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WEIGHTS FOR COVERING GROUPS OF SYMMETRIC AND ALTERNATING GROUPS, $p \neq 2$

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Introduction. In his fundamental paper [1] J. L. Alperin introduced the idea of a weight in modular representation theory of finite groups G. Let p be a prime. A p-subgroup R is called a radical subgroup of G if $R = O_p\big(N_G(R)\big)$. An irreducible character φ of $N_G(R)$ is called a weight character if φ is trivial on R and belongs to a p-block of defect zero of $N_G(R)/R$. The G-conjugacy class of the pair (R, φ) is a weight of G. Let b be the p-block of $N_G(R)$ containing φ, and let B be a p-block of G. A weight (R, φ) is a B-weight for the block B of G if $B = b^G$, which means that B and b correspond under the Brauer homomorphism. Alperin's conjecture on weights asserts that the number $l^*(B)$ of B-weights of a p-block B of a finite group G equals the number $l(B)$ of modular characters of B.

At present, a theoretical proof of Alperin's conjecture seems to be inaccessible. However, its truth has been proved for several classes of groups. In [2] J. L. Alperin and P. Fong have verified it for the p-blocks of the symmetric and the general linear groups, where $p \neq 2$.

It is the purpose of this article to show that for odd primes p Alperin's weight conjecture holds for the p-blocks B of the covering groups $S^+(n)$ or $A^+(n)$ of the finite symmetric groups $S(n)$ or alternating groups $A(n)$ of degree n, respectively; see Corollaries 5.3 and 5.5.

Recently, the second author [13] has determined the number $l(B)$ of modular characters of a p-block B of $S^+(n)$, $A^+(n)$, and $A(n)$. Using the methods of our joint paper [11] we construct in Section 4 all B-weights, (R, φ) of B having the same radical p-subgroup R; see Theorem 4.11. This result and a counting technique of Alperin and Fong [2] enable us in Section 5 to compute the number $l^*(B)$ of all B-weights of B, see Theorems 5.2 and 5.4. In each case it turns out that $l(B) = l^*(B)$, which is the assertion of Alperin's conjecture.

In Section 1 we restate some subsidiary and known results about irreducible modular characters of covering groups. By Alperin and Fong [2] we may assume that B is a spin block of $S^+(n)$ or $A^+(n)$ with width w. In Section 3 we reduce the conjecture to the case where B is the principal spin block of $S^+(pw)$, which has a Sylow p-subgroup X of $S^+(pw)$ as a defect group, see Reduction Theorem 3.4. Now let R be any radical subgroup of

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$S^+(pw)$ contained in X. In Section 2 the group structure of the normalizer N^+ of R in $S^+(pw)$ is determined. With these subsidiary results, the above mentioned theorems are proved in Sections 4 and 5.

Concerning our terminology and notation we refer to Feit [5], Gorenstein [6] and James and Kerber [9].

1. **Preliminaries.** Throughout this paper p is an *odd* prime. A large amount of notation and many introductory results from our paper [11] are needed here. We give a condensed version of the most important concepts in order to make this paper more self-contined and refer the reader to [11], § 1-2 for further details.

We consider the covering group $\hat{S}(n) = S^+(n)$ of the symmetric group $S(n)$ defined by the generators and relations

$$\hat{S}(n) = \left[a_1, a_2, \ldots, a_{n-1}, z \;\middle|\; \begin{array}{ll} z^2 = 1, a_i^2 = z, (a_i a_{i+1})^3 = z \\ {[a_i, a_j] = z} & \text{if } |i-j| \geq 2 \end{array} \right].$$

The other covering group of $S(n)$, which plays a minor role, is denoted by $\tilde{S}(n)$. We let π be the canonical epimorphism

$$\pi : S^+(n) \longrightarrow S(n) \text{ with kernel } \ker \pi = \langle z \rangle .$$

When H is a subgroup of $S(n)$ we define

$$H^+ = \pi^{-1}(H), \quad H^- = \pi^{-1}\big(H \cap A(n)\big).$$

Moreover $S^-(n) = A(n)^+ = A^-(n)$ is the covering group of $A(n)$. The exceptional 6-fold covers of $A(6)$ and $A(7)$ are denoted by C_6 and C_7, respectively. When $H \subseteq S(n)$ and P is a normal p-subgroup of H, then P may also be considered as normal p-subgroup of H^+. In this situation we often write H^+/P as $[H/P]^+$ for notational convenience.

$I(G)$ and $I(B)$ denote the sets of ordinary irreducible characters of the group G or of a p-block B of G, respectively. The corresponding sets of irreducible Brauer characters are denoted by $\mathrm{IBr}(G)$ and $\mathrm{IBr}(B)$. Moreover, $D_0(G)$ is the set of irreducible characters of p-defect 0 of G. When $H \subseteq S(n)$ and ε is a sign, a character of H^ε, which does not have z in its kernel, is called a *spin character* of H^ε. We let

$$\mathrm{SI}(H^\varepsilon) \subseteq I(H^\varepsilon), \quad \mathrm{SIBr}(H^\varepsilon) \subseteq \mathrm{IBr}(H^\varepsilon)$$

be the subsets of spin characters and

$$\mathrm{SD}_0(H^\varepsilon) = \mathrm{SI}(H^\varepsilon) \cap D_0(H^\varepsilon).$$

A p-block B of H^ε is called a *spin block* if $I(B) \subseteq \mathrm{SI}(H^\varepsilon)$. The *principal* spin block is the one containing the principal spin characters. Two characters $\chi, \psi \in I(H^\varepsilon)$ (or $\in \mathrm{IBr}(H^\varepsilon)$) are called *associate* if

$$\chi^{H^+} = \psi^{H^+} \ (\varepsilon = -1) \text{ or } \chi_{H^-} = \psi_{H^-} \ (\varepsilon = 1).$$

If χ has only itself as an associate character we call χ *selfassociate* (s.a.) and put $\chi^a = \chi$. Otherwise, χ is called *non-selfassociate* (n.s.a.) and we let χ^a be the unique character $\neq \chi$ which is associate to χ. Each spin character χ has a *sign,* which is given by

$$\sigma(\chi) = \begin{cases} 1 & \text{if } \chi = \chi^a \\ -1 & \text{if } \chi \neq \chi^a \end{cases}.$$

We define $\mathrm{SD}_0(H^\varepsilon)_+$ and $\mathrm{SD}_0(H^\varepsilon)_-$ to be the set of s.a. characters and the set of *pairs* of n.s.a. characters in $\mathrm{SD}_0(H^\varepsilon)$, respectively. Thus, if

$$d_0(H^\varepsilon)_\sigma = |\,\mathrm{SD}_0(H^\varepsilon)_\sigma)|$$

then

$$d_0(H^\varepsilon) = d_0(H^\varepsilon)_+ + 2d_0(H^\varepsilon)_-.$$

Since p is odd, we get easily the following

LEMMA 1.1. *If $H^+ \neq H^-$ then for any signs ε, σ*

$$d_0(H^\varepsilon)_\sigma = d_0(H^{-\varepsilon})_{-\sigma}.$$

Suppose now that the subgroups $H_1, \ldots, H_u$ of $S(n)$ operate on disjoint sets, i.e., that for all i, j, $1 \leq i \leq j \leq u$ any element of $\{1, \ldots, n\}$ is fixed by at least one of the groups H_i, H_j. Then $H_1, \ldots, H_u$ form a direct product $H = H_1 \times \cdots \times H_u$ and

$$H^+ = H_1^+ \hat{\times} \cdots \hat{\times} H_u^+,$$

where $\hat{\times}$ denotes a twisted central product defined by Humphreys [7].

LEMMA 1.2. *There is a surjective map $\hat{\otimes}$*

$$\mathrm{SI}(H_1^+) \times \cdots \times \mathrm{SI}(H_u^+) \longrightarrow \mathrm{SI}(H^+)$$
$$(\chi_1, \ldots, \chi_u) \longrightarrow \chi_1 \hat{\otimes} \cdots \hat{\otimes} \chi_u.$$

The basic properties of the map $\hat{\otimes}$ are listed in [11], Proposition 1.2.

LEMMA 1.3. *For each sign σ,*

$$d_0(H^+)_\sigma = \sum_{\{(\sigma_1, \ldots, \sigma_u)\}} d_0(H_1^+)_{\sigma_1} \cdots d_0(H_u^+)_{\sigma_u},$$

where $(\sigma_1, \ldots, \sigma_u)$ runs through all u-tuples of signs satisfying $\sigma_1\sigma_2 \cdots \sigma_u = \sigma$.

The labels of characters and blocks in $S^\varepsilon(n)$ are described in [12] and [14]. To each block B there is associated a non-negative integer $w(B)$, called the *width* of B. (In our paper [11], it was called the weight of B, but the name is changed to avoid confusion). Moreover, each block has a *core* $\gamma(B)$, which is a partition of a special type (a *p-bar core,* if B is a spin block, a p-core otherwise). We have

$$n = w(B)p + |\gamma(B)|.$$

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Furthermore, a spin block B has a *sign* $\delta(B)$ (see [11], § 1).

Let $H \subseteq S(n)$. A block B of H^ε is called *proper* if it contains s.a. and n.s.a. characters. Examples of proper blocks are spin blocks of positive defect (i.e. positive width) in $S^+(n)$ and ordinary blocks of positive defect (width) in $S^+(n)$ with a symmetric core.

Let B^* be the unique block of $H^{-\varepsilon}$ ($\neq H^\varepsilon$) covering B (when $\varepsilon = -1$) or covered by B (when $\varepsilon = 1$). We call B and B^* corresponding blocks. If B is proper we let $l_+(B)$ and $l_-(B)$ be the number of s.a. and the number of *pairs* of n.s.a. Brauer characters of B, respectively. The following result follows immediately.

LEMMA 1.4. *$l_\sigma(B) = l_{-\sigma}(B^*)$ for each sign σ.*

When λ is a partition, λ^0 denotes its conjugate (dual) partition. If $\lambda = \lambda^0$, then λ is called *symmetric*. When $r, w \in \mathbb{N}$ we let $K(r,w) = \{(\lambda_1, \ldots, \lambda_r) \mid \lambda_i$ partition and $\sum_i |\lambda_i| = w\}$ and $k(r,w) = |K(r,w)|$. If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in K(r,w)$ let $\boldsymbol{\lambda}^0 = (\lambda_r^0, \lambda_{r-1}^0, \ldots, \lambda_1^0)$. An r-tuple $\boldsymbol{\lambda}$ of partitions is called self-dual, if $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0$. The set of all such self-dual $\boldsymbol{\lambda}$ is denoted by

$$K^s(r,w) = \{\boldsymbol{\lambda} \in K(r,w) \mid \boldsymbol{\lambda} = \boldsymbol{\lambda}^0\}$$

and $k^s(r,w) = |K^s(r,w)|$.

