi_\lambda) \to 0$; where ϕ_λ is the distinguished homomorphism. This implies $\|\phi_{x_n}-\phi_\lambda\| \to 0$. Therefore, $\phi_\lambda \in X^*$ because X^* is closed. Hence, we have $\phi_\lambda=\phi_{x_0}$, $x_0 \in X$. As before, $\lambda=x_0$, a contradiction. So (e)$\Rightarrow$(c); Q.E.D.

Incidentally in [1], there is an analytic criterion for X_λ being a peak fibre.

X_λ is a peak fibre if and only if $\sum_{n=1}^{\infty} 2^n \gamma(E_n(\lambda)-X)=\infty$ where $E_n(\lambda)=\{z: 2^{-n-1}<|z-\lambda|<2^{-n}\}$, λ a boundary point; γ represents the analytic capacity [4].

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