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MARKOFF TYPE INEQUALITIES FOR CURVED MAJORANTS

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Abstract

Let $p_n(x)$ be a real algebraic polynomial of degree n, and consider the L_p norms on $I = [-1, 1]$. A classical result of A. A. Markoff states that if $\|p_n\|_\infty \le 1$, then $\|p_n'\|_\infty \le n^2$. A generalization of Markoff's problem, first suggested by P. Turán, is to find upper bounds for $\|p_n{}^{(j)}\|_p$ if $|p_n(x)| \le \psi(x)$, $x \in I$. Here $\psi(x)$ is a given function, a *curved majorant*. In this paper we study extremal properties of $\|p_n'\|_2$ and $\|p_n''\|_2$ if $p_n(x)$ has the parabolic majorant $|p_n(x)| \le 1 - x^2$, $x \in I$. We also consider the problem, motivated by a well-known result of S. Bernstein, of maximising $\|(1 - x^2)p_n''\|_2$ if $\|p_n\|_\infty \le 1$.

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1. Introduction

The majorization of the derivatives of polynomials is an old problem. In 1889, A. A. Markoff [9] gave the following estimate for the derivative of a polynomial on a finite interval. If $p_n(x)$ is a real algebraic polynomial of degree n that satisfies

$$\max_{-1\le x\le 1} |p_n(x)| = 1, \tag{1.1}$$

then

$$\max_{-1\le x\le 1} |p_n'(x)| \le n^2. \tag{1.2}$$

Equality holds in (1.2) only at the end points and only for $p_n(x) = \pm T_n(x)$, where $T_n(x)$ denotes the nth Chebyshev polynomial ($T_n(x) = \cos n\theta$, $\cos\theta = x$). Later Erdös [4], Lorentz [8], Erdös and Varma [5] and Szabados and Varma [15] showed that by restricting the form of the polynomials, substantially better bounds for the derivatives can be obtained.

In the case of the Markoff inequality, the condition (1.1) ensures that the graph of the polynomial $p_n(x)$ is contained in the square $-1 \le x \le 1$, $-1 \le y \le 1$. In 1970, Turán raised the problem of obtaining results of the Markoff type if the graph of $y = p_n(x)$ is contained in the disc $x^2 + y^2 \le 1$ (circular majorant) or within the region $|y| \le 1 - x^2$ (parabolic majorant). Indeed, he suggested generalizing the normalization by requiring that $|p_n(x)| \le \psi(x)$, $-1 \le x \le 1$,for a given curved majorant $\psi(x) \ge 0$.

Important contributions to the problem of P. Turán have been made by Rahman [12], Pierre and Rahman [10, 11] and Rahman and Schmeisser [13]. In the case of circular majorants, Rahman [12] proved the following result.

THEOREM A. *If $p_n(x)$ is a real algebraic polynomial of degree n such that*

$$|p_n(x)| \le (1 - x^2)^{1/2}, \qquad -1 \le x \le 1, \tag{1.3}$$

then

$$\max_{-1 \le x \le 1} |p_n'(x)| \le 2(n-1). \tag{1.4}$$

Recently Varma [17] has obtained an analogue of Theorem A in the L_2 norm. His results may be stated as follows.

THEOREM B. *Let $p_{n+1}(x)$ be any real algebraic polynomial of degree $n+1$ satisfying*

$$|p_{n+1}(x)| \le (1 - x^2)^{1/2}, \qquad -1 \le x \le 1. \tag{1.5}$$

Then for $n \ge 2$, we have

$$\int_{-1}^{1} (p_{n+1}'(x))^2 (1 - x^2)^{1/2}\, dx \le \int_{-1}^{1} (q_0'(x))^2 (1 - x^2)^{1/2}\, dx, \tag{1.6}$$

and

$$\int_{-1}^{1} (p_{n+1}'(x))^2\, dx \le \frac{2n^2(2n^2-1)}{4n^2-1} + 2 + 4\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right), \tag{1.7}$$

where $q_0(x) = (1 - x^2)U_{n-1}(x)$, $U_n(x) = \sin(n+1)\theta / \sin\theta$ and $x = \cos\theta$.

The first aim of this paper is to continue the investigation of Turán's problem in the L_2 norm for real algebraic polynomials of degree n that have the parabolic majorant

$$|p_n(x)| \le 1 - x^2, \qquad -1 \le x \le 1.$$

We shall prove the following two theorems.

