ON THE EQUIVALENCE OF CANCELLATIVE EXTENSIONS OF COMMUTATIVE CANCELLATIVE SEMIGROUPS BY GROUPS

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In [1] D. W. Miller and the author established necessary and sufficient conditions for the existence of a cancellative (ideal) extension of a commutative cancellative semigroup by a cyclic group with zero. The purpose of this paper is to extend these results to cancellative extensions by any finitely generated Abelian group with zero and to establish in this general case conditions under which two such extensions are equivalent.

1. Existence of extensions

If T is a commutative cancellative semigroup and S is a cancellative extension of T by a group with zero G^0, then $S = G \cup T$ where T is an ideal in S. Since any idempotent in the cancellative semigroup S is necessarily an identity for S it follows that T cannot contain an idempotent. Furthermore (see [1]) S is necessarily commutative and hence G must be also. All of this then reduces the problem of the existence of a cancellative extension of a commutative cancellative semigroup T by a group with zero G^0 to a consideration of the case where T does not contain an idempotent and G^0 is Abelian. The following theorem establishes necessary and sufficient conditions for the existence of such an extension in case G is finitely generated.

THEOREM 1. *Let T be a commutative cancellative semigroup without idempotent and let G be a finitely generated Abelian group. Suppose $g_1, \cdots, g_n$ is a basis for G and let $m_i = o(g_i)$ for $i = 1, \cdots, n$*[1]. *(Allowing the possibility that $m_i = \infty$ for some i.) Then there exists a cancellative semigroup $S = G \cup T$ if and only if there exist n distinct pairs of elements a_i, b_i, $(i = 1, \cdots, n)$ in T such that*

(I) $$a_i T = b_i T \quad \text{for} \quad i = 1, \cdots, n.$$

(II) $$\prod_{i=1}^{n} a_i^{u_i} = \prod_{i=1}^{n} b_i^{u_i} \Rightarrow \begin{cases} u_i = 0 & \text{if } m_i = \infty. \\ m_i | u_i & \text{if } m_i < \infty. \end{cases}$$

[1] $o(g_i)$ denotes the order of g_i.

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PROOF. Suppose a cancellative extension $S = G \cup T$ exists. (Recall that S is necessarily commutative.) Let $a_1, \cdots, a_n$ be any n distinct elements of T and let $g_1, \cdots, g_n$ be a basis for the group G such that $o(g_i) = m_i$ for each i. For each i choose $b_i = g_i a_i$. Then for any $t \in T$,

$$a_i t = a_i g_i g_i^{-1} t = b_i g_i^{-1} t \in b_i T$$

so $a_i T \subseteq b_i T$. Also $b_i t = a_i g_i t \in a_i T$ so $b_i T \subseteq a_i T$ proving (I). Furthermore since $b_i = g_i a_i$ it follows from cancellation that if

$$\prod_{i=1}^{n} a_i^{u_i} = \prod_{i=1}^{n} b_i^{u_i} \quad \text{then} \quad \prod_{i=1}^{n} g_i^{u_i} = e,$$

the identity element in G. Since $g_1, \cdots, g_n$ is a basis for G, (II) follows.

Conversely suppose there exist n distinct pairs of elements a_i, b_i $(i = 1, \cdots, n)$ of T satisfying (I) and (II). It is shown in [1] that the desired extension S of T by G^0 exists if the group of quotients Q of T contains a subgroup G', isomorphic to G, satisfying $G'T' \subseteq T'$ where T' is the natural isomorph of T in Q. So let G' be the subgroup of Q generated by the elements $(a_1, b_1), \cdots, (a_n, b_n)$. Condition (II) guarantees that these elements are independent and that (a_i, b_i) has order m_i for $i = 1, \cdots, n$. Hence G' is isomorphic to G.

It remains to show that $G'T' \subseteq T'$ and to do so it is sufficient to establish that $(a_i, b_i)(zt, t) \in T'$ for each i, where (zt, t) is a typical element of T'. By (I) there exists $w \in T$ such that $a_i z = b_i w$. Hence

$$(a_i, b_i)(zt, t) = (a_i zt, b_i t) = (b_i wt, b_i t) \in T',$$

completing the proof.

