ESTIMATING $|\alpha - p/q|$

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Abstract

An estimate for $q^2|\alpha - p/q|$ is obtained by considering the relation between the continued fractions for $\alpha$ and $p/q$. This leads to an extension of the standard result "$q^2|\alpha - p/q| < 1$ implies that for some $n, p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where $i = 0, 1$ or $a_{n+1} - 1$".


1. Introduction

We shall take $\alpha$ to be a positive real irrational number and $p/q$ to be a rational approximation to $\alpha$, written in reduced form (so that $p$ and $q$ are relatively prime positive integers). We write both $\alpha$ and $p/q$ as simple continued fractions

$$\alpha = [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots],$$
$$p/q = [a_0, a_1, \ldots, a_n, b_1, \ldots, b_r],$$

where, without loss of generality, $b_r \geq 2$. We assume $p/q$ is not a convergent $p_m/q_m$ to $\alpha$, so we may require $b_1 \neq a_{n+1}$ and $r > 1$.

There are a number of results on $|\alpha - p/q|$. For example there are two classical results (see Lang (1966)):

CR1. If $q^2|\alpha - p/q| < \frac{1}{2}$ then $p/q$ is a convergent to $\alpha$.

CR2. If $q^2|\alpha - p/q| < 1$ then, for some $n, p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where $i = 0, 1$ or $a_{n+1} - 1$.

More recently I had need to know what could be said if $q^2|\alpha - p/q| < 2$. In Worley (1977) it was shown that in this case $p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where either $i$ is an integer, $0 < i < a_{n+1} - 1$, or $i$ is a rational number of the form $j + \frac{1}{2}$ where $0 < j < a_{n+1} - 1$ and $j$ is integral.

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The purpose of this paper is to strengthen and generalize this last result. To do so we make use of the following results.

**Lemma 1** (Niven and Zuckerman (1972), Theorem 7.3). For any positive real number \(x\),
\[
[a_0, a_1, \ldots, a_n, x] = \frac{(xp_n + p_{n-1})}{(xq_n + q_{n-1})}.
\]

**Lemma 2** (Niven and Zuckerman (1972), Theorem 7.5).
\[
|p_nq_{n-1} - p_{n-1}q_n| = 1.
\]

**Lemma 3** (Worley (1973), Lemma 1). Let
\[
P_m/Q_m = \left[ 0, a_{m+2}, a_{m+3}, \ldots, a_{m+k} \right],
\]
where the continued fraction is to be interpreted as 0/1 if \(k = 1\). Then
\[
q_{m+k} = q_mQ_m(\left[ a_{m+1}, \ldots, a_{m+k} \right] + q_{n-1}/q_n).
\]

In the notation introduced earlier, Lemma 3 shows that
\[
q = q_n d([b_1, \ldots, b_r] + q_{n-1}/q_n),
\]
where \(d\) is the denominator of \([b_1, \ldots, b_r]\).

By combining the previous three lemmas we obtain

**Lemma 4.** If \(p/q\) is not a convergent to \(\alpha\), then
\[
q^2|\alpha - p/q| > \begin{cases} d^2(\beta - \beta^2/\gamma) & \text{if } \beta < \gamma, \\ d^2(\beta - \gamma)(\beta + 1)/(\gamma + 1) & \text{if } \beta > \gamma, \end{cases}
\]
where \(\beta = [b_1, \ldots, b_r]\) and \(\gamma = [a_{n+1}, a_{n+2}, \ldots]\) using the notation introduced earlier.

**Proof.** By Lemmas 1 and 2
\[
|\alpha - p/q| = \frac{|\gamma p_n + p_{n-1} - \beta p_n + p_{n-1}|}{|\gamma q_n + q_{n-1} - \beta q_n + q_{n-1}|} = \frac{|\gamma - \beta|}{(\gamma q_n + q_{n-1})(\beta q_n + q_{n-1})}.
\]

Using the comment after the statement of Lemma 3 this gives
\[
q^2|\alpha - p/q| = \frac{d^2|\gamma - \beta|(\beta + q_{n-1}/q_n)}{(\gamma + q_{n-1}/q_n)}.
\]

Since \((\beta + x)/(\gamma + x)\) is, for \(0 < x < 1\), an increasing function of \(x\) if \(\beta < \gamma\) and a decreasing function if \(\beta > \gamma\) the lemma follows.
2. The main results

By close analysis of the bounds given in Lemma 4, we obtain the following theorem, which includes the classical results CR1 and CR2 as the special cases $k = \frac{1}{2}$ and $k = 1$.