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THEOREM 1. *Suppose $n \geq 1$, and let $p_{n+2}(x)$ be any real algebraic polynomial of degree $n+2$ such that*

$$|p_{n+2}(x)| \leq 1 - x^2, \qquad -1 \leq x \leq 1. \tag{1.8}$$

Then we have

$$\int_{-1}^{1} (p''_{n+2}(x))^2\,dx \leq \int_{-1}^{1} (f''_0(x))^2\,dx, \tag{1.9}$$

where $f_0(x) = \pm(1-x^2)T_n(x)$, $T_n(x) = \cos n\theta$ and $x = \cos\theta$. Further, equality in (1.9) *occurs if and only if $p_{n+2}(x) = f_0(x)$.*

THEOREM 2. *Suppose $n \geq 1$, and let $p_{n+2}(x)$ be any real algebraic polynomial of degree $n+2$ having all its zeros in $[-1, 1]$. Suppose also that*

$$|p_{n+2}(x)| \leq 1 - x^2, \qquad -1 \leq x \leq 1. \tag{1.10}$$

Then we have

$$\int_{-1}^{1} (p'_{n+2}(x))^2\,dx \leq \int_{-1}^{1} (f'_0(x))^2\,dx, \tag{1.11}$$

with equality if and only if $p_{n+2}(x) = f_0(x)$.

We make the following remarks concerning Theorems 1 and 2.

REMARK 1. For the parabolic majorant, the corresponding problems in the uniform norm have been solved by Pierre and Rahman [11] and Rahman and Schmeisser [13].

REMARK 2. Problems of this type also occur in approximation theory, most notably in the work of Dzyadyk [3].

For the second aim of this paper, we recall a well known inequality of S. Bernstein [1]. According to this result, if $p_n(x)$ is a real algebraic polynomial of degree n or less that satisfies

$$|p_n(x)| \leq 1, \qquad -1 \leq x \leq 1, \tag{1.12}$$

then

$$(1-x^2)^{p/2}|p_n^{(p)}(x)| \leq p^{p/2}n(n-1)\dots(n-p+1). \tag{1.13}$$

In the case $p = 1$ equality can occur only if $p_n(x) = \pm T_n(x)$, where $T_n(x)$ is the nth Chebyshev polynomial of the first kind. Motivated by this result, one of us [18] has proved that if $p_n(x)$ is a real polynomial of degree n or less satisfying (1.12), then

$$\int_{-1}^{1} (1-x^2)(p'_n(x))^2\,dx \leq \int_{-1}^{1} (1-x^2)(T'_n(x))^2\,dx. \tag{1.14}$$

In this paper, we shall prove the following theorem which is suggested by (1.14) and the case $p = 2$ of (1.13).

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THEOREM 3. *Suppose $n \geq 2$, and let $q_n(x)$ be any real algebraic polynomial of degree n or less such that*

$$|q_n(x)| \leq 1, \qquad -1 \leq x \leq 1. \tag{1.15}$$

Then

$$\int_{-1}^{1} (1-x^2)^2 (q_n''(x))^2 \, dx \leq \int_{-1}^{1} (1-x^2)^2 (T_n''(x))^2 \, dx, \tag{1.16}$$

with equality if and only if $q_n(x) = \pm T_n(x)$.

2. Inequalities for trigonometric polynomials

For the proofs of Theorems 1 and 2 we shall need the following inequalities concerning real even trigonometric polynomials. We state them as follows.

LEMMA 2.1. *For $n \geq 1$, let $t_n(\theta)$ be any real even trigonometric polynomial of degree n, such that $|t_n(\theta)| \leq 1, 0 \leq \theta \leq \pi$. Then we have*

$$\int_0^{\pi} (t_n''(\theta))^2 \sin\theta \, d\theta \leq n^4 \int_0^{\pi} \cos^2 n\theta \sin\theta \, d\theta = n^4 \left(1 - \frac{1}{4n^2-1}\right) \tag{2.1}$$

and

$$\int_0^{\pi} (t_n'''(\theta))^2 \sin\theta \, d\theta \leq n^6 \int_0^{\pi} \sin^2 n\theta \sin\theta \, d\theta = n^6 \left(1 + \frac{1}{4n^2-1}\right), \tag{2.2}$$

with equality if and only if $t_n(\theta) = \pm\cos n\theta$.

LEMMA 2.2. *For $n \geq 1$, let $t_n(\theta)$ be any real even trigonometric polynomial of degree n, all of whose zeros are real. Further, suppose $|t_n(\theta)| \leq 1, 0 \leq \theta \leq \pi$. Then we have*

$$\int_0^{\pi} (t_n'(\theta))^2 \sin^3\theta \, d\theta \leq n^2 \int_0^{\pi} \sin^2 n\theta \sin^3\theta \, d\theta, \tag{2.3}$$

with equality if and only if $t_n(\theta) = \pm\cos n\theta$.

PROOF OF LEMMA 2.1. The proofs of (2.1) and (2.2) are similar to that of the integral inequality

$$\int_0^{\pi} (t_n'(\theta))^2 \sin\theta \, d\theta \leq n^2 \left(1 + \frac{1}{4n^2-1}\right), \tag{2.4}$$

which was established in an earlier work [18]. Thus we will prove (2.1) only.