DEFINITION. The extension constructed in the above proof will be called *the extension of T by G^0 associated with $a_1, b_1; \cdots; a_n, b_n$.*

2. Equivalence of extensions

If T is a semigroup and A is a semigroup with zero then extensions S_1 and S_2 of T by A are called *equivalent* if there is an isomorphism of S_1 onto S_2 which maps T onto itself. An extension S of T will be called an *M-extension* if T is a maximal ideal in S and is unique with this property. It is easy to observe (or see [1]) that any cancellative extension of a cancellative semigroup by a group with zero is an M-extension.

LEMMA 2.1. *Let T be a semigroup and A a semigroup with zero. If S_1 and S_2 are M-extensions of T by A then these extensions are equivalent if and only if they are isomorphic.*

PROOF. Let α be an isomorphism of S_1 onto S_2. Since T is the unique maximal

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ideal in each of S_1 and S_2 necessarily $T\alpha = T$. Hence S_1 and S_2 are equivalent. The converse is immediate from the definition of equivalence.

From the remarks preceding the lemma we have the following corollary.

COROLLARY. *If S_1 and S_2 are cancellative extensions of a cancellative semigroup by a group with zero then S_1 and S_2 are equivalent if and only if they are isomorphic.*

Let T be a commutative cancellative semigroup and let Q be the group of quotients of T. We identify T with its natural isomorph in Q, i.e. elements of T will be denoted by $(t, 1)$. (This is not to imply that T has an identity but rather we use $(t, 1)$ as opposed to (ta, a) for notational convenience.)

LEMMA 2.2. *Let T be a commutative cancellative semigroup and let Q be its group of quotients.*

(i) *Every automorphism α of T has a unique extension to an automorphism φ of Q, namely*

$$(a, b)\varphi = (a, 1)\alpha(b, 1)\alpha^{-1} \quad \textit{for all} \quad (a, b) \quad \textit{in} \quad Q.$$

(ii) *More generally if S_1 and S_2 are subsemigroups of Q each of which contains T and β is an isomorphism of S_1 onto S_2 such that $T\beta = T$ then β has a unique extension to an automorphism of Q.*

PROOF. (i) Let α be an automorphism of T and define φ as above. To show that φ is well defined let $(a, b) = (c, d)$. Then $(a, 1)(d, 1) = (b, 1)(c, 1)$. Since α is an automorphism of T,

$$[(a, 1)\alpha][(d, 1)\alpha] = [(b, 1)\alpha][(c, 1)\alpha].$$

Equivalently

$$[(a, 1)\alpha][(b, 1)\alpha]^{-1} = [(c, 1)\alpha][(d, 1)\alpha]^{-1}$$

which says $(a, b)\varphi = (c, d)\varphi$. Hence φ is well defined.

For any (x, y) in Q there exists $(a, 1)$, $(b, 1)$ in T such that $(a, 1)\alpha = (x, 1)$ and $(b, 1)\alpha = (y, 1)$. Then

$$(a, b)\varphi = [(a, 1)\alpha][(b, 1)\alpha]^{-1} = (x, 1)(y, 1)^{-1} = (x, y).$$

Hence φ maps Q onto Q.

Now

$$\begin{aligned}[(a_1, b_1)(a_2, b_2)]\varphi &= (a_1 a_2, b_1 b_2)\varphi = [(a_1 a_2, 1)\alpha][(b_1 b_2, 1)\alpha]^{-1} \\ &= [(a_1, 1)\alpha][(a_2, 1)\alpha][(b_1, 1)\alpha]^{-1}[(b_2, 1)\alpha]^{-1} = (a_1, b_1)\varphi(a_2, b_2)\varphi\end{aligned}$$

so φ is a homomorphism.