In [13] the second author computed the number of modular characters of a p-block of the covering group of $S^\varepsilon(n)$. In particular, he showed the following two results:

PROPOSITION 1.5. *Let B be a block of $S(n)$ of width $w(B) = w > 0$ and core $\gamma(B)$.*
(1) If $\gamma(B)$ is nonsymmetric and B^* is the block of $A(n)$ covered by B, then

$$l(B) = l(B^*) = k(p-1, w).$$

(2) If $\gamma(B)$ is symmetric, then

$$l_-(B) = \frac{1}{2}[k(p-1,w) - l_+(B)],$$

where

$$l_+(B) = k^s(p-1,w) = \begin{cases} k\big((p-1)/2, w'\big) & \text{if } w = 2w' \\ 0 & \text{if } w \text{ is odd.} \end{cases}$$

PROPOSITION 1.6. *Let B be a spin block of $S^\varepsilon(n)$ of width $w(B) = w > 0$ and with sign $\delta(B) = \delta$. Then for every sign σ,*

$$l_\sigma(B) = \begin{cases} k\big((p-1)/2, w\big) & \text{if } \sigma \in \delta = (-1)^w, \\ 0 & \text{otherwise.} \end{cases}$$

2. **Normalizers of radical subgroups.** In this section the group structure of the normalizers of the radical p-subgroups in the covering $S^+(n)$ of the symmetric groups $S(n)$ is determined.

The proofs of these subsidiary results depend on the following constructions and lemmas of Alperin and Fong [2] describing the structure of the normalizers of the radical p-subgroups of $S(n)$.

Let $S(n) = S(V)$ be the symmetric group of degree n acting on a set V with $n = |V|$ elements. For each positive integer c, let A_c be an elementary abelian p-subgroup of $S(n)$ with order $|A_c| = p^c$, embedded regularly as a subgroup of $S(p^c)$. It is well known that $C_{S(p^c)}(A_c) = A_c$, and $N_{S(p^c)}(A_c)/A_c \cong \mathrm{GL}(c,p)$.

For each sequence $\tau = (c_1, c_2, \ldots, c_{s(\tau)})$ of positive integers, let $A_\tau = A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_{s(\tau)}}$, and $d(\tau) = \sum_{i=1}^{s(\tau)} c_i$. With this notation Alperin and Fong [2] have shown

LEMMA 2.1. *a) A_τ is embedded uniquely up to conjugacy as a transitive subgroup of $S\big(p^{d(\tau)}\big)$.*

(b) $N_{S(p^{d(\tau)})}(A_\tau)/A_\tau \cong \mathrm{GL}(c_1,p) \times \mathrm{GL}(c_2,p) \times \cdots \times \mathrm{GL}(c_{s(\tau)},p)$.

The group A_τ is called a *basic p-subgroup* of $S(p^{d(\tau)})$ with degree $\deg(A_\tau) = p^{d(\tau)}$ and length $l(A_\tau) = s(\tau)$.

Lemmas (2A) and (2B) of Alperin and Fong [2] are restated as

LEMMA 2.2. *Let $\mathcal{C}$ be the set of sequences $\tau = (c_1, c_2, \ldots, c_{s(\tau)})$ of positive integers. Let R be a radical p-subgroup of $G = S(n) = S(V)$. Then the following assertions hold:*

a) There exist decompositions

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_u$$
$$R = R_0 \times R_1 \times R_2 \times \cdots \times R_u$$

such that R_0 is the identity subgroup of $S(V_0)$, and for each $i \in \{1, 2, \ldots, u\}$ $R_i \neq 1$ is a basic p-subgroup A_τ of $S(V_i)$ for some sequence $\tau \in \mathcal{C}$.

b) For each $\tau \in \mathcal{C}$ let $V(\tau) = \cup_i V_i$, $R(\tau) = \prod_i R_i$, where i runs over all the indices i such that $R_i = A_\tau$. Let $\zeta(\tau)$ be the multiplicity of A_τ in $R(\tau)$. Then ζ is a function $\mathcal{C} \to \mathbb{N} \cup \{0\}$ satisfying $\sum_\tau \zeta(\tau) p^{d(\tau)} \leq n$ and the following assertions hold:

$$R = R_0 \times \prod_\tau R(\tau),$$
$$N_G(R) = S(V_0) \times \prod_\tau N_{S\left(V(\tau)\right)}\big(R(\tau)\big)$$
$$N_G(R)/R = S(V_0) \times \prod_\tau N_{S\left(V(\tau)\right)}\big(R(\tau)\big)/R(\tau).$$

ζ is called the multiplicity function of R.

c) If V_τ denotes the underlying set of A_τ in V then

$$N_{S\left(V(\tau)\right)}\big(R(\tau)\big) \cong [N_{S(V_\tau)}(A_\tau)] \wr S\big(\zeta(\tau)\big),$$
$$N_{S\left(V(\tau)\right)}\big(R(\tau)\big)/R(\tau) \cong [N_{S(V_\tau)}(A_\tau)/A_\tau] \wr S\big(\zeta(\tau)\big).$$

d) For each $r \in \mathcal{C}$ A_r is a basic p-subgroup of $S(p^{d(r)})$ with length $l(A_r) = s(r)$ and degree $\deg(A_r) = p^{d(r)}$, *and*

$$R \cong \prod_{d \geq 1} \prod_{\{r, d(r)=d\}} (A_r)^{\zeta(r)}.$$

e) The G-conjugacy class of the radical p-subgroup R is uniquely determined by the multiplicity function $\zeta : \mathcal{C} \longrightarrow \mathbb{N} \cup \{0\}$, i.e.,

$$R =_G R_\zeta = \prod_{d \geq 1} \prod_{\{r | d(r)=d\}} (A_r)^{\zeta(r)}.$$

PROOF. a) follows at once from (2A) of [2]. Assertions b) and c) are restatements of (2B) of [2]. Certainly, d) is a consequence of a). The final statement follows from d) and Lemma 2.1a).

DEFINITION. For every radical p-subgroup R with multiplicity function ζ the number

$$w(R) = \sum_{d \geq 1} \sum_{\{r | d(r)=d\}} \zeta(r) p^{d-1}$$

is called the *width* of R.

We now turn to the covering groups $S^+(n)$ of $S(n)$. The semidirect product of the groups H and N is denoted by $N \rtimes H$, where N is assumed to be normal.

LEMMA 2.3. *Let $p \neq 2$ and let c be a positive integer. Then the following assertions hold:*

a) $\mathrm{GL}(c,p) \cong \mathrm{SL}(c,p) \rtimes C$, *where*

$$C = \left\langle m = \begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \mathrm{GL}(c,p) \;\middle|\; \alpha \in \mathrm{GF}(p)^* \text{ with } O(\alpha) = p-1 \right\rangle.$$

b) m is an odd permutation of $S(p^c)$ having p^{c-1} fixed points and p^{c-1} orbits of length $p-1$.

c) $\mathrm{SL}(c,p)$ *consists of even permutations of $S(p^c)$.*

d) $\mathrm{GL}(c,p)^+ \cong \mathrm{SL}(c,p) \rtimes C^+$, *where*

$$C^+ = \begin{cases} C \times \mathbb{Z}/2\mathbb{Z} & \text{if } p \equiv 1 \pmod 8,\ p \equiv 3 \pmod 8 \text{ and } c \text{ is even,} \\ & \quad p \equiv 7 \pmod 8 \text{ and } c \text{ is odd,} \\ \mathbb{Z}/2(p-1)\mathbb{Z} & \text{otherwise.} \end{cases}$$

PROOF. a) holds trivially as $\det(m^i) \neq 1$ for $1 \leq i \leq p-1$.

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b) The matrix m operates on the GF(p)-vector space $A_c \cong \mathrm{GF}(p)^c$ by matrix multiplication. Therefore, m has p^{c-1} fixed points and $\frac{p^c - p^{c-1}}{(p-1)} = p^{c-1}$ orbits of length $(p-1)$.

c) holds because $\mathrm{SL}(c,p)$ and the alternating group $A(p^c)$ are both perfect subgroups of the symmetric group $S(p^c)$.

d) Since p is odd, the Schur multiplier of $\mathrm{SL}(c,p)$ is trivial by [4], p. XVI. Hence $\mathrm{SL}(c,p)^+ \cong \mathrm{SL}(c,p) \times \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{GL}(c,p)^+ \cong \mathrm{SL}(c,p) \rtimes C^+$. The assertions on the structure of C^+ follow from b) and Lemma 3.6 of [11].

LEMMA 2.4. *For each sequence* $\tau = (c_1, c_2, \dots, c_s)$ *of positive integers* c_i *with* $d = \sum_{i=1}^{s} c_i$ *the following assertion holds:*

$$[N_{S(p^d)}(A_\tau)]^+ / A_\tau \cong \mathrm{GL}(c_1,p)^+ \overset{\vee}{|} \mathrm{GL}(c_2,p)^+ \overset{\vee}{|} \cdots \overset{\vee}{|} \mathrm{GL}(c_s,p)^+,$$

where $\overset{\vee}{|}$ *denotes the (untwisted) central product.*

PROOF. By Lemma 2.3 $\mathrm{GL}(c_i,p) = \mathrm{SL}(c_i,p) \rtimes C_i$, where $C_i = \langle m_i \rangle$ is generated by an odd permutation m_i of $S(p^{c_i})$ having $p^{c_i - 1}$ fixed points and $p^{c_i - 1}$ orbits of length $p - 1$. Therefore,

$$(m_i)^+(m_j)^+ = (m_j)^+(m_i)^+ \text{ for } i \neq j$$

by the proof of Lemma 3.7 of [11].

