The tables of John Wallis and the discovery of his product for π

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Introduction

In the year 1656 John Wallis published his *Arithmetica Infinitorum*, [1], in which he displayed many ideas that were to lead to the integral calculus of Newton. In this work we find the celebrated infinite product of Wallis which gives π,

\[ \frac{2}{\pi} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{15}{16} \cdots \]  

(1)

Earlier in 1593, Vieta [2] found another infinite product which gives π

\[ \frac{2}{\pi} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2}} \cdots \]

But, since Wallis does not mention it, we suppose that he was unaware of it. (Remarkably, these two seemingly different products are special cases of a more general formula [3]e.) The thoughts that lead Wallis to (1) are quite surprising and ingenious. It is the purpose of this paper to show to modem readers the brilliance of Wallis' thinking in his discovery of (1).

Using modern notation, we can say that Wallis knew the integration formula

\[ \int_0^1 x^p \, dx = \frac{c^{p+1}}{p+1}, \]  

(2)

and could use it for values of \( p \) that were both integers and fractions. Wallis wanted to find some convenient expression for the area bound by the unit circle 'in terms of integers', and (again in modern notation), he wanted to evaluate the integral

\[ \int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4}. \]  

(3)

The binomial theorem for fractional exponents had not yet been discovered, and so, knowing only (2), Wallis had no direct way of evaluating the integral (3). Instead, Wallis used a brilliant method of interpolation. He reasoned that the value of the integral (3) was between the two integrals

\[ \int_0^1 (1 - x^2)^0 \, dx \]  

and

\[ \int_0^1 (1 - x^2)^1 \, dx, \]

and, of course, he could evaluate both of these. To achieve this interpolation, he created a table of values of the reciprocal integrals \( 1 / \int_0^1 (1 - x^Q)^P \, dx \) for special choices of \( P \) and \( Q \) for which the integral would reduce to (2). (The reason for the reciprocal was to obtain more integer values in the table, as will be revealed.) A very careful study of this table led Wallis to tease out his product (1).
The modern reader will likely find Wallis' description of his method of discovery in his book [1] tough going. Nunn has given a very good explanation in [4] and [5] using many of Wallis' original notations and thoughts. It is our hope to present an even more accessible description of how Wallis reasoned using modern notation and a careful step by step description of his tables. Stedall [6] and Dunka [7] give further historical information. We do not follow Wallis' every thought, for this would obscure the essential beauty of the process by which he made his interpolations based on the scant evidence available in the table. We take the liberty to show the overall manner of discovery without following Wallis exactly.

We hope that young readers, trained largely in the rigours of mathematics, will find Wallis' numerous daring guesses an insight into the creative thought process.

How Wallis did it, in modern notation

1. Following Wallis, we first compute a table of the reciprocal integral \( \frac{1}{1^0} (1 - x^{1/Q})^P dx \) for integer \( P \) and \( Q \). We recognise these numbers as binomial coefficients and record them as such in Table 1. No doubt Wallis also made this observation; however it was of little use to him because of the binomial theorem for fractional exponents not being known. The formula \( \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \) was also unknown, but we

<table>
<thead>
<tr>
<th></th>
<th>( P = 0 )</th>
<th>( P = \frac{1}{2} )</th>
<th>( P = 1 )</th>
<th>( P = \frac{3}{2} )</th>
<th>( P = 2 )</th>
<th>( P = \frac{5}{2} )</th>
<th>( P = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = 0 )</td>
<td>( \binom{0}{0} )</td>
<td>( \binom{1}{0} )</td>
<td>( \binom{2}{0} )</td>
<td>( \binom{3}{0} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = \frac{1}{2} )</td>
<td>( \frac{4}{\pi} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = 1 )</td>
<td>( \binom{1}{1} )</td>
<td>( \binom{2}{1} )</td>
<td>( \binom{3}{1} )</td>
<td>( \binom{4}{1} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = \frac{3}{2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = 2 )</td>
<td>( \binom{2}{2} )</td>
<td>( \binom{3}{2} )</td>
<td>( \binom{4}{2} )</td>
<td>( \binom{5}{2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = \frac{5}{2} )</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q = 3 )</td>
<td>( \binom{3}{3} )</td>
<td>( \binom{4}{3} )</td>
<td>( \binom{5}{3} )</td>
<td>( \binom{6}{3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 1:** The reciprocal integral \( \frac{1}{1^0} (1 - x^{1/Q})^P dx \) for integer \( P \) and \( Q \) with values expressed as binomial coefficients.
shall see that Wallis discovers this expression (Table 3) without making the
connection to the binomial theorem. The binomial theorem for fractional
exponents would have to wait for Newton, who credits reading Wallis for
his discovery.