Finally if $(a_1, b_1)\varphi = (a_2, b_2)\varphi$ then

$$[(a_1, 1)\alpha][(b_1, 1)\alpha]^{-1} = [(a_2, 1)\alpha][(b_2, 1)\alpha]^{-1}$$

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from which it follows that $(a_1b_2, 1)\alpha = (a_2b_1, 1)\alpha$. Since α is $1-1$ $(a_1b_2, 1) = (a_2b_1, 1)$ so $a_1b_2 = a_2b_1$. Hence $(a_1, b_1) = (a_2, b_2)$ so φ is $1-1$.

To show the uniqueness of φ let η be an automorphism of Q such that η is an extension of α. Then

$$(a, b)\eta = [(a, 1)(b, 1)^{-1}]\eta = [(a, 1)\eta][(b, 1)\eta]^{-1} = [(a, 1)\alpha][(b, 1)\alpha]^{-1}.$$

Hence $\eta = \varphi$.

(ii) Let β be an isomorphism of S_1 onto S_2 which maps T onto T. Then $\bar{\beta}$, the restriction of β to T, is an automorphism of T and hence by (i) has a unique extension φ to an automorphism of Q. It remains to show that φ is an extension of β. Let $(a, b) \in S_1$. Then

$$[(a, b)\beta][(b, 1)\beta] = (ab, b)\beta = (a, 1)\beta.$$

Hence

$$(a, b)\beta = [(a, 1)\beta][(b, 1)\beta]^{-1} = (a, b)\varphi,$$

completing the proof of the lemma.

Before we can formulate the main theorem it is necessary to determine how one can tell whether a given set of elements in a finitely generated Abelian group G is a generating set for G or not.

Definition. Let $m_1, \cdots, m_n$ be positive integers or the symbol ∞. An $n \times n$ matrix $X = (x_{ij})$ over the integers will be called *right* $(m_1, \cdots, m_n)$*-invertible* if there exists an $n \times n$ matrix $Y = (y_{ij})$ over the integers such that

$$\sum_{k=1}^{n} x_{ik}y_{kj} \equiv \begin{cases} 1 \bmod m_j & \text{if } i = j \\ 0 \bmod m_i & \text{if } i \neq j \end{cases}$$

where we interpret $a \equiv b \bmod \infty$ to mean $a = b$. Equivalently if the i-th row of XY is reduced modulo m_i one obtains the usual identity matrix.

Lemma 2.3. *Let G be a finitely generated Abelian group with basis $a_1, \cdots, a_n$ and let $m_i = o(a_i)$ for $i = 1, \cdots, n$. Then the elements*

$$b_i = a_1^{x_{1i}}a_2^{x_{2i}} \cdots a_n^{x_{ni}} \qquad i = 1, \cdots, n$$

generate G if and only if the matrix $X = (x_{ij})$ is right $(m_1, \cdots, m_n)$-invertible.

Proof. Suppose $b_1, \cdots, b_n$ generate G. Then for each j there exist integers y_{ij} such that

$$a_j = b_1^{y_{1j}}b_2^{y_{2j}} \cdots b_n^{y_{nj}}. \tag{1}$$

So

$$\begin{aligned} a_j &= (a_1^{x_{11}} \cdots a_n^{x_{n1}})^{y_{1j}} \cdots (a_1^{x_{1n}} \cdots a_n^{x_{nn}})^{y_{nj}} \\ &= a_1^{\Sigma x_{1k}y_{kj}} \cdots a_n^{\Sigma x_{nk}y_{kj}} \end{aligned}$$

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where each sum ranges from $k = 1$ to $k = n$. Hence

$$\sum_{k=1}^{n} x_{ik}\, y_{kj} \equiv \begin{cases} 1 \bmod m_j & \text{if } i = j \\ 0 \bmod m_i & \text{if } i \neq j \end{cases}. \tag{2}$$

Conversely suppose there exists a matrix $Y = (y_{ij})$ which satisfies (2). Reversing the above argument shows that the equations (1) hold and hence $b_1, \cdots, b_n$ generate G.

We are now ready to state the main theorem on the equivalence of extensions.