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Let

$$I_n = \int_0^\pi (t_n''(\theta))^2 \sin\theta\, d\theta, \tag{2.5}$$

and note that, by two integrations by parts, we have

$$\begin{aligned}\int_0^\pi t_n'''(\theta) t'_n(\theta) \sin\theta\, d\theta &= -\int_0^\pi t_n''(\theta) \left[t_n''(\theta)\sin\theta + t_n'(\theta)\cos\theta\right] d\theta \\ &= -I_n - \frac{1}{2}\int_0^\pi (t_n'(\theta))^2 \sin\theta\, d\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}2I_n &= \int_0^\pi \left[(t_n''(\theta))^2 - t_n'(\theta)t_n'''(\theta)\right]\sin\theta\, d\theta - \frac{1}{2}\int_0^\pi (t_n'(\theta))^2 \sin\theta\, d\theta \\ &= \left(\frac{1}{2} - \frac{1}{2n^2}\right)\int_0^\pi \left[(t_n''(\theta))^2 + n^2(t_n'(\theta))^2\right]\sin\theta\, d\theta \\ &\quad + \frac{1}{2}\int_0^\pi \left[(t_n''(\theta))^2 + \left(t_n'''(\theta)/n\right)^2\right]\sin\theta\, d\theta + \frac{1}{2n^2} I_n \\ &\quad - \frac{1}{2n^2}\int_0^\pi \left[n^2 t_n'(\theta) + t_n'''(\theta)\right]^2 \sin\theta\, d\theta.\end{aligned} \tag{2.6}$$

Now, if $\tau_n(\theta)$ is a real trigonometric polynomial of degree n such that $|\tau_n(\theta)| \le 1$ for all θ, then by the Szegö inequality [16] we have

$$(\tau_n'(\theta))^2 + n^2(\tau_n(\theta))^2 \le n^2, \qquad 0 \le \theta \le 2\pi. \tag{2.7}$$

Equality holds in (2.7) at a given θ if and only if $\tau_n(\theta) = \cos(n\theta + \alpha)$ for some constant α (in which case equality holds for *all* θ). Also, by Bernstein's inequality [1], we have

$$\left|\tau_n'(\theta)/n\right| \le 1, \quad \left|\tau_n''(\theta)/n^2\right| \le 1, \qquad 0 \le \theta \le 2\pi. \tag{2.8}$$

If (2.7) is applied to the functions $(\tau_n'(\theta))/n$ and $(\tau_n''(\theta))/n^2$, we obtain

$$(\tau_n''(\theta))^2 + n^2(\tau_n'(\theta))^2 \le n^4, \qquad 0 \le \theta \le 2\pi, \tag{2.9}$$

and

$$(\tau_n''(\theta))^2 + \left(\tau_n'''(\theta)/n\right)^2 \le n^4, \qquad 0 \le \theta \le 2\pi. \tag{2.10}$$

Equality holds in (2.9) if and only if $(\tau_n'(\theta))/n$ is of the form $\cos(n\theta + \alpha)$, and holds in (2.10) if and only if $(\tau_n''(\theta))/n^2$ also has this form.

For the proof of (2.1), we note that if $t_n(\theta)$ is a real even trigonometric polynomial of degree n such that $|t_n(\theta)| \le 1$ for $0 \le \theta \le \pi$, then (2.9) and (2.10) hold true for

$\tau_n = t_n$ (and equality holds in each of (2.9) and (2.10) if and only if $t_n(\theta) = \pm\cos n\theta$). Thus (2.6) gives

$$I_n\left(2 - \frac{1}{2n^2}\right) \le \left(\frac{1}{2} - \frac{1}{2n^2}\right)2n^4 + \frac{1}{2}2n^4 = 2n^4 - n^2, \tag{2.11}$$

which is equivalent to (2.1). Further, by the above comments, equality holds in (2.1) if and only if $t_n(\theta) = \pm\cos n\theta$.

PROOF OF LEMMA 2.2. Define

$$J_n = \int_0^\pi (t_n'(\theta))^2 \sin^3\theta\, d\theta. \tag{2.12}$$

Then, as in the proof of Lemma 2.1, we obtain

$$\begin{aligned} J_n &= \frac{1}{2}\int_0^\pi \left[(t_n'(\theta))^2 - t_n(\theta)t_n''(\theta)\right]\sin^3\theta\, d\theta + \frac{3}{2}\int_0^\pi (t_n(\theta))^2 \sin\theta\, d\theta \\ &\qquad -\frac{9}{4}\int_0^\pi (t_n(\theta))^2\sin^3\theta\, d\theta \\ &= \frac{1}{2}\Bigg[\left(\frac{1}{2} - \frac{1}{2n^2}\right)\int_0^\pi \left[(t_n'(\theta))^2 + n^2(t_n(\theta))^2\right]\sin^3\theta\, d\theta \\ &\qquad + \frac{1}{2}\int_0^\pi \left[\left(t_n''(\theta)/n\right)^2 + (t_n'(\theta))^2\right]\sin^3\theta\, d\theta + \frac{J_n}{2n^2} \\ &\qquad - \frac{1}{2n^2}\int_0^\pi \left[t_n''(\theta) + n^2 t_n(\theta)\right]^2 \sin^3\theta\, d\theta\Bigg] \\ &\quad - \frac{2}{n^2}\int_0^\pi \left[n^2(t_n(\theta))^2 + (t_n'(\theta))^2\right]\sin^3\, d\theta + \frac{2}{n^2}J_n + \frac{3}{2}\int_0^\pi (t_n(\theta))^2\sin\theta\, d\theta. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} J_n\left(1 - \frac{9}{4n^2}\right) = \left[\frac{1}{4}\left(1 - \frac{1}{n^2}\right) - \frac{2}{n^2}\right]&\int_0^\pi \left[(t_n'(\theta))^2 + n^2(t_n(\theta))^2\right]\sin^3\theta\, d\theta \\ &+ \frac{1}{4}\int_0^\pi \left[\left(t_n''(\theta)/n\right)^2 + (t_n'(\theta))^2\right]\sin^3\theta\, d\theta \\ &- \frac{1}{4n^2}\int_0^\pi \left[t_n''(\theta) + n^2t_n(\theta)\right]^2\sin^3\theta\, d\theta \\ &+ \frac{3}{2}\int_0^\pi (t_n(\theta))^2\sin\theta\, d\theta. \end{aligned} \tag{2.13}$$