We will use the symbol \( \{Q, P\} \) to denote the entry in a cell of the table
where \( Q \) is the row and \( P \) is the column. Notice that \( \left\{ \frac{1}{2}, \frac{1}{2} \right\} = \frac{4}{\pi} \) since we
know that \( \int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4} \).

We also notice that the values obtained are symmetric. The number
\( \{P, Q\} \) is the same as the number \( \{Q, P\} \).

We see that in the \( Q \) th row and \( P \) th column we find the entry \( \{Q, P\} = \left( \frac{P + Q}{Q} \right) \).

2. Wallis can also calculate the reciprocal integral when \( P \) is an integer and
\( Q \) is an integer plus one half. See Table 2. Using the symmetry observed
above, and assuming that it remains true throughout the table we can enter
values into cells where \( P \) is an integer plus one half and \( Q \) is an integer as
also shown in Table 2. This is very important. Wallis cannot directly
evaluate the reciprocal integral \( \int_0^1 \frac{1}{(1 - x^{1/Q})^P} \, dx \) when \( P \) is a fraction.
However, because of the symmetry, he now conjectures the exact numerical
values that are in many cells which would otherwise remain unknown.

<table>
<thead>
<tr>
<th>( Q = 0 )</th>
<th>( Q = \frac{1}{2} )</th>
<th>( Q = 1 )</th>
<th>( Q = \frac{3}{2} )</th>
<th>( Q = 2 )</th>
<th>( Q = \frac{5}{2} )</th>
<th>( Q = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P = 0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( P = \frac{1}{2} )</td>
<td>( \frac{4}{\pi} )</td>
<td>( \frac{3}{2} )</td>
<td>( 15 )</td>
<td>( 8 )</td>
<td>105</td>
<td>( \frac{48}{105} )</td>
</tr>
<tr>
<td>( P = 1 )</td>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>2</td>
<td>( \frac{5}{2} )</td>
<td>3</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>( P = \frac{3}{2} )</td>
<td>1</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{35}{8} )</td>
<td>6</td>
<td>( \frac{63}{8} )</td>
<td>10</td>
</tr>
<tr>
<td>( P = 2 )</td>
<td>1</td>
<td>( \frac{15}{8} )</td>
<td>3</td>
<td>( \frac{35}{8} )</td>
<td>6</td>
<td>63</td>
</tr>
<tr>
<td>( P = \frac{5}{2} )</td>
<td>1</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{63}{8} )</td>
<td>( \frac{693}{48} )</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>( P = 3 )</td>
<td>1</td>
<td>4</td>
<td>( \frac{315}{48} )</td>
<td>( \frac{693}{48} )</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 2: Extending the table first by integration to fractional \( Q \) then to fractional \( P \)
using symmetry.

This leaves empty only the cells where neither \( P \) nor \( Q \) is an integer.
3. Next we follow Wallis and rewrite the numbers in cells where $P$ and $Q$ are integers as expressions that reveal how we can proceed, left to right, from one cell with integer $P$ to the next. Wallis would have discovered these expressions after an exhaustive study of the entries in the table and a series of guesses, many of which, no doubt, would fail. As mentioned in step 1, Wallis has conjectured
\[
\binom{n}{k} = \frac{n \cdot n - 1 \cdot n - 2 \cdot \ldots \cdot n - k + 1}{1 \cdot 2 \cdot 3 \ldots k},
\]
without recognising it.

