ON A RELATIONSHIP BETWEEN RECORD VALUES AND ROSS'S MODEL OF ALGORITHM EFFICIENCY

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Recently Ross (1981), (1983), Chapter 4.6) has developed a simple Markov chain model for an average-case analysis of the simplex algorithm in linear programming. Characteristically, this algorithm moves through the extreme points of the feasible region in such a way that only those points are successively considered which improve the actual value of the gain function (see e.g. Hadley (1962)). If we assume the N (say) extreme points to be arranged in such a way that the first point gives the largest and the Nth point the smallest value of the gain function, then the steps of the algorithm can appropriately be described by a finite Markov chain $S_1, \ldots, S_N$ with state space \{1, $\ldots$, N\} such that

\[
\begin{align*}
P(S_1 = k) &= \frac{1}{N}, & 1 \leq k \leq N \\
\text{and } P(S_{n+1} = k \mid S_n = i) &= \frac{1}{i-1}, & 1 \leq k < i \leq N
\end{align*}
\]

with 1 being an absorbing state. For this model Ross (1981), (1983) has shown that if $T_N$ denotes the number of steps required to reach state 1 for the first time then $T_N$ is approximately (for large N) Poisson distributed over N with mean $\log N$. Here we shall demonstrate that this result can also be obtained by record value theory. In fact, if \{X_n; n \in \mathbb{N}\} is an i.i.d. sequence of random variables following a uniform distribution over \{1, $\ldots$, N\}, then \{S_n; 1 \leq n \leq N\} is identically distributed with the lower record value sequence \{X_{U_n}; 1 \leq n \leq N\} where

\[
U_1 = 1, \quad U_{n+1} = \begin{cases} \min \{k; X_k < X_{U_n}\} & \text{if } X_{U_n} > 1, \\ U_n & \text{otherwise.} \end{cases}
\]

This follows readily by arguments as in Shorrock (1972). Especially, $T_N$ is identically distributed with $T = \min \{n; X_{U_n} = 1\}$.

Unfortunately, distribution theory for records from discrete distributions is rather cumbersome; however, to obtain the asymptotic results as indicated, we can use a continuous approximation in the following way. Obviously, nothing is seriously changed if we assume the random variables \{X_n; n \in \mathbb{N}\} to be uniformly distributed over \{1/N, $\ldots$, (N-1)/N, 1\} except that now $T = \min \{n; X_{U_n} = 1/N\} = \min \{n; X_{U_n} < 2/N\}$. But for large N, we may approximately assume the $X_n$'s to be uniformly distributed over the unit interval; then $T$ is close to the stopping time $T^* = \min \{n; X_{U_n} < 2/N\}$ where now \{U_n; n \in \mathbb{N}\} is the associated record time sequence. But as is known from record value theory (see Shorrock (1972)), \{-log X_{U_n}; n \in \mathbb{N}\} forms the arrival time sequence of a unit-rate Poisson process implying that $T^*$ follows exactly a Poisson distribution with mean $\log N + 1 = \log 2$. This gives the desired result. Moreover, the above arguments suggest that for the original Markov chain \{S_1, $\ldots$, S_N\} and large

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\[ N\{-\log S_n/N; 1 \leq n \leq N\} \text{ behaves approximately as the first } N \text{ arrival times } Z_1, \ldots, Z_N \text{ of a unit rate Poisson process, or equivalently,} \]

\[
S_n \approx \text{int} (N \exp (-Z_n)) + 1, \ 1 \leq n \leq N.
\]

References