Next, on applying (2.7) and (2.9) to $\tau_n(\theta) = t_n(\theta)$, we obtain (for $n \ge 3$),

$$J_n\left(1 - \frac{9}{4n^2}\right) \le \left(\frac{1}{4} - \frac{9}{4n^2}\right)n^2\int_0^\pi \sin^3\theta\, d\theta + \frac{n^2}{4}\int_0^\pi \sin^3\theta\, d\theta + \frac{3}{2}\int_0^\pi (t_n(\theta))^2\sin\theta\, d\theta, \tag{2.14}$$

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with equality if and only if $t_n(\theta) = \pm\cos n\theta$. Now, because $t_n(\theta)$ is a real even trigonometric polynomial with $|t_n(\theta)| \le 1, 0 \le \theta \le \pi$, and such that all its zeros are real, we have

$$\int_0^\pi (t_n(\theta))^2 \sin\theta \, d\theta \le \int_0^\pi \cos^2 n\theta \sin\theta \, d\theta. \tag{2.15}$$

The above statement is a consequence of G. K. Kristiansen [7, Corollary 1, p. 305]. Hence

$$J_n\left(1 - \frac{9}{4n^2}\right) \le \left(\frac{n^2}{2} - \frac{9}{4}\right)\frac{4}{3} + \frac{3}{2}\left(1 - \frac{1}{4n^2-1}\right).$$

From this we obtain (2.3) for $n \ge 3$.

To complete the proof of Lemma 2.2, it remains to establish (2.3) for $n = 1, 2$. Now if we define $g_n(x) = t_n(\cos^{-1} x), -1 \le x \le 1$, then $g_n(x)$ is an algebraic polynomial of degree n with all its zeros in $[-1, 1]$, and $|g_n(x)| \le 1$ for $-1 \le x \le 1$. Further,

$$J_n = \int_0^\pi (t_n'(\theta))^2 \sin^3\theta \, d\theta = \int_{-1}^1 (g_n'(x))^2 (1-x^2)^2 \, dx.$$

For $n = 1$, write $g_1(x) = ax+b$, so $J_1 = 16a^2/15$. Since $|g_1(x)| \le 1$ for $-1 \le x \le 1$, it is known (see, for example, Rivlin [14, p. 108]) that $|a| \le 1$, with equality if and only if $g_1(x) = \pm T_1(x)$. Hence $J_1 \le 16/15$, and equality holds if and only if $g_1(x) = \pm T_1(x)$. Thus the lemma is true for $n = 1$.

For $n = 2$, write $g_2(x) = ax^2+bx+c$, and note that $J_2 = 16(4a^2+7b^2)/105$. The quadratic g_2 has both zeros in $[-1, 1]$, and so if $a > 0$, we have $0 \le a+b+c \le 1$, $0 \le a-b+c \le 1$, and $-1 \le c - b^2/4a \le 0$. Thus $c \ge -1 + b^2/4a$, and so $0 \le a+b \le 2 - b^2/4a$, $0 \le a-b \le 2 - b^2/4a$. These last two inequalities give $a+|b| \le 2-b^2/4a$ (so $a \le 2$), and hence $4a^2+4a|b|+b^2 \le 8a$. Therefore, $2a+|b| \le 2\sqrt{2}\sqrt{a}$, and so $b^2 \le 4a(\sqrt{2}-\sqrt{a})^2$. Thus $J_2 \le 128(4a^2 - 7\sqrt{2}a^{3/2} + 7a)/105$. Now, $4a^2 - 7\sqrt{2}a^{3/2} + 7a$ is increasing on $[0, 2]$, and since $0 \le a \le 2$, then $J_2 \le 256/105$. Further, equality holds if and only if $a = 2$, so $b = 0$, $c = -1$, and hence $g_2(x) = T_2(x)$. The case $a < 0$ leads similarly to $J_2 \le 256/105$, with equality if and only if $g_2(x) = -T_2(x)$. Hence Lemma 2.2 is proved.