<table>
<thead>
<tr>
<th></th>
<th>$P = 0$</th>
<th>$P = \frac{1}{2}$</th>
<th>$P = 1$</th>
<th>$P = \frac{3}{2}$</th>
<th>$P = 2$</th>
<th>$P = \frac{5}{2}$</th>
<th>$P = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q = 0$</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1.2}{1}$</td>
<td>1</td>
<td>$\frac{1.2.3}{1.2.3}$</td>
</tr>
<tr>
<td>$Q = \frac{1}{2}$</td>
<td>1</td>
<td>$\frac{4}{\pi}$</td>
<td>$\frac{3}{2}$</td>
<td>15</td>
<td>$\frac{35}{8}$</td>
<td>105</td>
<td>$\frac{315}{48}$</td>
</tr>
<tr>
<td>$Q = 1$</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1.2}{1}$</td>
<td>5</td>
<td>$\frac{1.2.3}{1}$</td>
<td>7</td>
<td>$\frac{1.2.3.4}{1.2.3}$</td>
</tr>
<tr>
<td>$Q = \frac{3}{2}$</td>
<td>1</td>
<td>$\frac{5}{2}$</td>
<td>35</td>
<td>63</td>
<td>$\frac{3.4}{1}$</td>
<td>$\frac{693}{48}$</td>
<td>$\frac{1.2.3}{1.2.3}$</td>
</tr>
<tr>
<td>$Q = 2$</td>
<td>1</td>
<td>$\frac{15}{8}$</td>
<td>$\frac{1.3}{1}$</td>
<td>$\frac{35}{8}$</td>
<td>$\frac{1.2.3}{1}$</td>
<td>693</td>
<td>$\frac{4.5.6}{1.2.3}$</td>
</tr>
<tr>
<td>$Q = \frac{5}{2}$</td>
<td>1</td>
<td>$\frac{7}{2}$</td>
<td>$\frac{63}{8}$</td>
<td>$\frac{693}{48}$</td>
<td>$\frac{1.2.3}{1}$</td>
<td>$\frac{1.2.3}{1.2.3}$</td>
<td></td>
</tr>
<tr>
<td>$Q = 3$</td>
<td>1</td>
<td>$\frac{105}{48}$</td>
<td>$\frac{1.4}{1}$</td>
<td>$\frac{315}{48}$</td>
<td>$\frac{4.5}{1}$</td>
<td>$\frac{693}{48}$</td>
<td>$\frac{4.5.6}{1.2.3}$</td>
</tr>
</tbody>
</table>

**TABLE 3:** Replacing cells with integer $P$ by ‘growth revealing expressions’

4. Wallis now considers the expressions that should appear in intermediate cells above and below those just entered. For this purpose it seems natural to double every value just entered, for then a sequence of increasing integers will fit in the numerators as we descend a column. (We note that the general expression he found can be expressed as
\[
\{Q, P\} = 1 \cdot 2Q + 2 \cdot 2Q + 4 \cdot 2Q + 6 \cdot 2Q + 2P.
\]

5. These expressions immediately suggest how we can write the numbers in adjacent cells above and below those just entered. This completes all the columns where $P$ is an integer in Table 5.