Furthermore, Lemma 2.3 asserts that $\mathrm{SL}(c_i,p)$ consists of even permutations of $S(p^{c_i})$. As p is odd, $\mathrm{SL}(c_i,p)$ is generated by even permutations x_i of odd order. Now let $i \neq j$, and assume that x_i^+ and x_j^+ are preimages of odd order in $S^+(p^{c_i})$ and $S^+(p^{c_j})$, respectively, such that $[x_i^+, y_j^+] = z$. Then $(x_i^+)^{-1}(y_j^+)(x_i^+) = y_j z$ has even order, a contradiction. Thus $[x_i^+, y_j^+] = 1$ for $i \neq j$. Hence $\mathrm{GL}(c_i,p)^+ = \mathrm{SL}(c_i,p)C_i^+$ and $\mathrm{GL}(c_j,p)^+ = \mathrm{SL}(c_j,p)C_j^+$ commute elementwise for $i \neq j$. This completes the proof.

DEFINITION. For $x \in S(n)$ and any positive integer k the *k-fold diagonalization* of x in $S(nk)$ is denoted by $\Delta_k x$.

For example, if $x = (1,3,4) \in S(5)$ then

$$\Delta_3 x = (1,3,4)(6,8,9)(11,13,14) \in S(15).$$

With the notation of Lemma 2.2 and Section 1 the following subsidiary result holds.

LEMMA 2.5. *Let* $p \neq 2$. *Let* R *be a radical* p*-subgroup of* $S(n)$ *with multiplicity function* ζ. *Then*

$$S(\zeta(\tau))^+ \cong [\Delta_{p^{d(\tau)}} S(\zeta(\tau))]^+ \cong \begin{cases} \hat{S}(\zeta(\tau)) & \text{if } p^{d(\tau)} \equiv 1 \pmod 4, \\ \tilde{S}(\zeta(\tau)) & \text{if } p^{d(\tau)} \equiv 3 \pmod 4. \end{cases}$$

PROOF. Since $p^{d(\tau)}$ is odd, the result follows immediately from Lemma 3.5 of [11].

As in Section 1 the twisted Humphreys product of two or finitely many groups is denoted by $\hat{\times}$ or $\hat{\prod}$, respectively. The Humphreys product of u copies of a group U is denoted by $\hat{\prod}_u U$. With this and the notation of Lemma 2.2 we have

PROPOSITION 2.6. *Let $p \neq 2$. If R is a radical p-subgroup of the covering group $G^+ = S^+(n)$ of $S(n)$ with multiplicity function ζ, then*

a) $N_{G^+}(R) = S^+(V_0) \hat{\times} \hat{\prod_{\mathcal{T}}} [N_{S(V(\mathcal{T}))}(R(\mathcal{T}))]^+$

b) $N_{G^+}(R)/R = S^+(V_0) \hat{\times} \hat{\prod_{\mathcal{T}}} [N_{S(V(\mathcal{T}))}(R(\mathcal{T}))]^+/R(\mathcal{T})$

c) $\left[N_{S(V(\mathcal{T}))}(R(\mathcal{T}))\right]^+ \cong \left[\left[N_{S(V_{\mathcal{T}})}(A_{\mathcal{T}})\right] \wr S(\zeta(\mathcal{T}))\right]^+$

d) $\left[N_{S(V(\mathcal{T}))}(R(\mathcal{T}))\right]^+/R(\mathcal{T}) \cong \left[\left[N_{S(V_{\mathcal{T}})}(A_{\mathcal{T}})/A_{\mathcal{T}}\right] \wr S(\zeta(\mathcal{T}))\right]^+$

e) If $M_{\mathcal{T}}$ denotes the base subgroup of the wreath product $\left[N_{S(V_{\mathcal{T}})}(A_{\mathcal{T}})/A_{\mathcal{T}}\right] \wr S(\zeta(\mathcal{T}))$, then $\left[N_{S(V(\mathcal{T}))}(R(\mathcal{T}))\right]^+/R(\mathcal{T}) = M_{\mathcal{T}}^+ S^+(\zeta(\mathcal{T}))$, $M_{\mathcal{T}}^+ \cap S(\zeta(\mathcal{T}))^+ = \langle z \rangle$, $M_{\mathcal{T}}^+ \cong \hat{\prod_{\zeta(\mathcal{T})}} \left[N_{S(p^{d(\mathcal{T})})}(A)\mathcal{T}/A_{\mathcal{T}}\right]^+$,

$$\left[S(\zeta(\mathcal{T}))\right]^+ \cong \begin{cases} \hat{S}(\zeta(\mathcal{T})) & \text{if } p^{d(\mathcal{T})} \equiv 1 \pmod 4, \\ \tilde{S}(\zeta(\mathcal{T})) & \text{if } p^{d(\mathcal{T})} \equiv 3 \pmod 4. \end{cases}$$

f) If $\mathcal{T} = (c_1, c_2, \ldots, c_{s(\mathcal{T})})$ and $s = s(\mathcal{T})$, then

$$\left[N_{S(p^{d(\mathcal{T})})}(A_{\mathcal{T}})\right]^+/A_{\mathcal{T}} \cong \mathrm{GL}(c_1,p)^+ \overset{\vee}{|} \cdots \overset{\vee}{|} \mathrm{GL}(c_s,p)^+,$$

where $\overset{\vee}{|}$ denotes the (untwisted) central product.

PROOF. Assertions a), b), c) and d) follow immediately from the remarks in Section 1 and Lemma 2.2. Lemma 2.5 implies e). The final statement f) is a restatement of Lemma 2.4. This completes the proof.

3. **Reduction Theorem.** Let B be a proper block of $S^\varepsilon(n)$ of positive width. A B-weight (R, φ) is called s.a. (n.s.a.) if the character φ is s.a. (n.s.a.) as a character of $N_{S^\varepsilon(n)}(R)$. We let $l_+^*(B)$ be the number of s.a. B-weights and $l_-^*(B)$ the number of pairs of n.s.a. B-weights. In the last section Alperin's weight conjecture will be verified by showing $l_\sigma(B) = l_\sigma^*(B)$ for any sign σ.

PROPOSITION 3.1. *Let B and B^* be corresponding blocks of $S^\varepsilon(n)$ and $S^{-\varepsilon}(n)$, respectively. Let σ be a sign. Then*

$$l_\sigma^*(B) = l_{-\sigma}^*(B^*).$$

PROOF. Assume that B is a block of $S^+(n)$ and let (R, φ) be a B-weight. Lemma 2.3 and Proposition 2.6 imply $|N_{S^+(n)}(R) : N_{S^-(n)}(R)| = 2$. By a result of Blau ([11], Lemma 2.3) (R, φ^*) is a B^*-weight whenever φ^* is a constituent of the restriction of φ to $N_{S^-(n)}(R)$. Since all B^*-weights may be obtained in this way, the result follows in the case $\varepsilon = 1$. Now Lemma 1.4 completes the proof.

NOTATION. Let B be a proper spin block of $S^{+}(n)$, $w(B) = w > 0$. Let (R_{ζ}, φ) be a B-weight. Thus

$$N_{S^{+}(n)}(R) = S^{+}(V_0)\hat{\times}\hat{\prod_{r}}\big[N_{S(V(r))}\big(R(r)\big)\big]^{+}$$

in the notation of Section 2. By Lemma 1.2 we may write $\varphi = \varphi_0\hat{\otimes}\varphi_1$, where $\varphi_0 \in \mathrm{SI}\big(S^{+}(V_0)\big)$, $\varphi_1 \in \mathrm{SI}\Big(\hat{\prod_{r}}\big[N_{S(V(r))}\big(R(r)\big)\big]^{+}\Big)$.

Since φ has defect 0 as a character of $N_{S^{+}(n)}(R)/R$, Proposition 1.2(1) of [11] implies that $\varphi_0 \in \mathrm{SD}_0\big(S^{+}(V_0)\big)$. With this notation we state:

PROPOSITION 3.2. *Let B be a spin block of $S^{+}(n)$ with sign $\delta(B)$, positive width $w(B)$ and p-bar core $\gamma(B)$. Let (R, φ) be a B-weight with radical p-subgroup R of width $w(R)$. Then:*

(1) $w(B) = w(R)$

(2) φ_0 *is an irreducible defect zero spin character of* $S^{+}(V_0)$ *labelled by* $\gamma(B)$.

(3) $\sigma(\varphi_0) = \delta(B)$.

PROOF. By the general remarks in [2], Section 1, there exists a block b of $RC_{S^{+}(n)}(R)$ with R as defect group, such that $b^{G} = B$. Thus (R, b) is a self centralizing B-subpair in the sense explained in [3], Section 3.8(e). Moreover, by the proposition proved there, the core of B has to be a partition of $n - w(R)p$, which proves (1). (2) is a consequence of the description of the inclusion of subpairs given in [3], Theorem A. (3) follows from the definitions.

THEOREM 3.3 (REDUCTION THEOREM). *Let B be a spin block of $S^{\varepsilon}(n)$ of positive width w and sign $\delta(B) = \delta$. If σ is a sign then*

$$l_{\sigma}^{*}(B) = l_{\sigma}^{*}(B_0),$$

where B_0 is the principal spin block of $S^{\varepsilon\delta}(pw)$.

PROOF. Let B^{*} be the block of $S^{-\varepsilon}(n)$ corresponding to B and B_0^{*} be the block of $S^{-\varepsilon\delta}(wp)$ corresponding to B_0. By Proposition 3.1

$$l_{\sigma}^{*}(B) = l_{-\sigma}^{*}(B^{*}), \quad l_{\sigma}^{*}(B_0) = l_{-\sigma}^{*}(B_0^{*}).$$

We may therefore assume that $\varepsilon = 1$, so that B is a spin block of $S^{+}(n)$. Let (R, φ) be a B-weight. In the notation above $\varphi = \varphi_0\hat{\otimes}\varphi_1$, where φ_0 is a spin character labelled by $\gamma(B)$. Moreover, by Proposition 3.2(1) R may be considered as a radical subgroup of $S^{+}(pw)$. Thus (R, φ_1) is a weight in $S^{+}(pw)$. Since only the principal spin block B_0^{+} of $S^{+}(pw)$ has width w, (R, φ_1) is a B_0^{+}-weight. Conversely, if (R, φ_1) is a B_0^{+}-weight, then $(R, \varphi_0\hat{\otimes}\varphi_1)$ is a B-weight. Using [11], Proposition 1.2(1), we see that

$$\sigma(\varphi_0\hat{\otimes}\varphi_1) = \sigma(\varphi_0)\sigma(\varphi_1) = \delta(B)\sigma(\varphi_1).$$

If $\delta(B) = 1$, $B_0 = B_0^{+}$ and the map $(R, \varphi_1) \longrightarrow (R, \varphi_0\hat{\otimes}\varphi_1)$ induces a sign preserving bijection between the sets of the weights of B_0 and of B. If $\delta(B) = -1$, then $B_0^{*} = B_0^{+}$. If

(R,φ_1) is a s.a. B_0^+-weight then $(R,\varphi_0\hat{\otimes}\varphi_1)$ and $(R,\varphi_0^a\hat{\otimes}\varphi_1)$ is a pair of n.s.a. B-weights. If (R,φ_1) and (R,φ_1^a) is a pair of n.s.a. B_0^+-weights then $\varphi_0\hat{\otimes}\varphi_1 = \varphi_0\hat{\otimes}\varphi_1^a$ and $(R,\varphi_0\otimes\varphi_1)$ is a s.a. B-weight. This shows that $l_\sigma^*(B_0^*) = l_\sigma^*(B_0^+) = l_{-\sigma}(B)$. Since $l_\sigma^*(B_0^*) = l_{-\sigma}^*(B_0)$, the result follows in this case, too.

