NOTE ON A DIFFERENCE-PRODUCT INEQUALITY

by A. C. AITKEN
(Received 2nd March, 1962)

1. L. J. Mordell has recently considered (**1**) the squared modulus of a complex difference-product, namely

$$\Delta = \prod_{r>s\geqq 1} |z_r - z_s|^2, \quad \text{.............................(1)}$$

under the conditions

$$|z_r| = 1, \quad r = 1, 2, \ldots, n, \quad \text{.............................(2)}$$

and also under the quite different condition

$$\sum_{r=1}^{n} |z_r|^2 = n. \text{.......................................(3)}$$

He proves that under (2) the maximum of Δ is n^n, and is attained when and only when the z_r are vertices of a regular n-gon on the unit circle.

He proves also that the same holds under condition (3) for the special case $n = 3$, pointing out, by a counter-example due to J. H. H. Chalk, that the corresponding result for $n > 5$ cannot hold, since values of the z_r can then be exhibited for which $\Delta > n^n$. Chalk's example is interesting; z_1 is the origin, the remaining $n-1$ points z_r make a regular $(n-1)$-gon about this centre, but the radius is $\left(\dfrac{n}{n-1}\right)^{\frac{1}{2}}$. It is in fact easy to show that for this configuration we have

$$\Delta = n^{\frac{1}{2}n(n-1)}/(n-1)^{\frac{1}{2}(n-1)(n-2)},$$

and it is no trouble to show that this exceeds n^n when $n > 5$.

We give alternative proofs of the results in question. With respect to (1) under (2) we may without loss of generality take

$$z_1 = 1, \quad z_r = \exp i\theta_{r-1}, \quad r = 2, 3, \ldots, n. \quad \text{..................(4)}$$

Consider now the alternant matrix

$$U = [u_{rs}], \quad u_{rs} = \exp\{i(s-1)\theta_{r-1}\}. \text{........................(5)}$$

Its determinant is the difference-product of the problem. Also $\overline{U}'U$, of determinant Δ, is positive definite Hermitian with diagonal elements all equal to n. Thus by a typical property the maximum of Δ is the product n^n of all the diagonal elements, and is attained only when all non-diagonal elements vanish. These, above the diagonal, are the sums-of-powers symmetric functions of the z_r, namely s_j, $j = 1, 2, \ldots, n-1$; below the diagonal, their conjugates. If these s_j all vanish, so also, being isobaric in them, do the corresponding elementary symmetric functions. Hence the z_r are the roots of $z^n - 1 = 0$, that is, the nth roots of 1, and the theorem is established.

2. Under the other, and more difficult condition (3), the proof of the result for the case $n = 3$ could similarly be set out in terms of complex numbers, but is easier to see geometrically. Let ABC be a plane triangle of fixed shape and O any point in its plane. Condition (3) then implies that

$$OA^2 + OB^2 + OC^2 = 3, \qquad \text{(6)}$$

and we have to prove that $BC.CA.AB$ is maximum under (6) when and only when ABC is equilateral and O is its centroid. But given any triangle ABC it is elementary that $OA^2 + OB^2 + OC^2$ is minimum when and only when O is the centroid. Reciprocally then, among triangles similar to ABC and such that (6) holds, the one of greatest area, and therefore such that $BC.CA.AB$ is maximum, is that which has O placed at its centroid. But again, by familiar properties of medians, $BC^2 + CA^2 + AB^2 = 9$. It follows that $BC.CA.AB$ is maximum when $BC = CA = AB$, and the second result is thus proved.

REFERENCE

(1) L. J. MORDELL, On a discriminant inequality, *Can. J. Math.* xii (1960), 699-704.