3. Proof of Theorem 1

If $p_{n+2}(x)$ is a real algebraic polynomial of degree $n+2$ satisfying the condition (1.8), we can write

$$p_{n+2}(x) = (1-x^2)q_n(x), \tag{3.1}$$

where $q_n(x)$ is a real algebraic polynomial of degree n such that

$$|q_n(x)| \le 1, \qquad -1 \le x \le 1. \tag{3.2}$$

From (3.1) we have

$$p''_{n+2}(x) = (1-x^2)q''_n(x) - 4xq'_n(x) - 2q_n(x),$$

and so

$$\begin{aligned}\int_{-1}^{1}(p''_{n+2}(x))^2\,dx &= \int_{-1}^{1}(1-x^2)^2(q''_n(x))^2\,dx + 16\int_{-1}^{1}x^2(q'_n(x))^2\,dx \\ &\quad + 4\int_{-1}^{1}(q_n(x))^2\,dx - 8\int_{-1}^{1}x(1-x^2)q'_n(x)q''_n(x)\,dx \\ &\quad +16\int_{-1}^{1}xq_n(x)q'_n(x)\,dx - 4\int_{-1}^{1}(1-x^2)q_n(x)q''_n(x)\,dx.\end{aligned}\tag{3.3}$$

Now, on using integration by parts, we obtain the three identities

$$\begin{aligned}16\int_{-1}^{1}xq_n(x)q'_n(x)\,dx &= 8(q_n{}^2(1)+q_n{}^2(-1)) - 8\int_{-1}^{1}(q_n(x))^2\,dx, \\ -8\int_{-1}^{1}x(1-x^2)q'_n(x)q''_n(x)\,dx &= 4\int_{-1}^{1}(1-3x^2)(q'_n(x))^2\,dx, \\ -4\int_{-1}^{1}(1-x^2)q_n(x)q''_n(x)\,dx &= 4\int_{-1}^{1}(1-x^2)(q'_n(x))^2\,dx + 4\int_{-1}^{1}(q_n(x))^2\,dx \\ &\quad - 4(q_n{}^2(1)+q_n{}^2(-1)).\end{aligned}\tag{3.4}$$

These identities (3.4) enable (3.3) to be simplified to

$$\int_{-1}^{1}(p''_{n+2}(x))^2\,dx = \int_{-1}^{1}(1-x^2)^2(q''_n(x))^2 + 8\int_{-1}^{1}(q'_n(x))^2\,dx + 4(q_n{}^2(1)+q_n{}^2(-1)).\tag{3.5}$$

Next, we set

$$t_n(\theta) = q_n(\cos\theta) = q_n(x).\tag{3.6}$$

Clearly, $t_n(\theta)$ is a purely cosine polynomial of degree n. Further, from (3.2) it follows that

$$|\,t_n(\theta)| \le 1, \qquad 0 \le \theta \le \pi.\tag{3.7}$$

From (3.6) we have

$$t''_n(\theta) = (1-x^2)q''_n(x) - xq'_n(x).\tag{3.8}$$

Therefore, we can write

$$\begin{aligned}\int_{0}^{\pi}(t''_n(\theta))^2\sin\theta\,d\theta &= \int_{-1}^{1}(1-x^2)^2(q''_n(x))^2\,dx + \int_{-1}^{1}x^2(q'_n(x))^2\,dx \\ &\quad -2\int_{-1}^{1}x(1-x^2)q'_n(x)q''_n(x)\,dx \\ &= \int_{-1}^{1}(1-x^2)^2(q''_n(x))^2\,dx + \int_{-1}^{1}(1-2x^2)(q'_n(x))^2\,dx.\end{aligned}\tag{3.9}$$

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On using (3.5) and (3.9) we have

$$\int_{-1}^{1}(p''_{n+2}(x))^2\,dx = \int_0^\pi (t''_n(\theta))^2\sin\theta\,d\theta + 4({t_n}^2(0)+{t_n}^2(\pi)) \tag{3.10}$$
$$+9\int_0^\pi \frac{(t'_n(\theta))^2}{\sin\theta}\,d\theta - 2\int_0^\pi (t'_n(\theta))^2\sin\theta\,d\theta.$$

Now, a simple calculation shows that

$$-\int_0^\pi (t'_n(\theta))^2\sin\theta\,d\theta = -\frac{1}{n^4}\int_0^\pi \left[n^2 t'_n(\theta)+t'''_n(\theta)\right]^2\sin\theta\,d\theta \tag{3.11}$$
$$+\frac{1}{n^4}\int_0^\pi (t'''_n(\theta))^2\sin\theta\,d\theta + \frac{2}{n^2}\int_0^\pi t'''_n(\theta)t'_n(\theta)\sin\theta\,d\theta.$$

However,

$$\int_0^\pi t'''_n(\theta)t'_n(\theta)\sin\theta\,d\theta = -\int_0^\pi (t''_n(\theta))^2\sin\theta\,d\theta - \frac{1}{2}\int_0^\pi (t'_n(\theta))^2\sin\theta\,d\theta. \tag{3.12}$$