**Theorem 1.** If $\alpha$ is irrational, $k > \frac{1}{2}$, and $p/q$ is a rational approximation to $\alpha$ (in reduced form) for which

$$q^2|\alpha - p/q| < k$$

then either $p/q$ is a convergent $p_n/q_n$ to $\alpha$ or $p/q$ has one of the following forms:

\begin{align*}
\text{(i)} & \quad \frac{p}{q} = \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}}, \quad ab < 2k, \\
\text{(ii)} & \quad \frac{p}{q} = \frac{ap_n - bp_{n-1}}{aq_n + bq_{n-1}}, \quad ab < 2k, \\
\text{(iii)} & \quad \frac{p}{q} = \frac{ap_{n+1} + bp_{n-1}}{aq_{n+1} + bq_{n-1}}, \quad ab < k, a_{n+1} = 1,
\end{align*}

where $a$ and $b$ are positive integers.

To obtain the symmetry of this result, there has been a slight sacrifice of detail. Some applications may require the slightly stronger version:

**Theorem 2.** If $\alpha$ is irrational, $k > \frac{1}{2}$, and $p/q$ is a rational approximation to $\alpha$ (in reduced form) for which

$$q^2|\alpha - p/q| < k$$

then either $p/q$ is a convergent $p_n/q_n$ to $\alpha$ or $p/q$ has one of the following forms:

\begin{align*}
\text{(i)'} & \quad \frac{p}{q} = \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}}, \quad a > b \quad \text{and} \quad ab < 2k, \quad \text{or} \\
\text{(ii)'} & \quad \frac{p}{q} = \frac{ap_n - bp_{n-1}}{aq_n - bq_{n-1}}, \quad a < b \quad \text{and} \quad ab < k + a^2/a_{n+1}, \\
\text{(iii)'} & \quad \frac{p}{q} = \frac{ap_{n+1} + bp_{n-1}}{aq_{n+1} + bq_{n-1}}, \quad a > b \quad \text{and} \quad ab(1 - b/2a) < k,
\end{align*}

where $a$ and $b$ are positive integers.

Theorem 1 follows from Theorem 2 on writing $p_{n+1} = p_n + p_{n-1}, q_{n+1} = q_n + q_{n-1}$ if $p/q$ has the form (i)' where $a < b$ and $a_{n+1} = 1$. In all other cases of (i)' and (ii)' we have $ab < 2k$.

**Proof of Theorem 2.** We suppose $q^2|\alpha - p/q| < k$ and $p/q$ is not a convergent to $\alpha$. We write $\alpha$ and $p/q$ in the form described in Section 1 and
compare \( k \) with the bounds given by Lemma 4. We also make use of the fact that the graph of \( f_y(x) = x - x^2/y \) is a parabola with maximum at \( x = \frac{1}{2}y \) and axis of symmetry the line \( x = \frac{1}{2}y \). From this it follows that if \( m \leq \min(\beta, \gamma - \beta) \) then

\[
\beta - \beta^2/\gamma \geq m(1 - m/\gamma).
\]

It is convenient to break the proof into separate cases.

Case 1. \( r = 1, \beta = b_1 < \gamma \). In this case we have \( d = 1 \), and setting \( m = \min(b_1, a_{n+1} - b_1) \) we obtain from (1) and Lemma 4 that

\[
k > q^2|\alpha - p/q| > \beta - \beta^2/\gamma > m(1 - m/\gamma).
\]

Since \( \gamma > a_{n+1} > 2m \) we conclude that \( m < 2k \). On observing that if \( \beta = b_1 = m \) then

\[
p/q = (mp_n + p_{n-1})/(mq_n + q_{n-1})
\]

and if \( \beta = b_1 = a_{n+1} - m \) then

\[
p/q = (p_{n+1} - mp_n)/(q_{n+1} - mq_n).
\]

The proof of Theorem 2 is complete in this case.