THEOREM 3.4. *Let $p \neq 2$. To prove the weight conjecture for all spin p-blocks of $S^\varepsilon(n)$, it suffices to do so for the principal spin p-block of $S^+(pw)$, $w \in \mathbb{N}$.*

PROOF. By Proposition 3.3 and Proposition 1.6 it suffices to prove the result for the principal spin blocks of $S^\varepsilon(wp)$, $w \in \mathbb{N}$. But the result for $\varepsilon = -1$ follows from the corresponding result for $\varepsilon = 1$ by Proposition 3.1 and Lemma 1.4.

We turn to the case of the alternating groups.

NOTATION. Let B be a block of $S(n)$ of positive width $w(B) = w > 0$. Let (R,φ) be a B-weight. As before we may write $\varphi = \varphi_0 \otimes \varphi_1$, where $\varphi_0 \in D_0\big(S(V_0)\big)$ and $\varphi_1 \in I\Big(\prod_r N_{S\left(V(r)\right)}\big(R(r)\big)\Big)$.

As already noted in [11] with this notation the following result holds.

PROPOSITION 3.5. *Let B be a block of $S(n)$ of positive width $w(B) = w > 0$. Let (R,φ) be a B-weight. Then:*

(1) $w(B) = w(R)$

(2) φ_0 is an irreducible defect zero character of $S(V_0)$ labelled by $\gamma(B)$.

THEOREM 3.6. *Let p be odd. To prove the weight conjecture for all p-blocks of $A(n)$, it suffices to do so for the principal p-block of $A(pw)$, $w \in \mathbb{N}$.*

PROOF. Let (R,φ) be a B-weight in $S(n)$, where B is a block of $S(n)$ of width $w = w(B) > 0$ covering the block B^* of $A(n)$. Write $\varphi = \varphi_0\otimes\varphi_1$ as above. As $\varphi^a = \varphi_0^a\otimes\varphi_1^a$ it follows that φ is s.a. if and only if both φ_0 and φ_1 are s.a.

Suppose first that $\gamma(B)$ is non-symmetric. Then φ_0 is n.s.a., since φ_0 is labelled by $\gamma(B)$. This means that the restriction φ^* of φ to $N_{A(n)}(R)$ is irreducible. Therefore, it is clear that the map $(R,\varphi) \longrightarrow (R,\varphi^*)$ is a bijection between the sets of B-weights and B^*-weights. Thus $l^*(B) = l^*(B^*)$. By Proposition 1.5 $l(B) = l(B^*)$. Since $l(B) = l^*(B)$ by Alperin and Fong [2] the weight conjecture is true for blocks of $A(n)$ with non symmetric core.

Suppose next that $\gamma(B)$ is symmetric. Thus φ_0 is s.a. Hence φ s.a. if and only if φ_1 is s.a. Moreover, if B_0 is the principal block of $S(wp)$ then (R,φ_1) is a B_0-weight. Using Proposition 3.5 we see that the map $(R,\varphi) \longrightarrow (R,\varphi_1)$ is a bijection between the sets of weights of B and B_0 preserving s.a. and n.s.a. weights. Thus $l_\sigma^*(B) = l_\sigma^*(B_0)$. Similarly, $l_\sigma(B) = l_\sigma(B_0)$, by Proposition 1.5(2). Now B_0 covers the principal block B_0^* of $A(wp)$. Therefore, by Proposition 3.1 and Lemma 1.4 we get $l_\sigma^*(B^*) = l_\sigma^*(B_0^*)$, $l_\sigma(B^*) = l_\sigma(B_0^*)$, which proves our claim.

4. **Construction and parametrization of the weight characters.** In this section we construct all irreducible weight characters φ having the same radical p-subgroup R with $\zeta\colon \mathcal{C} \longrightarrow \mathbb{N} \cup \{0\}$ as multiplicity function. Again let $d(r) = \sum_{i=1}^{s(r)} c_i$ for all $r = (c_1, c_2, \ldots, c_{s(r)}) \in \mathcal{C}$. By the results of Sections 1 and 2 it suffices to determine the p-blocks of defect zero in

$$N_r^+ = \left[N_{S(p^{d(r)})}(A_r)/A_r \wr S\big(\zeta(r)\big)\right]^+ \text{ for all } r \in \mathcal{C},$$

where A_r denotes a basic p-subgroup of $S^+(p^{d(r)})$ with length $s(r)$ and degree $p^{d(r)}$.

Let M_r be the base subgroup of the wreath product $N_r = N_{S(p^{d(r)})}(A_r)/A_r \wr S\big(\zeta(r)\big)$. Then by Proposition 2.6

$$N_r^+ = M_r^+ \cdot S^+\big(\zeta(r)\big), \quad M_r^+ \cap S^+\big(\zeta(r)\big) = \langle z \rangle, \text{ and}$$

$$M_r^+ = \hat{\prod_{\zeta(r)}} \left[N_{S(p^{d(r)})}(A_r)/A_r\right]^+ \lhd N_r^+,$$

where $\hat{\prod_{m}} U$ denotes the Humphreys product of m copies of the group U.

The defect zero characters θ of M_r^+ are easily determined by means of Lemmas 2.3 and 2.4. In order to find the irreducible constituents of their induced characters $\theta^{N_r^+}$ the following subsidiary results and notations are needed.

As in [7] let $\mathcal{G}$ denote the class of finite groups G^+ with central involution $z \neq 1$ and a homomorphism $s\colon G^+ \longrightarrow \mathbb{Z}/2\mathbb{Z}$ with $s(z) = 0$. Let G be the quotient group $G^+/\langle z \rangle$ and let π be the natural epimorphism $G^+ \longrightarrow G$. An irreducible representation $\rho\colon G^+ \longrightarrow \mathrm{GL}(n, F)$ is called a spin representation of G^+, if $\rho(z) = -I_n$, where $I_n \in \mathrm{GL}(n, F)$ denotes the identity matrix.

Certainly, $S^+(n) \in \mathcal{G}$, where for each $x \in S^+(n)$

$$s(x) = \begin{cases} 1 & \text{if } \pi(x) \in S(n) \text{ is an odd permutation} \\ 0 & \text{if } \pi(x) \in A(n) \end{cases}$$

In this context the homomorphism $s\colon S^+(n) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is also denoted by δ.

In [7], p. 450, Humphreys constructed for each pair of groups $G_i^+ \in \mathcal{G}$, $i = 1, 2$, a uniquely determined group $G_1^+ \hat{\times} G_2^+ \in \mathcal{G}$ with involution z.

For the sake of an easy reference the following result is stated.

LEMMA 4.1. *Let $G_i^+ = (G_i^+, s_i, z_i) \in \mathcal{G}$, $i = 1, 2$. Suppose that $G_i = \pi(G_i^+)$ has a perfect normal subgroup H_i and a cyclic subgroup C_i such that $H_i \cap C_i = 1$, and $G_i = H_iC_i$. If H_i has trivial Schur multiplicator $H^2(H_i, \mathbb{C}) = 1$ and $s_i(H_i \times \langle z_i \rangle) = 0$ for $i = 1, 2$, then $G_i^+ = H_i \rtimes C_i^+$ for $i = 1, 2$, and*

$$G_1^+ \hat{\times} G_2^+ = (H_1 \times H_2) \rtimes (C_1^+ \hat{\times} C_2^+).$$

PROOF. Consider the direct product $G_1^+ \times G_2^+$ with twisted multiplication

$$(*) \qquad (g_1, g_2)(g_1', g_2') = (z_1^{s_1(g_1')s_2(g_2)} g_1 g_1', g_2 g_2').$$

Let Z be the subgroup $\langle 1_1, 1_2), (z_1, z_2)\rangle$. Then by [7] $G_1^+ \hat{\times} G_2^+ = (G_1^+ \times G_2^+)/Z$.

Since $H_i' = H_i$ and $H^2(H_i, \mathbb{C}) = 1$ we have $G_i^+ = H_i \rtimes C_i^+$. Therfore using (*) the final assertion $G_1^+ \hat{\times} G_2^+ = (H_1 \times H_2) \rtimes (C_1^+ \hat{\times} C_2^+)$ follows.

DEFINITION [7]. If P is an irreducible spin representation of $G^+ \in \mathcal{G}$, then its associate spin representation P^a of G^+ is defined by

$$P^a(g) = (-1)^{s(g)} P(g) \text{ for every } g \in G^+.$$

P is called self associate (s.a.) if $P = P^a$, and non self associate (n.s.a.) otherwise.

Since the covering groups $\hat{S}(n)$, $\tilde{S}(n)$ of the symmetric group $S(n)$ belong to $\mathcal{G}$, this definition is easily seen to be a generalization of the corresponding one given in Section 1 for the s.a. or n.s.a. irreducible spin representations of $S^+(n)$.

DEFINITION [7]. Let M_i be a n.s.a. irreducible spin representation of $G_i^+ = (G_i^+, s_i, z_i) \in \mathcal{G}$, $i = 1, 2$. Then the spin representation $M_1 \hat{\otimes} M_2$ of $G_1^+ \hat{\times} G_2^+$ is defined by $(M_1 \hat{\otimes} M_2)(g_1 \hat{\times} g_2) = \big(M_1(g_1) + (-1)^{s_2(g_2)} M_1^a(g_1)\big) \otimes M_2(g_2)$ for all $g_i \in G_i^+$, $i = 1, 2$.

