BULL. AUSTRAL. MATH. SOC. 16A70
VOL. 21 (1980), 43-46.

A COMMUTATIVITY THEOREM FOR DIVISION RINGS

HAZAR ABU-KHUZAM AND ADIL YAQUB

The following theorem is proved: *Let* D *be a division ring such that for all* x, y *in* D *there exists a positive integer* $n = n(x, y)$ *for which* $(xy)^n - (yx)^n$ *is in the center of* D . *Then* D *is commutative.* This theorem also holds for semisimple rings.

It is well known [1] that a division ring D with the property that, for all x, y in D , $xy - yx$ is in the center of D must be commutative. Our objective is to generalize this theorem by assuming instead that $(xy)^{n(x,y)} - (yx)^{n(x,y)}$ is always in the center. Indeed, we prove the following:

THEOREM 1. *Let* D *be a division ring such that for all* x, y *in* D *there exists a positive integer* $n = n(x, y)$ *for which* $(xy)^n - (yx)^n$ *is in the center* Z *of* D . *Then* D *is commutative.*

We also show that this theorem holds for semisimple rings. As usual, for any a, b in R , $[a, b] = ab - ba$.

Proof of Theorem 1. Let x, y be any nonzero elements of D . By hypothesis, there exists a positive integer $n = n\left(xy^{-1}, y\right)$ such that

$$\left(\left(xy^{-1}\right)y\right)^n - \left(y\left(xy^{-1}\right)\right)^n \in Z .$$

This implies that $x^n - yx^ny^{-1} \in Z$ and hence

Received 2 August 1979.

https://doi.org/10.1017/S000497270001128X Published online by Cambridge University Press

$$\left(x^n-yx^ny^{-1}\right)y = y\left(x^n-yx^ny^{-1}\right) .$$

Therefore, $x^ny - yx^n = yx^n - y^2x^ny^{-1}$, and hence

$$\left(x^ny-yx^n\right)y = \left(yx^n-y^2x^ny^{-1}\right)y = yx^ny - y^2x^n = y\left(x^ny-yx^n\right) .$$

We have thus shown that

(1) y commutes with $[x^n, y]$ $\left(x, y \in D, n = n(x, y) \ge 1\right)$.

We now distinguish two cases.

CASE 1. Characteristic of $D = p > 0$.

By (1) and induction, we see that

(2) $[x^n, y^k] = ky^{k-1}[x^n, y]$, for all positive integers k .

Let $k = p$ in (2). Then, since D is of characteristic p ,

$$[x^n, y^p] = 0 \text{ , for all } x, y \text{ in } D .$$

Hence, by a well known theorem of Herstein [2], D is commutative.

CASE 2. Characteristic of D is zero.

By (1),

(3) y commutes with $[x^n, y]$ $\left(n = n(x, y) \ge 1\right)$.

By (1) again,

(4) x^n commutes with $[x^n, y^m]$ $\left(m = m(x^n, y) \ge 1\right)$.

By (3) and induction, we see that

(5) $[x^n, y^m] = my^{m-1}[x^n, y]$.

Combining (4) and (5), we obtain

$$[x^n, my^{m-1}[x^n, y]] = 0 = m[x^n, y^{m-1}[x^n, y]] .$$

Since D is of characteristic zero, we get

$$[x^n, y^{m-1}[x^n, y]] = 0 ,$$

and thus

(6) x^n commutes with $y^{m-1}[x^n, y]$.

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Combining (3) and (6), we conclude that

(7) yx^n commutes with $y^{m-1}[x^n, y]$.

Now, by (1),

(8) yx^n commutes with $[yx^n, (y^m)^k]$ $(k = k(yx^n, y^m) \geq 1)$.

Moreover, as is readily verified,

(9) $$[yx^n, y^{mk}] = y[x^n, y^{mk}] .$$

But, by (1), $[x^n, y^{mk}] = mky^{mk-1}[x^n, y]$, and hence by (9),

(10) $$[yx^n, y^{mk}] = mky^{mk}[x^n, y] .$$

So, by (8) and (10),

(11) yx^n commutes with $mky^{mk}[x^n, y]$.

Since D is of characteristic zero, (11) implies that

(12) yx^n commutes with $y^{mk}[x^n, y]$.

Now suppose for the moment that $[x^n, y] \neq 0$. Then, by (7),

(13) yx^n commutes with $[x^n, y]^{-1}y^{-(m-1)}$.

Combining (12) and (13), we conclude that

(14) yx^n commutes with y^{mk-m+1} .

Let $l = mk - m + 1$. Clearly $l \geq 1$ and hence by (14), $(yx^n)y^l = y^l(yx^n)$. Therefore

(15) $$x^ny^l = y^lx^n \quad (x, y \in D, l \geq 1) .$$

Clearly (15) holds if $[x^n, y] = 0$. Hence, by Herstein's Theorem [2], D is commutative. This proves the theorem.

Next we condiser the semisimple case. Thus suppose that R is a semisimple ring such that, for all x, y in R , there exists a positive integer $n = n(x, y)$ for which

(16) $$(xy)^n - (yx)^n \in Z \text{ [= center of } R] .$$

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Note that the property in (16) is inherited by all subrings and all homomorphic images of R. Note also that *no complete matrix ring* D_m *over a division ring* D, *with* $m > 1$, *satisfies the property in* (16), as a consideration of $x = E_{11}$, $y = E_{11} + E_{12}$ shows. Using these facts and the structure theory of rings, we see that Theorem 1 holds for semisimple rings as well. We omit the details.

References

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[2] I.N. Herstein, "A commutativity theorem", *J. Algebra* 38 (1976), 112-118.

Department of Mathematics,
University of California,
Santa Barbara,
California 93106,
USA.

https://doi.org/10.1017/S000497270001128X Published online by Cambridge University Press