Therefore, we have

$$\left(-1+\frac{1}{n^2}\right)\int_0^\pi (t'_n(\theta))^2\sin\theta\,d\theta = -\frac{1}{n^4}\int_0^\pi \left[n^2 t'_n(\theta)+t'''_n(\theta)\right]^2\sin\theta\,d\theta \tag{3.13}$$
$$+\frac{1}{n^4}\int_0^\pi (t'''_n(\theta))^2\sin\theta\,d\theta$$
$$-\frac{2}{n^2}\int_0^\pi (t''_n(\theta))^2\sin\theta\,d\theta.$$

On using (3.10) and (3.13) if follows that (for $n \geq 2$),

$$\int_{-1}^{1}(p''_{n+2}(x))^2\,dx = \left(1-\frac{4}{n^2-1}\right)\int_0^\pi (t''_n(\theta))^2\sin\theta\,d\theta + 4({t_n}^2(0)+{t_n}^2(\pi)) \tag{3.14}$$
$$+9\int_0^\pi \frac{(t'_n(\theta))^2}{\sin\theta}\,d\theta + \frac{2}{n^2(n^2-1)}\int_0^\pi (t'''_n(\theta))^2\sin\theta\,d\theta$$
$$-\frac{2}{n^2(n^2-1)}\int_0^\pi \left[n^2 t'_n(\theta)+t'''_n(\theta)\right]^2\sin\theta\,d\theta.$$

Thus, by applying the results of Lemma 2.1 (where equality holds if and only if $t_n(\theta) = \pm\cos n\theta$) and a well known result of B. D. Bojanov [2],

$$\int_0^\pi \frac{(t'_n(\theta))^2}{\sin\theta}\,d\theta \leq 2n^2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right), \tag{3.15}$$

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to (3.14), we obtain (for $n \geq 3$)

$$\begin{aligned}\int_{-1}^{1}(p''_{n+2}(x))^2\,dx &\leq \left(1-\frac{4}{n^2-1}\right)n^4\left(1-\frac{1}{4n^2-1}\right)+8\\ &\quad +18n^2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)+\frac{2n^6}{n^2(n^2-1)}\left(1+\frac{1}{4n^2-1}\right)\\ &= \int_{-1}^{1}(f''_0(x))^2\,dx.\end{aligned}$$

Here equality holds if and only if $p_{n+2}(x) = f_0(x) = \pm(1-x^2)T_n(x)$. Thus the theorem is established for $n \geq 3$.

For $n = 1$, write $p_3(x) = (ax+b)(1-x^2)$, where $a \neq 0$ and $|ax+b| \leq 1$, $-1 \leq x \leq 1$. Then

$$\int_{-1}^{1}(p''_3(x))^2\,dx = 8(3a^2+b^2).$$

From $(a-b)^2 \leq 1$, $(a+b)^2 \leq 1$, it follows that $a^2+b^2 \leq 1$, and hence $8(3a^2+b^2) \leq 24(a^2+b^2) \leq 24$. Further, equality holds if and only if $|a| = 1$, $b = 0$, so $p_3(x) = \pm(1-x^2)T_1(x)$.

For $n = 2$, write $p_4(x) = (ax^2+bx+c)(1-x^2)$, where $g(x) = ax^2+bx+c$ is such that $a \neq 0$ and $|g(x)| \leq 1$, $-1 \leq x \leq 1$. Then

$$I = \int_{-1}^{1}(p''_4(x))^2\,dx = \frac{8}{5}\left(16a^2+15b^2+5(a+c)^2\right).$$

The conditions $|g(1)| \leq 1$, $|g(-1)| \leq 1$, give $(a+b+c)^2 \leq 1$, $(a-b+c)^2 \leq 1$, and hence $(a+c)^2+b^2 \leq 1$. Thus $I \leq 8(16a^2+10b^2+5)/5$. From $|g(1)-g(0)| \leq 2$, $|g(-1)-g(0)| \leq 2$, it follows that $|a+b| \leq 2$, $|a-b| \leq 2$, and hence $|a|+|b| \leq 2$ (so $|a| \leq 2$). Thus $16a^2+10b^2 = 10(|a|+|b|)^2+6a^2-20|a||b| \leq 64$, and so $I \leq 552/5$. Furthermore, equality holds if and only if $|a| = 2$, $b = 0$, and $|a+c| = 1$, conditions which imply that $g(x) = \pm T_2(x)$. Hence the theorem is proved.