The spin representation $M_1 \hat{\otimes} M_2$ is called the Humphreys product of M_1 and M_2. It is an irreducible spin representation of $G_1^+ \hat{\times} G_2^+$ by Theorem 2.4 of [7].

LEMMA 4.2. *Suppose that the groups $G_i^+ = H_i C_i^+ \in \mathcal{G}$, $i = 1, 2$, satisfy the hypothesis of Lemma 4.1. Let θ_i be a G_i^+-stable irreducible representation of H_i, and let λ_i be a linear spin representation of G_i^+ for $i = 1, 2$. Then the following assertions hold:*

a) $P_i = \theta_i \otimes \lambda_i$ is a n.s.a. irreducible spin representation of G_i^+.

b) $P_i^a = \theta_i \otimes \lambda_i^a$.

c) $P_1 \hat{\otimes} P_2 = (\theta_1 \otimes \theta_2) \times (\lambda_1 \hat{\otimes} \lambda_2)$ is an irreducible spin representation of $G_1^+ \hat{\times} G_2^+ = (H_1 \times H_2) \rtimes (C_1^+ \hat{\times} C_2^+)$.

PROOF. As $\ker s_i$ is a proper subgroup of G_i^+, each linear character λ_i of G_i^+ is n.s.a. by Theorem 1.1 of [7]. Since $G_i^+/H_i = C_i^+$ or $G_i^+/(H_i \times \langle z \rangle) = C_i$ is cyclic, the stable irreducible representation θ_i of H_i can be extended to an irreducible representation of G_i^+ by Corollary 11.22 of Isaacs [8], p. 186. Hence a) follows from Corollary 6.17 of [8], p. 85, because $H_i \times \langle z \rangle \subseteq \ker s_i$ by hypothesis.

b) is an immediate consequence of a).

By Lemma 4.1 each $g_i \in G_i^+ = H_i \rtimes C_i^+$ has a unique representation $g_i = h_i c_i$ with $h_i \in H_i$ and $c_i \in C_i^+$, $i = 1, 2$. Since H_i is perfect, it follows that $\ker \lambda_i \geq H_i$. By a), Corollary 6.17 of [8], p. 86, and the definition of $P_1 \hat{\otimes} P_2$ the following equations holds.

$$\begin{aligned}
(P_1 \hat{\otimes} P_2)(g_1 \hat{\times} g_2) &= [P_1(g_1) + (-1)^{s_2(g_2)} P_1^a(g_1)] \otimes P_2(g_2) \\
&= [(\theta_1 \otimes \lambda_1)(h_1 c_1) + (-1)^{s_2(h_2 c_2)} (\theta_1 \otimes \lambda_1^a)(h_1 c_1)] \otimes (\theta_2 \otimes \lambda_2)(h_2 c_2) \\
&= [\theta_1(h_1)\lambda_1(c_1) + (-1)^{s_2(c_2)} \theta_1(h_1) \lambda_1^a(c_1)] \otimes \theta_2(h_2)\lambda_2(c_2) \\
&= \theta_1(h_1)[\lambda_1(c_1) + (-1)^{s_2(c_2)} \lambda_1^a(c_1)] \otimes \theta_2(h_2)\lambda_2(c_2) \\
&= \theta_1(h_1) \otimes \theta_2(h_2) \otimes [\lambda_1(c_1) + (-1)^{s_2(c_2)} \lambda_1^a(c_1)] \otimes \lambda_2(c_2) \\
&= [(\theta_1 \otimes \theta_2)(h_1 \times h_2)] \otimes [(\lambda_1 \hat{\otimes} \lambda_2)(c_1 \hat{\times} c_2)],
\end{aligned}$$

because λ_1 and λ_1^a are linear characters. Now Lemma 4.1 completes the proof.

The following subsidiary result is proved in our paper [11]. In order to restate it the following notation is needed.

For every positive integer t let $s = \left[\frac{t}{2}\right]$. In [14], p. 450, I. Schur constructed t complex $2^s \times 2^s$ matrices F_i, $1 \leq i \leq t$ satisfying the following relations

$$F_i^2 = E, \; F_iF_j = -F_jF_i \text{ for } i \neq j, \tag{4.3}$$

where E denotes the $2^s \times 2^s$ identity matrix.

With these matrices F_i we constructed in [11] a selfassociate spin representation D of the covering group S_t^+ with degree 2^s as follows.

LEMMA 4.4. *Let* $D_i = (-1)^{t-i-1}\sqrt{-\frac{1}{2}}(F_{t-i} + F_{t-i+1})$ *for* $1 \leq i \leq t-1$. *Let* $D\colon S_t^+ \longrightarrow \mathrm{GL}(2^s, \mathbb{C})$ *be defined by*

$$D(a_i) = \begin{cases} D_i & \text{if } S_t^+ = \hat{S}_t \\ \sqrt{-1}D_i & \text{if } S_t^+ = \tilde{S}_t \end{cases} \text{ for } 1 \leq i \leq t-1.$$
$$D(z) = -E \text{ in each case,}$$

where $\pi(a_i) = (i, i+1) \in S(t)$. *Then D is a s.a. spin representation of the covering group* S_t^+ *of the symmetric group* $S(t)$ *with degree* 2^s. *If t is odd, then D is the principal spin representation of* S_t^+, *and if t is even, then D is the direct sum of the principal spin representation and its associate representation.*

PROOF. See Lemma 4.2 of [11].

LEMMA 4.5. *Let* $\mathfrak{r} = (c_1, c_2, \ldots, c_s) \in \mathcal{C}$ *be a sequence of positive integers. Let* $A_{\mathfrak{r}}$ *be a basic p-subgroup of* $S(p^d)$ *of length s and degree* p^d. *Let* $S_t^+ = \{a_1, a_2, \ldots, a_{t-1}, z\}$ *be a covering group of* $S(t)$, *where* $\pi(a_i) = (i, i+1) \in S(t)$. *Then:*

a) *Each element* $u \in N_{\mathfrak{r},t}^+ = [(N_{S(p^d)}(A_{\mathfrak{r}})/A_{\mathfrak{r}} \wr S(t)]^+$ *can be represented by a* $(t+1)$*-tuple* $\mu = (x_1, x_2, \ldots, x_t, a)$, *where* $x_i \in [N_{S(p^d)}(A_{\mathfrak{r}})/A_{\mathfrak{r}}]^+$, $a \in S_t^+$ *and* $(x_1, x_2, \ldots, x_t) \in M_{\mathfrak{r},t}^+$, *where* $M_{\mathfrak{r},t}$ *is the base subgroup of* $N_{\mathfrak{r},t}$.

b) *The multiplication of the group* $N_{\mathfrak{r},t}^+$ *is given by*

$$(x_1, x_2, \ldots, x_t, a_i)(y_1, y_2, \ldots, y_t, a) = (x_1y_1^*, \ldots, x_ty_t^*, a_ia)z^e,$$

where

$$y_j^* = \begin{cases} y_j & \text{if } j \neq i, i+1 \\ y_{i+1} & \text{if } j = i \\ y_i & \text{if } j = i+1 \end{cases}$$

and $e = \sum_{1 \leq j < k \leq t} d(y_j^*)\delta(x_k) + \sum_{\substack{s \notin \{i,i+1\} \\ 1 \leq s \leq t}} \delta(y_s^*) + \delta(y_i^*)\delta(y_{i+1}^*)$.

PROOF. See Lemma 3.10 of [11].

LEMMA 4.6. *Let $r = (c_1, c_2, \ldots, c_s) \in C$ be a sequence of positive integers. Let A_r be a basic p-subgroup of $S(p^{d(r)})$ of length $s(r) = s$ and degree $p^d = p^{d(r)}$. For every positive integer t let $M_{r,t}$ be the base subgroup of the wreath product*

$$N_{r,t} = N_{S(p^d)}(A_r)/A_r \wr S(t) = M_{r,t} \rtimes S(t).$$

Then the following assertions hold:

a) $[N_{S(p^d)}(A_r)/A_r]^+ = \mathrm{GL}(c_1,p)^+ \overset{\vee}{|} \mathrm{GL}(c_2,p)^+ \overset{\vee}{|} \cdots \overset{\vee}{|} \mathrm{GL}(c_s,p)^+$

b) $\mathrm{GL}(c_i,p)^+ = \mathrm{SL}(c_i,p) \rtimes C_i^+$, $1 \leq i \leq s$, *where C_i is a cyclic group of order* $p-1$.

c) *Each irreducible defect zero spin representation θ of $[N_{S(p^d)}(A_r)/A_r]^+$ is of the form*
$$\theta = \bigotimes_{i=1}^{s} (\mathrm{St}_i \otimes \lambda_i) = (\bigotimes_{i=1}^{s} \mathrm{St}_i) \otimes \lambda,$$
where St_i denotes the Steinberg representation of $\mathrm{SL}(c_i,p)$, λ_i is a n.s.a. linear spin representation of C_i^+, and $\lambda = \otimes_{i=1}^{s} \lambda_i$.

d) *$[N_{S(p^d)}(A_r)/A_r]^+$ has $e(r) = \frac{1}{2}(p-1)^s$ pairs of n.s.a. irreducible defect zero spin representations θ, and $d_0\big([N_{S(p^d)}(A_r)/A_r]^+\big)_+ = 0$.*

e) *Each $N_{r,t}^+$-stable irreducible defect zero spin representation of $M_{r,t}^+$ is the t-fold Humphreys power $\hat{\otimes}_t \theta$ of a n.s.a. irreducible defect zero representation θ of $[N_{S(p^d)}(A_r)/A_r]^+$.*

PROOF. a) holds by Lemma 2.4.

b) is a restatement of Lemma 2.3d).

c) By Steinberg's tensor product theorem each irreducible defect zero representation θ of $\mathrm{GL}(c_i,p)^+$ is of the form $\theta = \mathrm{St}_i \otimes \lambda_i$, where St_i denotes the Steinberg representation of $\mathrm{SL}(c_i,p)$, and λ_i is a linear representation of $\mathrm{GL}(c_i,p)^+$. From Lemma 2.3 follows that θ is a spin representation if and only if λ_i is a spin representation. Thus c) holds.