THEOREM 2. *Let T be a commutative cancellative semigroup without idempotent and G a finitely generated Abelian group with basis $g_1, \cdots, g_n$ where $o(g_i) = m_i$ for $i = 1, \cdots, n$. Let $a_1, b_1; \cdots; a_n, b_n$; and $c_1, d_1; \cdots; c_n, d_n$; be two sets of n distinct pairs of elements of T satisfying conditions* (I) *and* (II) *of Theorem* 1.

If S_1 and S_2 are the associated cancellative extensions of T by G^0 then S_1 and S_2 are equivalent if and only if there is an automorphism α of T such that

$$(a_j\alpha)d_1^{x_{1j}} \cdots d_n^{x_{nj}} = (b_j\alpha)c_1^{x_{1j}} \cdots c_n^{x_{nj}} \qquad \text{for } j = 1, \cdots, n \tag{$*$}$$

where $X = (x_{ij})$ is a right $(m_1, \cdots, m_n)$-invertible matrix.

PROOF. Identify T with its natural isomorph in its group of quotients Q. Let G_1 be the subgroup of Q with basis $(a_1, b_1), \cdots, (a_n, b_n)$ and G_2 the subgroup of Q with basis $(c_1, d_1), \cdots, (c_n, d_n)$. Then $S_1 = G_1 \cup T$ and $S_2 = G_2 \cup T$.

If α is an automorphism of T satisfying $(*)$ then by Lemma 2.2 α has a unique extension to an automorphism of Q, namely the mapping φ defined by

$$(a, b)\varphi = [(a, 1)\alpha][(b, 1)\alpha]^{-1} \qquad \text{for all } (a, b) \text{ in } Q.$$

We then have for $j = 1, \cdots, n$

$$(a_j, b_j)\varphi = [(a_j, 1)\alpha][(b_j, 1)\alpha]^{-1} = (c_1, d_1)^{x_{1j}} \cdots (c_n, d_n)^{x_{nj}},$$

the last equality following from $(*)$. Now since $X = (x_{ij})$ is right $(m_1, \cdots, m_n)$-invertible Lemma 2.3 guarantees that the n elements $(a_j, b_j)\varphi$, $j = 1, \cdots, n$, generate G_2 and hence φ maps G_1 onto G_2. The restriction of φ to S_1 is then an isomorphism of S_1 onto S_2. Hence S_1 and S_2 are equivalent by Lemma 2.1.

Conversely suppose S_1 and S_2 are equivalent and let β be an isomorphism of S_1 onto S_2 such that $T\beta = T$. If φ is the unique extension of β to an automorphism of Q and α is the restriction of β to T then α is an automorphism of T. Also, by Lemma 2.2,

$$(a_j, b_j)\varphi = [(a_j, 1)\alpha][(b_j, 1)\alpha]^{-1} \qquad j = 1, \cdots, n. \tag{3}$$

But the n elements $(a_1, b_1)\varphi, \cdots, (a_n, b_n)\varphi$ must be a basis for G_2. Consequently if we write

$$(a_j, b_j)\varphi = (c_1, d_1)^{x_{1j}} \cdots (c_n, d_n)^{x_{nj}} \qquad j = 1, \cdots, n \tag{4}$$

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it follows from Lemma 2.3 that the matrix $X = (x_{ij})$ is right $(m_1, \cdots, m_n)$-invertible. It now follows readily from (3) and (4) that α satisfies $(*)$.

It is worthwhile to note the special case where G is cyclic of order m_1. We then have $n = 1$ and the condition that $X = (x_{11})$ is right m_1-invertible just amounts to the condition that the congruence $x_{11} z \equiv 1 \bmod m_1$ is solvable for z, i.e. x_{11} and m_1 are relatively prime if m_1 is finite or $x_{11} = \pm 1$ if $m_1 = \infty$.

Reference

[1] Charles V. Heuer and Donald W. Miller, 'An extension problem for cancellative semigroups', *Trans. Amer. Math. Soc.* 122 (1966), 499—515.

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