Case 2. \( r > 1, \beta < \gamma \). To use (1) we write \( \beta \) as \( m + \rho/d \) if \( \beta \leq \frac{1}{2}a_{n+1} \) and as \( a_{n+1} - m - \rho/d \) if \( \beta > \frac{1}{2}a_{n+1} \), where we take \( 1 < \rho < d - 1 \). Using (1) and Lemma 4 we obtain

\[
k > q^2|\alpha - p/q| > d^2(\beta - \beta^2/\gamma)
\]

\[
> d^2(m + \rho/d)(1 - (m + \rho/d)/\gamma)
\]

\[
= d(dm + \rho)(1 - (m + \rho/d)/\gamma).
\]

If \( m > 1 \) we observe that, because \( \gamma > a_{n+1} > 2(m + \rho/d) \), it follows that \( d(dm + \rho) < 2k \). However if \( m = 0 \) we use the fact that \( \gamma > a_{n+1} > b_1 > 1 \), so \( \gamma > 2 \). This yields \( d\rho(1 - \rho/2d) < k \). Theorem 2 now follows in this case, on observing that if \( \beta = m + \rho/d \) then

\[
p/q = (ap_n + bp_{n-1})/(aq_n + bq_{n-1}),
\]

where \( a = dm + \rho \) and \( b = d \), and if \( \beta = a_{n+1} - m - \rho/d \) then

\[
p/q = (ap_{n+1} - bp_n)/(aq_{n+1} - bq_n),
\]

where \( a = d \) and \( b = dm + \rho \).

Case 3. \( r = 1, \beta = b_1 > \gamma \). In this case \( d = 1 \) and

\[
k > q^2|\alpha - p/q| > (\beta - \gamma)(\beta + 1)/(\gamma + 1) > \beta - \gamma.
\]

We observe that \( \gamma < a_{n+1} + 1/a_{n+2} \), so we have \( b_1 - a_{n+1} - 1/a_{n+2} < k \).

Writing \( b_1 = a_{n+1} + b \) we have

\[
p/q = (p_{n+1} + bp_n)/(q_{n+1} + bq_n),
\]

where \( b < k + 1/a_{n+2} \). Thus \( p/q \) has the form specified in (i).
Case 4. \( r > 1, \beta > \gamma \). In this case we write \( \beta = b_1 + \rho/d \) where \( 1 < \rho < d - 1 \). We have

\[
\begin{align*}
k > q^2|\alpha - p/q| &> d^2(\beta - \alpha)(\beta + 1)/(\gamma + 1) \\
&> d^2(\beta - \gamma) \\
&> d(db_1 + \rho - da_{n+1} - d/a_{n+2}).
\end{align*}
\]

Writing \( m = b_1 - a_{n+1} \) this gives \( d(dm + \rho) < k + d^2/a_{n+2} \). Since

\[
p/q = (dp_{n+1} + (dm + \rho)p_n)/(dq_{n+1} + (dm + \rho)q_n)
\]

the proof of Theorem 2 is now complete.

Since the particular case \( k = 2 \) arose earlier, and the result has a nice form in this case, we conclude with it.

**Corollary.** If \( q^2|\alpha - p/q| < 2 \) then either \( p/q \) is a convergent \( p_n/q_n \) to \( \alpha \) or \( p/q \) has one of the following two forms:

(i)'' \quad \frac{p}{q} = \frac{(ap_n + p_{n-1})}{(aqn + q_{n-1})}

\[ a = 1, 2, 3, a_{n+1} - 3, a_{n+1} - 2, a_{n+1} - 1, \]

(ii)'' \quad \frac{p}{q} = \frac{(ap_n + 2p_{n-1})}{(aqn + 2q_{n-1})} \quad a = 1 \text{ or } 2a_{n+1} - 1.

**Proof.** The possibilities given by Theorem 2 for form (i)' are \((a, b) = (1, 1), (2, 1), (3, 1), (1, 2)\) which correspond to the first three possibilities given for (i)'' and the first for (ii)''. The possibilities given by Theorem 2 for form (ii)' are \((a, b) = (1, 1), (1, 2), (1, 3), (2, 1)\) and correspond to the other possibilities given for (i)'' and (ii)''.

References


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