d) Now Lemma 4.2 asserts that $\mathrm{GL}(c_i,p)^+$ has $\frac{1}{2}(p-1)$ pairs of n.s.a. irreducible defect zero spin representations, each of which is of the form $\mathrm{St}_i \otimes \lambda_i$, where $\lambda_i \neq \lambda_i^a$. Since the center of $\mathrm{GL}(c_i,p)$ is in the kernel of St_i, it follows from a) that $[N_{S(p^d)}(A_r)/A_r]^+$ has $(p-1)^s$ irreducible defect zero spin representations θ, which are pairwise n.s.a. Thus $e(r) = \frac{1}{2}(p-1)^s$, and $d_0\big([N_{S(p^d)}(A_r)/A_r]^+\big)_+ = 0$.

e) By Proposition 2.6, $M_{r,t}^+$ is the t-fold Humphreys product
$$M_{r,t}^+ = \hat{\prod_t}[N_{S(p^d)}(A_r)/A_r]^+.$$
Therefore, Propositions 1.2 and 1.5 of [11] assert that each irreducible defect zero spin character μ of $M_{r,t}^+$ is of the form $\mu = \theta_1 \hat{\otimes} \theta_2 \hat{\otimes} \cdots \hat{\otimes} \theta_t$, where each θ_i is an irreducible defect zero spin character of $N_{S(p^d)}(A_r)/A_r$. Let $a_i \in S_t^+$ map onto the

transposition $\pi(a_i) = (i, i+1) \in S(t)$. Then by Lemma 3.11 of [11] S_t^+ operates on μ via

$$\mu^{a_i} = \theta_1^a \hat{\otimes} \theta_2^a \hat{\otimes} \cdots \hat{\otimes} \theta_{i-1}^a \hat{\otimes} \theta_{i+1} \hat{\otimes} \theta_i \hat{\otimes} \theta_{i+2}^a \hat{\otimes} \cdots \hat{\otimes} \theta_t^a.$$

Hence d) and Proposition 1.2 imply that μ is stable in $N_{r,t}^+$ if and only if $\theta_i = \theta$ for all $1 \leq i \leq t$. This completes the proof.

With the notation of (4.3) and of the previous lemmas we can now state

LEMMA 4.7. *Let $r = (c_1, c_2, \ldots, c_s) \in C$ be a sequence of positive integers. Let A_r be a basic p-subgroup of $S(p^d)$ of length s and degree p^d. Let $S_t^+ = \langle a_1, a_2, \ldots, a_{t-1}, z\rangle$ be a covering group of $S(t)$, where $\pi(a_i) = (i, i+1) \in S(t)$. Let $N_{r,t}^+ = [N_{S(p^d)}(A_r)/A_r \wr S(t)]^+ = M_{r,t}^+ S_t^+$, where $M_{r,t}$ denotes the base subgroup of the wreath product.*

Suppose that $\theta = (\otimes_{i=1}^s \mathrm{St}_i) \otimes \lambda$ is a n.s.a. irreducible defect zero spin representation of $[N_{S(p^d)}(A_r)/A_r]^+$. For every $(x_1, x_2, \ldots, x_t) \in M_{r,t}^+$ and every $a \in S_t^+$ let

$$D_\theta(x_1, x_2, \ldots, x_t, a) = \otimes_t(\bigotimes_{i=1}^s \mathrm{St}_i)(x_1, x_2, \ldots, x_t) \otimes \prod_{i=1}^t \lambda(x_i) F_t^{\delta(x_1)} \ldots F_1^{\delta(x_t)} D(a),$$

where $D\colon S_t^+ \longrightarrow \mathrm{GL}(2^{[\frac{t}{2}]}, \mathbb{C})$ is the spin representation of S_t^+ defined in Lemma 4.4, and where $\otimes_t\, \mu$ denotes the t-fold tensor power of the representation μ.

Then the following assertions hold:

a) D_θ is an irreducible spin representation of $N_{r,t}^+$ extending the t-fold Humphreys power $\hat{\otimes}_t \theta \in \mathrm{Irr}_{\mathbb{C}}(M_{r,t}^+)$ of θ.

b) If t is even, then D_θ is s.a.

c) If t is odd, then D_θ is n.s.a.

PROOF. By Lemma 4.6a) and b)

$$[N_{S(p^d)}(A_r)/A_r]^+ = [\mathrm{GL}(c_1, p) \overset{\vee}{|} \cdots \overset{\vee}{|} \mathrm{GL}(c_s, p)]^+,$$

and $\mathrm{GL}(c_i, p)^+ = \mathrm{SL}(c_i, p) \rtimes C_i^+$, $1 \leq i \leq s$, where C_i is a cyclic group of order $p-1$. Thus Lemma 4.2 implies that the t-fold Humphreys power

$$\begin{aligned} \hat{\otimes}_t \theta &= \hat{\otimes}_t[(\bigotimes_{i=1}^s \mathrm{St}_i) \otimes \lambda] = \otimes_t(\bigotimes_{i=1}^s \mathrm{St}_i) \otimes (\hat{\otimes}_t \lambda) \\ &\in \mathrm{SI}\big(\prod_t [\overset{\vee s}{|}_{i=1} \mathrm{SL}(c_i, p)] \rtimes \overset{\wedge}{\prod_t} [\overset{\vee s}{|}_{i=1} C_i^+]\big), \text{ and} \\ \hat{\otimes}_t \lambda &\in \mathrm{SI}(\overset{\wedge}{\prod_t}[\overset{\vee s}{|}_{i=1} C_i^+]). \end{aligned}$$

Furthermore, it is S_t^+-stable. Since λ is a n.s.a. linear representation of $\overset{\vee s}{|}_{i=1} C_i^+$, it follows from Lemma 4.6 and the proof of Lemma 4.3 of [11] that

$$D_\theta(x_1, x_2, \ldots, x_t, a) = \bigotimes_{j=1}^t \Big[\bigotimes_{j=1}^s \mathrm{St}_i\Big](x_j) \otimes \prod_{j=1}^t \lambda(x_j) F_t^{\delta(x_1)} \ldots F_1^{\delta(x_t)} D(a)$$

defines an irreducible spin representation of $N_{r,t}^+$ such that its restriction $D_{\theta|M_{r,t}^+} = \hat{\otimes}_t[\otimes_{i=1}^s(\mathrm{St}_i) \otimes \lambda]$. The remaining assertions b) and c) also follow from Lemma 4.3b) and c) of [11].

PROPOSITION 4.8. *Let $r = (c_1, c_2, \ldots, c_s) \in C$ be a sequence of positive integers. Let A_r be a basic p-subgroup of $S(p^d)$ of length s and degree p^d. Let $N^+_{r,t} = [N_{S(p^d)}(A_r)/A_r) \wr S(t)]^+ = M^+_{r,t} \cdot S^+_t$, where $M_{r,t}$ denotes the base subgroups of the wreath product. Then the following assertions hold:*

a) Each $N^+_{r,t}$-stable irreducible defect zero spin representation φ of $M_{r,t}$ is of the form $\varphi = \hat{\otimes}_t\theta$, where θ is an irreducible defect zero spin representation of $N_{S(p^d)}(A_r)/A_r$.

b) Each $N^+_{r,t}$-stable irreducible defect zero spin representation $\varphi = \hat{\otimes}_t\theta$ of $M^+_{r,t}$ can be extended to an irreducible spin representation D_θ of $N^+_{r,t}$, and every irreducible defect zero constituent V of $\varphi^{N^+_{r,t}}$ is of the form $V = D_\theta \otimes T$, where T is an irreducible defect zero representation of $N^+_{r,t}/M^+_{r,t} \cong S(t)$.

c) If t is odd then every irreducible constituent V of $\varphi^{N^+_{r,t}}$ is n.s.a.

d) If t is even then every irreducible constituent V of $\varphi^{N^+_{r,t}}$ is s.a.

PROOF. a) is a restatement of Lemma 4.6e). b) The existence of the extension D_θ of $\varphi = \hat{\otimes}_t$ is guaranteed by Lemma 4.7. Therefore, Corollary 6.17 of Isaacs [8], p. 85, asserts that every irreducible constituent V of $\varphi^{N^+_{r,t}}$ is of the form $V = D_\theta \otimes T$, where T is an irreducible representation of $N^+_{r,t}/M_{r\cdot t} \cong S(t)$. Now V belongs to a p-block of defect zero if and only if T does. Thus b) holds.

Assertions c) and d) follow from Proposition 4.4, a) and b) of [11], respectively.

LEMMA 4.9. *Let B be the principal spin block of $G^+ = S^+(wp)$. Let R be a radical p-subgroup of G^+ with width $w(R) = w$. If (R, φ) is a B-weight, then the irreducible defect zero spin character φ of $N_{G^+}(R)/R$ has sign $\sigma(\varphi) = (-1)^w$.*

PROOF. By Lemma 2.2c) the function ζ is uniquely determined by the radical subgroup R of G^+. Now Proposition 3.2 asserts that

$$w = w(B) = w(R) = \sum_{d\geq 1}\sum_{r\in C_d} \zeta(r)p^{d-1}, \text{ where } C_d = \{r \in C \mid d(r) = d\}.$$

Hence

$$(*) \qquad w \equiv \sum_{d\geq 1}\sum_{r\in C_d} \zeta(r) \bmod 2$$

because p is odd.

Furthermore, Lemma 2.2e) asserts that

$$R = \prod_{d\geq 1}\prod_{r\in C_d} (A_r)^{\zeta(r)}.$$

Now Proposition 2.6 implies that

$$N_{G^+}(R)/R = \hat{\prod_{d\geq 1}}\hat{\prod_{r\in C_d}} \left[\left(N_{S(p^d)}(A_r)/A_r\right) \wr S(\zeta(r))\right]^+.$$

Hence $\varphi \in \mathrm{SD}_0\left(N_{G^+}(R)/R\right)$ factors as

$$\varphi = \hat{\otimes}_{d\geq 1}[\hat{\otimes}_{r\in C_d}\varphi_r],$$

where φ_r is an irreducible defect zero spin character of the group $\big[\big(N_{S(p^d)}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)\big]^+$.