Wallis now fills in the cells where $Q$ is an integer, but $P$ is a fraction. Looking at the rows in the previous table where we used denominators 2, 4, 6, etc., Wallis now tries denominators 1, 3, 5, etc. By experimenting, he


\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& P = 0 & P = \frac{1}{2} & P = 1 & P = \frac{3}{2} & P = 2 & P = \frac{5}{2} \\
\hline
Q = 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
Q = \frac{1}{2} & 1 & \frac{4}{\pi} & 3 & 1.5 & 1.5 & 1.5 \\
\hline
Q = 1 & 1 & \frac{3}{2} & 1.5 & 1.5 & 1.5 & 1.5 \\
\hline
Q = \frac{3}{2} & 1 & 5 & 2 & 2 & 2 & 2 \\
\hline
Q = 2 & 1 & 15 & 8 & 6 & 6 & 6 \\
\hline
Q = \frac{5}{2} & 1 & 7 & 2 & 2 & 2 & 2 \\
\hline
Q = 3 & 1 & \frac{105}{48} & 8 & 315 & 315 & 315 \\
\hline
\end{array}
\]

TABLE 4: Replacing cells by improved 'growth revealing expressions'

finds that he can express the values conveniently this way. For example, in the row where \( Q = 1 \), the numbers proceed to the right in the form

\[
x = \frac{3 \cdot 5 \cdot 7}{1 \cdot 3 \cdot 5} \ldots
\]

where \( x = \frac{1}{2} \). The choice of this number \( x = \frac{1}{2} \) in Table 5 is made by

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& P = 0 & P = \frac{1}{2} & P = 1 & P = \frac{3}{2} & P = 2 & P = \frac{5}{2} \\
\hline
Q = 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
Q = \frac{1}{2} & 1 & \frac{4}{\pi} & 3 & 1.5 & 1.5 & 1.5 \\
\hline
Q = 1 & 1 & \frac{3}{2} & 1.5 & 1.5 & 1.5 & 1.5 \\
\hline
Q = \frac{3}{2} & 1 & 5 & 2 & 2 & 2 & 2 \\
\hline
Q = 2 & 1 & 15 & 8 & 6 & 6 & 6 \\
\hline
Q = \frac{5}{2} & 1 & 7 & 2 & 2 & 2 & 2 \\
\hline
Q = 3 & 1 & \frac{105}{48} & 8 & 315 & 315 & 315 \\
\hline
\end{array}
\]

TABLE 5: More growth revealing expressions in rows where \( Q \) is a fraction followed by new growth revealing expressions in columns where \( P \) is a fraction.
examining the corresponding numbers in Table 4, and by inspection, observing that this number always produces the correct entry in the row where \( Q = 1 \) and \( P \) is a fraction. The successive factors \( x = 1, \frac{1}{2}, \frac{3}{8} \) and \( \frac{5}{8} \) that appear first in the column where \( P = \frac{1}{2} \) are unusual and must be determined one at a time as each row is examined. The fact that the number \( x \) selected in column \( P = \frac{1}{2} \) produces the correct entry for cells to the right is remarkable, and convinces Wallis that he is on the right track. This is perhaps Wallis' most brilliant conjecture.

6. Looking at the column where \( P = \frac{1}{2} \). It seems clear from other expressions just discovered that the value \( \frac{4}{\pi} \) in cell \( \left\{ \frac{1}{2}, \frac{1}{2} \right\} \) should be written as \( \frac{2}{\pi} \cdot \frac{2}{1} \), and the remaining empty cells in that row with \( \frac{2}{\pi} \cdot \frac{2}{1} \) and \( \frac{2}{\pi} \cdot \frac{2}{1} \).