4. Proof of Theorem 2

Let $p_{n+2}(x)$ be any real algebraic polynomial of degree $n+2$ that satisfies the condition (1.10) and which has all its zeros in the interval $[-1, 1]$. Then we can write

$$p_{n+2}(x) = (1-x^2)q_n(x) \equiv \sin^2\theta\, t_n(\theta), \tag{4.1}$$

where $t_n(\theta)$ is a purely cosine trigonometric polynomial of degree n that has real coefficients and only real zeros. Further,

$$|t_n(\theta)| \leq 1, \qquad 0 \leq \theta \leq \pi. \tag{4.2}$$

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From (4.1) we obtain

$$p'_{n+2}(x) = (1 - x^2)q'_n(x) - 2xq_n(x)$$

and

$$\begin{aligned}
\int_{-1}^{1} (p'_{n+2}(x))^2\,dx &= \int_{-1}^{1} (1-x^2)^2 (q'_n(x))^2\,dx + 4\int_{-1}^{1} x^2 (q_n(x))^2\,dx \\
&\quad -4\int_{-1}^{1} x(1-x^2) q_n(x) q'_n(x)\,dx \\
&= \int_{-1}^{1} (1-x^2)^2 (q'_n(x))^2\,dx + 2\int_{-1}^{1} (1-x^2)(q_n(x))^2\,dx \\
&= \int_0^{\pi} (t'_n(\theta))^2 \sin^3\theta\,d\theta + 2\int_0^{\pi} (t_n(\theta))^2 \sin^3\theta\,d\theta \\
&= \int_0^{\pi} (t'_n(\theta))^2 \sin^3\theta\,d\theta + \frac{2}{n^2}\int_0^{\pi} \left[n^2 (t_n(\theta))^2 + (t'_n(\theta))^2\right] \sin^3\theta\,d\theta \\
&\quad -\frac{2}{n^2}\int_0^{\pi} (t'_n(\theta))^2 \sin^3\theta\,d\theta \\
&= \left(1 - \frac{2}{n^2}\right)\int_0^{\pi} (t'_n(\theta))^2 \sin^3\theta\,d\theta \\
&\quad +\frac{2}{n^2}\int_0^{\pi} \left[n^2 (t_n(\theta))^2 + (t'_n(\theta))^2\right] \sin^3\theta\,d\theta.
\end{aligned}$$

Thus, on applying Lemma 2.2 (where equality holds if and only if $t_n(\theta) = \pm\cos n\theta$) and the Szegö inequality (2.7), we conclude that (for $n \geq 2$),

$$\begin{aligned}
\int_{-1}^{1} (p'_{n+2}(x))^2\,dx &\leq \left(1 - \frac{2}{n^2}\right) n^2 \int_0^{\pi} \sin^2 n\theta \sin^3\theta\,d\theta + \frac{2}{n^2} n^2 \int_0^{\pi} \sin^3\theta\,d\theta \\
&= \int_0^{\pi} (f'_0(\cos\theta))^2 \sin\theta\,d\theta,
\end{aligned}$$

with equality if and only if $p_{n+2}(x) = f_0(x)$. Hence Theorem 2 is proved for $n \geq 2$.

For the case $n = 1$ of Theorem 2, we write

$$p_3(x) = (ax + b)(1 - x^2),$$

where $ax + b$ has its zero in $[-1, 1]$, and $|ax + b| \leq 1$, $-1 \leq x \leq 1$. If $a > 0$, these conditions give $-1 \leq b - a \leq 0 \leq b + a \leq 1$. Thus $0 \leq a + |b| \leq 1$, and $|b| - a \leq 0$.

Now

$$J = \int_{-1}^{1} (p_3'(x))^2\, dx = \frac{8}{15}(3a^2 + 5b^2).$$

From the above results we then have

$$3a^2 + 5b^2 = 3(a + |b|)^2 + 2|b|(|b| - 3a) \le 3.$$

Therefore $J \le 8/5$, with equality if and only if $a = 1, b = 0$, so $p_3(x) = (1-x^2)T_1(x)$. If $a < 0$, a similar argument gives $J \le 8/5$, with equality if and only if $p_3(x) = -(1 - x^2)T_1(x)$. Thus Theorem 2 is established.

5. Proof of Theorem 3

Let $q_n(x)$ be any real algebraic polynomial of degree n or less which satisfies

$$|q_n(x)| \le 1, \qquad -1 \le x \le 1, \tag{5.1}$$

and set $t_n(\theta) = q_n(\cos\theta)$. Then $t_n(\theta)$ is an even trigonometric polynomial of degree n or less such that

$$|t_n(\theta)| \le 1, \qquad 0 \le \theta \le \pi. \tag{5.2}$$

Now, by (3.9) we can write

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)^2 (q_n''(x))^2\, dx = \int_0^{\pi} (t_n''(\theta))^2 \sin\theta\, d\theta + \int_0^{\pi} \frac{(t_n'(\theta))^2}{\sin\theta}\, d\theta \\ - 2\int_0^{\pi} (t_n'(\theta))^2 \sin\theta\, d\theta.\end{aligned} \tag{5.3}$$

Also, for $n \ge 2$, it follows from (3.13) that

$$\begin{aligned}-2\int_0^{\pi} (t_n'(\theta))^2 \sin\theta\, d\theta = -\frac{2}{n^2(n^2-1)} \int_0^{\pi} \left[n^2 t_n'(\theta) + t_n'''(\theta)\right]^2 \sin\theta\, d\theta \\ + \frac{2}{n^2(n^2-1)} \int_0^{\pi} (t_n'''(\theta))^2 \sin\theta\, d\theta \\ - \frac{4}{n^2-1} \int_0^{\pi} (t_n''(\theta))^2 \sin\theta\, d\theta.\end{aligned} \tag{5.4}$$