The spin character φ_r has sign $\sigma(\varphi_r) = (-1)^{\zeta(r)}$ by assertions c) and d) of Proposition 4.8. Hence

$$\sigma(\varphi) = (-1)^u, \text{ where } u = \sum_{d\geq 1}\sum_{r\in C_d} \zeta(r).$$

From (∗) follows that $u \equiv w \bmod 2$. Hence $\sigma(\varphi) = (-1)^w$. This completes the proof.

PROPOSITION 4.10. *Let R be a radical p-subgroup of $G^+ = S^+(wp)$ with width $w(R) = \sum_{d\geq 1}\sum_{r\in C_d}\zeta(r)p^{d-1}$, where $C_d = \{\, r \in C \mid d(r) = d\}$. For each sequence $r = (c_1, c_2, \ldots, c_{s(r)}) \in C$ let $X(r)$ be the set of $\frac{1}{2}(p-1)^{s(r)}$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})$ of p-core partitions κ_i such that $\sum_{i=1}|\kappa_i| = \zeta(r)$, where $e(r) = \frac{1}{2}(p-1)^{s(r)}$. Let $N_r = \big(N_{S(p^{d(r)})}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)$. Then for each $r \in C$ there is a bijection between the sets $X(r)$ and $\mathrm{SD}_0(N_r^+)_\sigma$, where $\sigma = (-1)^{\zeta(r)}$. Furthermore, $d_0(N_r^+)_{-\sigma} = 0$.*

PROOF. Fix $r \in C$. Let $s = s(r)$, $d = d(r)$, $t = \zeta(r)$ and $e = e(r) = \frac{1}{2}(p-1)^{s(r)}$. Then $N_r = \big(N_{S(p^d)}(A_r)/A_r\big) \wr S(t) = M_r \rtimes S(t)$, where M_r is the base subgroup of the wreath product.

By Lemma 4.6 $\big[N_{S(p^d)}\big((A_r)/A_r\big)\big]^+$ has e pairs of n.s.a. irreducible defect zero spin representations θ, and $d_0\big(\big[N_{S(p^d)}(A_r)/A_r\big]^+\big)_+ = 0$. Then the representatives of these pairwise non associated characters θ can be denoted by $\theta_1, \theta_2, \ldots, \theta_e$.

Let $\mathrm{SD}_0(M_r^+)$ be the set of irreducible defect zero spin representations φ of M_r^+. In order to parametrize the N_r^+-orbits of $\mathrm{SD}_0(M_r^+)$ we consider the following set

$$\mathcal{A} = \Big\{(t_1, t_2, \ldots, t_e) \in \mathbb{N}^e \;\Big|\; \sum_{i=1}^{e} t_i = t\Big\}.$$

For each e-tuple $a = (t_1, t_2, \ldots, t_e) \in \mathcal{A}$ there is an irreducible defect zero spin representation of M_r^+ of the form $\theta_a = \mu_1 \hat{\otimes} \mu_2 \hat{\otimes} \cdots \hat{\otimes} \mu_e$, where each μ_i is a t_i-fold Humphreys power $\mu = \hat{\otimes}_{t_i}\theta_i$ of the irreducible defect zero spin representation θ_i of $[N_{S(p^d)}(A_r)/A_r]^+$. Using now Theorem 2.4 and Proposition 3.3 of Humphreys [7] and Lemma 3.11 of [11] it follows that $\mathbb{W} = \{\, \theta_a \mid a \in \mathcal{A}\}$ is a complete set of representatives of the N_r^+-orbits of $\mathrm{SD}_0(M_r^+)$.

For each $a \in \mathcal{A}$ let T_a be the inertial subgroup of θ_a in $N_r^+ = M_r^+ \cdot S^+(t)$. Then Lemma 4.6e) implies

$$T_a/M_r^+ \cong S(t_1) \times S(t_2) \times \cdots \times S(t_e)$$

Therefore Proposition 4.8b) and Clifford's theorem, see Theorem 7.16 of [10], imply that every irreducible defect zero spin representation χ_a of T_a is of the form $\theta_a \otimes \gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_e$, where γ_i is an irreducible defect zero representation of the symmetric group $S(t_i)$. Now the theorem of R. Brauer and G. de B. Robinson called the Nakayama Conjecture, see James and Kerber [9], p. 245, asserts that each such representation γ_i corresponds uniquely to a p-core partition κ_i of $t_i = |\kappa_i|$.

Furthermore, the sign of χ_a is

$$\sigma(\chi_a) = \sigma(\theta_a) = \prod_{i=1}^{e} \sigma(\mu_i) = (-1)^{\sum_{i=1}^{e} t_i} = (-1)^t = (-1)^{\zeta(r)}$$

for each $a \in \mathcal{A}$. Hence it follows that there is a bijection between $\mathcal{X}(r)$ and $\mathrm{SD}_0(N_r^+)_\sigma$, where $\sigma = (-1)^t = (-1)^{\zeta(r)}$. By Proposition 4.8c) and d) $d_0(N_r^+)_{-\sigma} = 0$. This completes the proof.

After all these preparations we now can show the main result of this section. Together with the Reduction Theorem 3.3 it gives for any spin block B of $S^+(n)$ with positive width w the number of B-weights (R, φ) having the same radical p-subgroup R.

THEOREM 4.11. *Let B be the principal spin block of $G^+ = S^+(wp)$. Let R be a radical p-subgroup of G^+ with multiplicity function ζ. Then the number of B-weights (R, φ) with radical subgroup R is given by:*

a) $d_0\big(N_{G^+}(R)/R\big) = d_0\big(N_{G^+}(R)/R\big)_+ + 2d_0\big(N_{G^+}(R)/R\big)_-$

b) For any sign σ

$$d_0\big(N_{G^+}(R)/R\big)_\sigma = \begin{cases} \prod_{r\in C} d_0(N_r^+)_{\sigma(r)} & \text{if } \sigma = (-1)^w \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma(r) = (-1)^{\zeta(r)}$ for every $r \in C$.

c) For each $r \in C$ $d_0(N_r^+)_{\sigma(r)}$ equals the number of $e(r)$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})$ of p-core partitions κ_i such that $\sum|\kappa_i| = \zeta(r)$, where $e(r) = \frac{1}{2}(p-1)^{s(r)}$.

PROOF. a) follows immediately from Section 1.

b) Proposition 2.6 asserts that

$$N_{G^+}(R)/R = \hat{\prod_{r\in C}}\Big[\big(N_{S(p^{d(r)})}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)\Big]^+$$

Therefore Lemmas 1.2 and 1.3 yield that each $\varphi \in \mathrm{SD}_0\big(N_{G^+}(R)/R\big)_\sigma$ is a Humphreys product of the form $\varphi = \hat{\bigotimes}_{r\in C}\varphi_r$, where for each $r \in C$

$$\varphi_r \in \mathrm{SD}_0\Big(\Big[N_{S(p^{d(r)})}(A_r)/A_r\Big) \wr S\big(\zeta(r)\big)\Big]^+\Big)_{\sigma(r)}$$

By Proposition 4.10 $\sigma(r) = (-1)^{\zeta(r)}$ and $d_0(N_r^+)_{-\sigma} = 0$. Furthermore, $\sigma(\varphi) = (-1)^w$ by Lemma 4.9. Since $\sigma(\varphi) = \prod_{r\in C}\sigma(r)$, Lemma 1.3 completes the proof of b).

c) is a consequence of Proposition 4.10. This completes the proof.

5. **Proof of Alperin's weight conjecture for $S^+(n)$ and $A^+(n)$.** In this section the number $l^*(B)$ of all B-weights of a p-block B of the covering groups $S^\varepsilon(n)$ of the symmetric and alternating groups is determined, where $p \neq 2$. In each case, it turns out that $l(B) = l^*(B)$, which verifies Alperin's weight conjecture for these groups.

LEMMA 5.1. *Let $\mathcal{C}$ be the set of all sequences $r = (c_1, c_2, \ldots, c_{s(r)})$ of positive integers c_i. Let $d(r) = \sum_{i=1}^{s(r)} c_i$ for each $r \in \mathcal{C}$, and for every natural number $d > 0$ let $\mathcal{C}_d = \{ r \in \mathcal{C} \mid d(r) = d\}$. Then $\sum_{r \in \mathcal{C}_d} (p-1)^{s(r)} = (p-1)p^{d-1}$.*

PROOF. By Alperin and Fong [2] there are $\binom{d(r)-1}{s(r)-1}$ basic subgroups A_r of degree $p^{d(r)}$ and length $l(A_r) = s(r)$.

Hence

$$\sum_{r \in \mathcal{C}_d} (p-1)^{s(r)} = \sum_{t \geq 1} \binom{d-1}{t-1} (p-1)^t$$
$$= (p-1) \sum_{t \geq 1} \binom{d-1}{t-1} (p-1)^{t-1}$$
$$= (p-1)[(p-1)+1]^{d-1} = (p-1)p^{d-1}$$

With the notation of Section 1 we now can state the main result of this paper.

THEOREM 5.2. *Let B be a spin block of $S^\varepsilon(n)$ with width $w(B) = w > 0$ and sign $\delta(B) = \delta$. Then for every sign σ the number $l_\sigma^*(B)$ of B-weights with sign σ is*

$$l_\sigma^*(B) = \begin{cases} k\big(\frac{1}{2}(p-1), w\big) & \textit{if } \sigma = \delta = (-1)^w \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, $l_\sigma(B) = l_\sigma^*(B)$ for each sign σ.