<table>
<thead>
<tr>
<th>( P )</th>
<th>( P = 0 )</th>
<th>( P = \frac{1}{2} )</th>
<th>( P = 1 )</th>
<th>( P = \frac{3}{2} )</th>
<th>( P = 2 )</th>
<th>( P = \frac{5}{2} )</th>
<th>( P = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = 0 )</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( Q = \frac{1}{2} )</td>
<td>1</td>
<td>( \frac{2}{\pi} )</td>
<td>( \frac{3}{\pi} )</td>
<td>( \frac{3}{\pi} )</td>
<td>( \frac{3}{\pi} )</td>
<td>( \frac{3}{\pi} )</td>
<td>( \frac{3}{\pi} )</td>
</tr>
<tr>
<td>( Q = 1 )</td>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
</tr>
<tr>
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<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{5}{2} )</td>
</tr>
<tr>
<td>( Q = 2 )</td>
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<td>( \frac{35}{8} )</td>
<td>( \frac{35}{8} )</td>
<td>( \frac{35}{8} )</td>
<td>( \frac{35}{8} )</td>
<td>( \frac{35}{8} )</td>
<td>( \frac{35}{8} )</td>
</tr>
<tr>
<td>( Q = \frac{5}{2} )</td>
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<td>( \frac{7}{2} )</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{7}{2} )</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>( Q = 3 )</td>
<td>1</td>
<td>( \frac{57}{8} )</td>
<td>( \frac{57}{8} )</td>
<td>( \frac{57}{8} )</td>
<td>( \frac{57}{8} )</td>
<td>( \frac{57}{8} )</td>
<td>( \frac{57}{8} )</td>
</tr>
</tbody>
</table>

TABLE 6: Completing the critical row with \( Q = \frac{1}{2} \)

We could now fill the remaining empty cells with similar expressions, but this is unnecessary for our purpose, which is to find an expression for \( \pi \).
7. Wallis has now surrounded the critical cell \( \left\{ \frac{1}{2}, \frac{1}{2} \right\} \) with expressions revealing the growth of numbers as we move through the table. In his final step, he examines this growth carefully.

Call the ratio of two successive entries in the row where \( Q = \frac{1}{2} \),

\[
R(n) = \frac{\left\{ \frac{1}{2}, n + \frac{1}{2} \right\}}{\left\{ \frac{1}{2}, n + 1 \right\}}
\] (4)

and notice that

\[
R(0) = \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{2}{\pi \frac{1}{3}},
\]
\[
R(1) = \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{2}{\pi \frac{1}{3} \cdot \frac{3}{5}},
\]
\[
R(2) = \frac{\frac{3}{5}}{\frac{3}{5}} = \frac{2}{\pi \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{3}{7}},
\]

and in general we have

\[
R(n) = \frac{2}{\pi \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{3}{7} \cdot \ldots \cdot \frac{(2n + 1)(2n + 3)}{(2n + 2)(2n + 3)}}
\] (5)

Thus it is clear that

\[
\frac{2}{\pi} = \frac{R(n)}{2 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n + 2)(2n + 2) R(n)}.
\]

What can we say about \( R(n) \) as \( n \) grows large? To answer this question Wallis examines another ratio in this row where \( Q = \frac{1}{2} \). He looks at the ratio of two entries with integer values of \( P \). He observes

\[
\left\{ \frac{1}{2}, 1 \right\} = 5 \left\{ \frac{1}{2}, 2 \right\} = 6 \left\{ \frac{1}{2}, 3 \right\} = 8 \left\{ \frac{1}{2}, 4 \right\} = \frac{8}{9}, \text{ etc.}
\]

and it is clear that

\[
\lim_{n \to \infty} \frac{\left\{ \frac{1}{2}, n \right\}}{\left\{ \frac{1}{2}, n + 1 \right\}} = 1.
\] (6)
From this limit Wallis feels safe in assuming also

\[
\lim_{n \to \infty} \left\{ \frac{1}{n} \left( \frac{1}{2} + \frac{\pi}{2} \right) \right\} = \lim_{n \to \infty} R(n) = 1.
\]

Returning to (5) and letting \( n \) tend to infinity, Wallis now has his product

\[
\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}.
\]

This completes our explanation of how Wallis conjectured his product.

Final remarks

The tables of Wallis did not end with the discovery of the Wallis product. His colleague Lord Brouncker used the tables to discover a sequence of continued fractions for \( \pi \), the first of which is

\[
\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \ldots}}}}.
\]

After seeing these tables, Newton conjectured the binomial theorem for fractional exponents. Euler credits Wallis for ideas that led to his discovery of the gamma function and its properties and further work in continued fractions.

References