On substituting (5.4) in (5.3) we obtain

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$$\int_{-1}^{1}(1-x^2)^2(q_n''(x))^2\,dx = \left(1-\frac{4}{n^2-1}\right)\int_0^{\pi}(t_n''(\theta))^2\sin\theta\,d\theta + \int_0^{\pi}\frac{(t_n'(\theta))^2}{\sin\theta}\,d\theta$$
$$+\frac{2}{n^2(n^2-1)}\int_0^{\pi}(t_n'''(\theta))^2\sin\theta\,d\theta$$
$$-\frac{2}{n^2(n^2-1)}\int_0^{\pi}\left[n^2t_n'(\theta)+t_n'''(\theta)\right]^2\sin\theta\,d\theta.$$

Then, on using (2.1), (2.2) and (3.15), we can conclude that (for $n \geq 3$),

$$\int_{-1}^{1}(1-x^2)^2(q_n''(x))^2\,dx \leq \frac{n^2-5}{n^2-1}n^4\left(1-\frac{1}{4n^2-1}\right)+2n^2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)$$
$$+\frac{2n^4}{n^2-1}\left(1+\frac{1}{4n^2-1}\right)$$
$$=\int_{-1}^{1}(1-x^2)^2(T_n''(x))^2\,dx,$$

with equality if and only if $q_n(x) = \pm T_n(x)$. Thus the theorem is true if $n \geq 3$.

To conclude the proof of Theorem 3, we note that the case $n = 2$ follows immediately from the fact that if $q_2(x) = ax^2+bx+c$, and $|q_2(x)| \leq 1$ for $-1 \leq x \leq 1$, then $|a| \leq 2$, with equality if and only if $q_2(x) = \pm T_2(x)$ (see Rivlin [14, p. 108]).

References

[1] S. Bernstein, 'Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné', *Acad. Roy. Belg. Cl. Sci. Mém. Collect. (2)* **4** (1912), 1–103.

[2] B. D. Bojanov, 'An extension of the Markov inequality', *J. Approx. Theory* **35** (1982), 181–190.

[3] V. K. Dzyadyk, 'Constructive characterization of functions satisfying the Lipschitz condition α ($0 < \alpha < 1$) on a finite segment of the real axis', *Izv. Akad. Nauk SSSR Ser. Mat.* **20** (1956), 623–642 (in Russian).

[4] P. Erdös, 'Extremal properties of derivatives of polynomials', *Ann. of Math. (2)* **41** (1940), 310–313.

[5] P. Erdös and A. K. Varma, 'An extremal problem concerning algebraic polynomials', *Acta Math. Hungar.* **47** (1986), 137–143.

[6] E. Hille, G. Szegö and J. D. Tamarkin, 'On some generalizations of a theorem of A. A. Markov', *Duke Math. J.* **3** (1937), 729–739.

[7] G. K. Kristiansen, 'Some inequalities for algebraic and trigonometric polynomials', *J. London Math. Soc. (2)* **20** (1979), 300–314.

[8] G. G. Lorentz, 'Degree of approximation by polynomials with positive coefficients', *Math. Ann.* **151** (1963), 239–251.

[9] A. A. Markoff, 'Sur une question posée par D. I. Mendeleieff', *Zap. Imp. Akad. Nauk, St. Petersburg* **62** (1889), 1–24.

[10] R. Pierre and Q. I. Rahman, 'On a problem of Turán about polynomials', *Proc. Amer. Math. Soc.* **56** (1976), 231–238.

[11] ——, 'On a problem of Turán about polynomials II', *Canad. J. Math.* **33** (1981), 701–733.
[12] Q. I. Rahman, 'On a problem of Turán about polynomials with curved majorants', *Trans. Amer. Math. Soc.* **163** (1972), 447–455.
[13] Q. I. Rahman and G. Schmeisser, 'Markoff type inequalities for curved majorants', in: *International Series of Numerical Math. 81* (Birkhäuser, Basel, 1987) pp. 169–183.
[14] T. J. Rivlin, *Chebyshev polynomials*, 2nd edition (Wiley, New York, 1990).
[15] J. Szabados and A. K. Varma, 'Inequalities for derivatives of polynomials having real zeros', in: *Approx. Theory III* (ed. E. W. Cheney) (Academic Press, New York, 1980) pp. 881–887.
[16] G. Szegö, 'Über einen Satz des Herrn Serge Bernstein', *Schriften der Königsberger Gelehrten Gesellschaft, Naturw. Klasse* **5** (1928), 59–70.
[17] A. K. Varma, 'Markoff type inequalities for curved majorants in L_2 norm', in: *Colloq. Math. Soc. János Bolyai: Approximation theory* (Kecskemét, Hungary, 1990) pp. 689–697.
[18] ——, 'On some extremal properties of algebraic polynomials', *J. Approx. Theory* **69** (1992), 48–54.

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