PROOF. We keep the notations of Lemma 5.1 and Theorem 4.11. By the Reduction Theorem 3.3 $l_\sigma^*(B) = l_\sigma^*(B_0)$, where B_0 is the principal spin block of $S^{\varepsilon\delta}(pw)$. Furthermore, Theorem 3.4 asserts that we may assume that $\varepsilon\delta = 1$, i.e., that B_0 is the principal spin block of $S^+(pw)$. By Lemma 2.2 for each B_0-weight (R, φ) of $G^+ = S^+(pw)$ there is a uniquely determined multiplicity function $\zeta : \mathcal{C} \longrightarrow \mathbb{N} \cup \{0\}$ such that the radical p-subgroup R has width $w(R) = \sum_{d \geq 1} \sum_{r \in \mathcal{C}_d} \zeta(r) p^{d-1}$. Furthermore, Proposition 3.2 asserts that $w(R) = w$. Now $N_{G^+}(R)/R = \hat{\prod}_{r \in \mathcal{C}} \Big[\big(N_{S(p^{d(r)})}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)\Big]^+$ by Proposition 2.6. Therefore the spin character φ of $N_{G^+}(R)/R$ has the Humphreys product decomposition $\varphi = \hat{\otimes}_{r \in \mathcal{C}} \varphi_r$, where $\varphi_r \in \mathrm{SD}_0\Big[\big(N_{S(p^{d(r)})}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)\Big]^+$ by Lemma 1.3. For each $d \geq 1$ let $\varphi_d = \hat{\otimes}_{r \in \mathcal{C}_d} \varphi_r$. Then $\varphi = \hat{\otimes}_{d \geq 1} \varphi_d$. For each $r \in \mathcal{C}$ let $e(r) = \frac{1}{2}(p-1)^{s(r)}$. By Theorem 4.11 there is a bijection between the characters $\varphi_r \in \mathrm{SD}_0\Big(\Big[\big(N_{S(p^{d(r)})}(A_r)/A_r\big) \wr S\big(\zeta(r)\big)\Big]^+\Big)$ and the $e(r)$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})$ of p-core partitions κ_i such that $\sum_{i=1}^{e(r)} |\kappa_i| = \zeta(r)$. Using Lemma 5.1 we see that for a fixed $d > 0$ $\sum_{r \in \mathcal{C}_d} e(r) = \frac{1}{2}(p-1)p^{d-1}$. Since $\varphi_d = \hat{\otimes}_{r \in \mathcal{C}_d} \varphi_r$, it follows that each character φ_d determines uniquely a $\frac{1}{2}(p-1)p^{d-1}$-tuple of p-core partitions κ_{dj} such that

$$\sum_j |\kappa_{dj}| = \sum_{r \in \mathcal{C}_d} \zeta(r) = a_d.$$

As $w = w(R) = \sum_{d \geq 1} \sum_{r \in \mathcal{C}_d} \zeta(r) p^{d-1} = \sum_{d \geq 1} a_d p^{d-1}$, it follows now from (1A) of Alperin and Fong [2] that φ determines uniquely an e-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_e)$ of partitions λ with $\sum_{i=1}^{e} |\lambda_i| = w$, where $e = \frac{1}{2}(p-1)$.

Since all the above steps of the proof can be reversed, we have shown that there is a bijection between the B_0-weights (R, φ) and the set of e-tuples of partitions λ_i such that $\sum |\lambda_i| = w$. By Lemma 4.9 each B_0-weight (R, φ) has the sign $\sigma(\varphi) = (-1)^w$. Hence by Section 1

$$l^*_\sigma(B_0) = \begin{cases} k(e,w) & \text{if } \sigma = (-1)^w \\ 0 & \text{otherwise.} \end{cases}$$

Thus the first assertion holds. Together with Proposition 1.6 it implies that $l_\sigma(B) = l^*_\sigma(B)$ for each sign σ. This completes the proof.

COROLLARY 5.3. *Let $p \neq 2$. Then Alperin's weight conjecture holds for all p-blocks B of the covering groups $S^+(n)$ of the symmetric groups.*

PROOF. If B is a spin block of $S^+(n)$, then $l(B) = l_+(B) + 2l_-(B)$ and $l^*(B) = l^*_+(B) + 2l^*_-(B)$. Hence $l(B) = l^*(B)$ by Theorem 5.2. If B is a block of $S(n)$, then $l(B) = l^*(B)$ by Theorem (2C) of Alperin and Fong [2]. Thus Corollary 5.3 holds.

It remains to prove Alperin's weight conjecture for the alternating groups. Therefore we show

THEOREM 5.4. *Let $p \neq 2$. Let B be a p-block of $A(n)$ with positive width w. Then $l_\sigma(B) = l^*_\sigma(B)$ for each sign σ.*

PROOF. By Theorem 3.6 we may assume that B is the principal p-block of $A(pw)$. It is covered by the principal p-block B_0 of $S(pw)$. Therefore Lemma 1.4 and Proposition 3.1 assert that for each sign σ we have

$$l_\sigma(B_0) = l_{-\sigma}(B) \text{ and } l^*_\sigma(B_0) = l^*_{-\sigma}(B).$$

Hence it suffices to show that $l_\sigma(B_0) = l^*_\sigma(B_0)$. As $l(B_0) = l_+(B_0) + 2l_-(B_0) = l^*_+(B_0) + 2l^*_-(B_0) = l^*(B_0)$, by Theorem (2C) of Alperin and Fong [2], it is enough to show that $l_+(B_0) = l^*_+(B_0)$.

The principal p-block B_0 of $S(pw)$ has the symmetric p-core $\emptyset$. Thus $l_+(B_0) = k^s(p-1, w)$ by Proposition 1.5. Therefore it remains to show that there is a bijection between the s.a. B_0-weights (R, φ) and the self-dual $(p-1)$-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_{p-1}) = (\lambda^0_{p-1}, \lambda^0_{p-2}, \ldots, \lambda^0_2, \lambda^0_1)$ of partitions λ_j satisfying $\sum_{j=1}^{p-1} |\lambda_j| = w$, because the number of these $(p-1)$-tuples equals $k^s(p-1, w)$ by definition.

Now let (R, φ) be a s.a. B_0-weight of $G = S(pw)$ with multiplicity function ζ. Then $w(R) = w$. By Lemma 2.2

$$N_G(R)/R = \prod_{\tau \in C} \left(N_{S(p^{d(\tau)})}(A_\tau)/A_\tau\right) \wr S\big(\zeta(\tau)\big)$$

Hence φ has a tensor product decomposition

$$\varphi = \otimes_{\tau \in C} \varphi_\tau, \text{ where } \varphi_\tau \in D_0\Big[\left(N_{S(p^{d(\tau)})}(A_\tau)/A_\tau\right) \wr S\big(\zeta(\tau)\big)\Big]$$

By Proposition 1.2 of [11] $\varphi = \varphi^a$ if and only if $\varphi_\tau = \varphi^a_\tau$ for all $\tau \in C$. Lemma 2.1b) asserts that for each $\tau = (c_1, c_2, \ldots, c_{s(\tau)}) \in C$

$$U_\tau = N_{S(p^{d(\tau)})}(A_\tau)/A_\tau = \prod_{i=1}^{s(\tau)} \mathrm{GL}(c_i, p).$$

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Therefore U_r has $e(r) = (p-1)^{s(r)}$ irreducible defect zero characters by Steinberg's tensor product theorem, which are denoted by $\theta_1, \theta_2, \ldots, \theta_{e(r)}$. Hence for each irreducible defect zero character θ of the base subgroup M_r of $N_r = \left(N_{S(p^{d(r)})}(A_r)/A_r\right) \wr S\left(\zeta(r)\right)$ there are integers $n_k \in \mathbb{N}$ such that $\theta = \otimes_{k=1}^{e(r)}(\otimes_{n_k}\theta_k)$ and $\zeta(r) = \sum_{k=1}^{e(r)} n_k$. Furthermore, by Theorem 4.3.34 of James-Kerber [9], p. 155, θ can be extended to its inertial subgroup $T(\theta)$ in N_r and $T(\theta)/M_r \cong \prod_k S(n_k)$.

By Theorem 7.16 of [10] for each s.a. irreducible defect zero character φ_r of N_r there is a s.a. irreducible defect zero character θ of M_r and an irreducible defect zero character μ of its inertial factor group $T(\theta)/M_r \cong \prod_k S(n_k)$ such that $\varphi_r = (\theta \otimes \mu)^{N_r}$. By the Nakayama Conjecture [9], p. 245, μ determines uniquely an $e(r)$-tuple $(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})$ of p-core partitions κ_k of $n_k = |\kappa_k|$ such that $\sum_{k=1}^{e(r)} |\kappa_k| = \zeta(r)$. By Lemma 2.3 and 4.2 none of the $e(r)$ characters θ_k of U_r is s.a. Hence the θ_k may be ordered such that $\theta_k^a = \theta_{e(r)+1-k}$. Since

$$\theta = \bigotimes_{k=1}^{e(r)}(\otimes_{|\kappa_k|}\theta_k) = \theta^a = \bigotimes_{k=1}^{e(r)}(\otimes_{|\kappa_k|}\theta_{e(r)+1-k})$$

it follows that

$$(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})^0 = (\kappa_{e(r)}^0, \kappa_{e(r)-1}^0, \ldots, \kappa_2^0, \kappa_1^0) = (\kappa_1, \kappa_2, \ldots, \kappa_{e(r)}).$$

In particular, each s.a. character φ_r, $r \in \mathcal{C}$, corresponds uniquely to a self-dual $e(r)$-tuple $(\kappa_1, \kappa_2, \ldots, \kappa_{e(r)})$ of p-core partitions κ_i satisfying $\sum |\kappa_i| = \zeta(r)$. Applying now Lemma 5.1 and assertion (1A) of Alperin and Fong [2] as in the proof of Theorem 5.2 it follows that there is a bijection between the s.a. B_0-weights (R, φ) and the self-dual $(p-1)$-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_{p-1})$ of partitions satisfying $\sum_{j=1}^{p-1} |\lambda_j| = w$. This completes the proof.

COROLLARY 5.5. *Let $p \neq 2$. Then Alperin's weight conjecture holds for all p-blocks B of the covering groups $A^+(n)$ of the alternating groups $A(n)$ and of the exceptional 6-fold covers C_6 and C_7 of $A(6)$ and $A(7)$, respectively.*

PROOF. For the blocks B of $A^+(n)$ the result holds by Theorems 5.2 and 5.4. Alperin's weight conjecture holds for any block B of any finite group G with a cyclic defect group $\delta(B)$ by Theorem 2.1 of Feit [5], p. 275. Since $|C_6| = 2^4 \cdot 3^3 \cdot 5$ and $|C_7| = 2^4 \cdot 3^3 \cdot 5 \cdot 7$, only the 3-blocks of $G \in \{C_6, C_7\}$ have to be checked. Now G contains a central subgroup Z of order 3 such that $G/Z \in \{A^+(6), A^+(7)\}$. By Lemma 4.5 of Feit [5], p. 204, there is a bijection between the 3-blocks of G and the ones of G/Z, which is weight preserving. Furthermore, corresponding blocks have the same number of modular characters by Corollary 2.13 of [5], p. 102. Now the conjecture holds for $A^+(6)$, $A^+(7)$ as remarked above. This completes the proof.

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